Applied Financial Economics Letters, 2005, **1**, 177–188

Stochastic market volatility models

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A new market-based approach to evaluating options on an asset is offered. The model corresponds to the real situations encountered in the market: option prices are not uniquely determined by their underlying asset but mainly by another factor, namely stochastic market volatility (or simply SMV). To begin constructing SMV, it is assumed that there exists a hedging portfolio which replicates perfectly the value of the underlying option. By 'perfectly', it is meant that the value of the hedging portfolio will always equal exactly to the option. The hedging portfolio takes asset price and SMV as its input, therefore, for a given asset price the correct value of SMV gives the correct value for the option. SMV presents the dynamics of options market. We provide the proof of existence and uniqueness of solutions for SMV.

I. Introduction

Black and Scholes' contribution (1973) to option pricing theory has won them plaudits for its formula for a European call option. Although the formula is widely known and has been successful in application, it fails to explain the structure of option prices across strike price and time-to-maturity. Researchers have developed option valuation models that relate the distribution of asset price to this structure of option prices and over time three classes of asset price models have emerged. These are stochastic volatility models, studied by Hull and White (1987) and Heston (1993) among others; discrete-time GARCH models employed by Engle and Mustafa (1992) and Duan (1995); and Lévy process models favoured by Barndorff-Nielsen (1997) and Madan *et al.* (1998). Although these models are often successful in explaining asset prices (for example they have explained that the volatility smile is often due to leptokurtic distributions of asset return) they frequently fails to explain the term structure of volatility. These models are often referred to as *smile models*.

Consequently, it has become usual for practitioners to quote an option price in terms of the so-called risk-neutral volatility or the *implied volatility* which is determined by the market. The Black–Scholes formula for the theoretical price $C(t)$ of a European call option at time t on an asset $S(t)$, strike K and maturing at time $T > t$ is given by

$$C(t) = S(t)N(h) - Ke^{-b(T-t)}N(k)$$

where b is the risk-free interest rate, N the standard normal distribution and

$$h = \frac{\ln(S/K) + (b + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}}$$
$$k = h - \sigma\sqrt{(T-t)}$$

the parameter σ is termed implied volatility when one matches $C(t)$ to the market price of the option. If the Black–Scholes model were correct, the implied volatility σ must be the asset volatility, however, this implied volatility is in fact often observed as a non-flat structure of strike K and time to maturity $T - t$, known as implied volatility surface.

Many researchers have attempted to formulate this implied volatility with or without introducing a non-traded source of risk. For instance, Dupire (1994, 1998) shows that it is possible to recover from market option prices a unique implied diffusion process for asset price. Local volatility models, otherwise known as deterministic volatility function

Applied Financial Economics Letters ISSN 1744–6546 print/ISSN 1744–6554 online
http://www.tandf.co.uk/journals
DOI: 10.1080/17446540500101986

models or implied tree models (implied binomial tree or implied trinomial tree models, depending on the type of tree), assume that future volatilities will be a deterministic function of the underlying asset and time. This function is implied by the current volatility skew and can be fully reflected in a suitably calibrated binomial or trinomial tree (Derman and Kani (1994), Dupire (1994) and Rubinstein (1994)).

A market-based approach is to directly model the implied volatility, in which the implied volatility is directly used as the state variable to describe the joint evolution of the market prices of options and their asset prices. This approach was adopted by a number of researchers, including Dumas *et al.* (1998) and Ait-Sahalia and Lo (2000). In the model of Cont and da Fonseca (2002), it is necessary to specify factors to model the joint dynamics of implied volatility across strikes and maturities. Arbitrage restrictions on stochastic models for implied volatility were considered in Schonbucher (1999), Ledoit and Santa Clara (1999) and Brace *et al.* (2002) and these works assumed that the underlying asset price is the main factor that influences the prices of options. Therefore, the dynamics of option prices must not be modelled alone without specifying the dynamics of their underlying asset.

Dumas *et al.* (1998) showed that while a model with a large number of parameters may calibrate well the implied volatility surface at a given time, the same model parameters may give a poor fit at future times. This creates the need for frequent re-calibration of the model.

The points discussed above have indicated that the dynamics of option prices are not simply described by the specification of the underlying asset, and furthermore, neither are they described by the many other factors that represent the correlation of option prices across strikes and maturities. These points motivate the study of the dynamics of the options market.

This study offers a market-based approach to modelling the dynamics of the options market. The model developed here is formed in terms of a state variable, namely stochastic market volatility. Stochastic market volatility (or simply SMV) represents the state of the options market which is familiar to options traders. Sufficient conditions are imposed on the dynamics of SMV so as to ensure the absence of arbitrage in the model. It is shown in the case of vanilla options, that SMV is the stochastic implied volatility of the Black–Scholes formula.

To begin constructing the dynamics of SMV it is assumed that there exists a hedging portfolio which replicates perfectly the value of an option. By 'perfectly', it is meant that the value of the hedging portfolio will always exactly equal to the value of option. The hedging portfolio takes asset price and SMV as its input, and therefore for a given asset price, the correct value of SMV gives the correct value for the option. SMV presents the way the market valuates options and determines the structure of option prices. The study provides the proof of the existence and uniqueness of solutions for SMV.

The model of SMV has some important advantages. First, market volatilities are observable since they are the market data of option prices. As opposed to market-based models in Schonbucher (1999) and Brace *et al.* (2002), there is no need to make any modelling assumptions on the process of the underlying asset. Second in a particular case, market volatilities are Black–Scholes implied volatilities, which are already used by practitioners when quoting option prices. Finally, there is no need to specify the factors that represent the joint dynamics of SMV across strikes and maturities as there is in Cont and da Fonseca (2002).

II. Stochastic Market Volatility Model

The goal, in this section, is to derive the dynamics of SMV in an arbitrage-free market as follows.

Definition 1. A contingent claim H is a stochastic variable of the form

$$H(u) = \phi(S, u), \quad u \in [t, T] \tag{1}$$

where ϕ is a real valued function. u denotes time prior maturity date T (or absolute maturity) so that $T - u$ is time-to-maturity (or relative maturity). S is the value of the underlying asset.

Definition 2. A portfolio strategy $\chi = (\xi, \eta)$ is a stochastic process and the value process L corresponding to the portfolio χ is defined by

$$L(u) = L(v, S, u) = \xi(v, u)S + \eta(v, u)B, \quad u \in [t, T] \tag{2}$$

where $\xi(v, u)$ is units of a traded asset S and $\eta(v, u)$ units of risk-free asset B, where ξ and η may be functions of asset price S. v is the *volatility* parameter for L.

Definition 3. A given contingent claim H is said to be *attainable* if there exists a portfolio χ such that

$$L(u) = H(u) \quad \text{for } u \in [t, T]$$

in which case the portfolio χ is a hedging portfolio or a replicating portfolio. The value process $L(u)$ has the final value

$$L(T) = H(T) = \phi(S, T) \tag{3}$$

Let χ be a hedging portfolio for a claim H and the value process L be a function of S and v as it evolves with time u and L admits a solution of the stochastic equation, namely the *backward* stochastic differential equation, of the form

$$L(t) + \int_t^T F(v,S,u)\,du + \int_t^T G(v,S,u)\,dW(u) = L(T) \tag{4}$$

where F and G are known and well-behaved functions. One may interpret this backward equation as the stochastic differential equation in the usual form

$$\begin{aligned} dL(u) &= F(u)\,du + G(u)\,dW(u) \\ &= F(v,S,u)\,du + G(v,S,u)\,dW(u) \end{aligned} \tag{5}$$

with a final value $L(T)$, where W is the standard Brownian motion, and G is continuously differentiable with respect to S and v. Let $v = \{v(u) : t \le u \le T\}$ be a solution of the stochastic differential equation (SDE)

$$\begin{aligned} dv(u) &= f\,du + g\,dW(u) \\ &= f\,(v(u),S,u)\,du + g(v(u),S,u)\,dW(u) \end{aligned} \tag{6}$$

where g is a known and continuously differentiable with respect to S and v and f is an arbitrary well-behaved function.

Now applying Itô–Venttsel theorem to SDEs 5 and 6 one obtains a new SDE for L.

$$\begin{aligned} dL(u) = &\left\{F + f\,\frac{\partial L}{\partial v} + \frac{1}{2}g^2\frac{\partial^2 L}{\partial v^2} + g\frac{\partial G}{\partial v}\right\}du \\ &+ \left\{G + g\frac{\partial L}{\partial v}\right\}dW(u) \end{aligned} \tag{7}$$

Standard arbitrage arguments (Black and Scholes, 1973; Merton, 1973) state that the market portfolio; L must earn its rate of return equal to the risk-free rate b, that is

$$F + f\,\frac{\partial L}{\partial v} + \frac{1}{2}g^2\frac{\partial^2 L}{\partial v^2} + g\frac{\partial G}{\partial v} = bL \tag{8}$$

Definition 4. A volatility v is said to be a volatility process of the portfolio L if $F = bL$ and the arbitrage condition (Equation 8) holds.

The interpretation of this definition is that the dynamics of the volatility can be pre-determined so that the portfolio L always earns a risk-free interest rate b. It is not hard to see that Equation 8 holds for arbitrarily known functions F, G, f and g, this property allows one to choose one in infinitely many drift functions f so that this equation holds. This is the property of infinitely many equivalent martingale measures under which the discounted values $e^{-bt}L(t)$ of options are martingales.

Following definition 4, a drift function f is proposed as follows

$$f = \frac{\partial v}{\partial L}\left\{F - g\frac{\partial G}{\partial v} - \frac{1}{2}g^2\frac{\partial^2 L}{\partial v^2} - bL\right\} \tag{9}$$

which clearly satisfies Equation 8. Therefore, the dynamics of v now becomes

$$\begin{aligned} dv(u) = &\frac{\partial v}{\partial L}\left\{F - g\frac{\partial G}{\partial v} - \frac{1}{2}g^2\frac{\partial^2 L}{\partial v^2} - bL\right\}(u)\,du \\ &+ g(v(u),S,u)\,dW(u) \end{aligned} \tag{10}$$

This is the formulation of volatility v based on the portfolio strategy χ in Equation 2. The definition of v is made as precise as follows.

Definition 5. A volatility v is said to be a *market* volatility of the portfolio L if its portfolio strategy χ is unique. The portfolio L is then said to be a market portfolio.

Suppose that there exists in the market a traded asset S, a risk-free asset B and a risk-neutral martingale measure Q under which S/B is a Q-martingale. Under Q, suppose that the asset price $S(t)$ and risk-free asset $B(t)$ are solutions to:

$$\begin{aligned} &dS(t) = \mu(S(t),t)\,dt + \sigma(S(t),t)\,dW^S(t) \\ \text{and}\quad &dB(t) = bB(t)\,dt \end{aligned} \tag{11}$$

respectively, where W^S is the standard Brownian motion which may be correlated with W^S with coefficient of correlation ρ, and b is the instantaneous risk-free rate.

The study has derived the dynamics of v and one may then introduce $S = S(u)$, $t \le u \le T$, as a price process into the dynamics of v so that Equation 10 can be rewritten in the form of Itô integrals

$$\begin{aligned} v(t) &+ \int_t^T f\,(v(u),S(u),u)\,du \\ &+ \int_t^T g(v(u),S(u),u)\,dW(u) = v(T) \end{aligned} \tag{12}$$

whose solution exists and is unique under certain conditions. The proof of existence and uniqueness for Equation 12 is provided in Section VII.

The relation between the underlying asset S and the market volatility v can be seen as follows. Let $\mathcal{F}_S(t)$ and $\mathcal{F}_v(t)$ be the filtrations generated by $S(t)$ and $v(t)$, respectively. In general, $\mathcal{F}_S(t)$ does not contain $\mathcal{F}_v(t)$ and vice versa. This property interprets an important fact: that option prices are not only determined by the underlying asset but also by other factors and that the implied dynamics of the underlying asset cannot be fully retrieved using market prices of options. In fact, this has been a problem of calibrating smile models to option prices.

This section ends with a word of criticism. It has been assumed that there actually exists an arbitrage-free market portfolio L that hedges a contingent claim on an asset S. This assumption is crucial for the procedure since one is constructing SMV based on the portfolio L and the asset S. If such a market portfolio does not exist then the dynamics of SMV cannot be formed.

III. Option Valuation Market Model

This section introduces an options valuation model with a stochastic market volatility. The model shows that the stochastic market volatility, in general, is not the volatility of the underlying asset and that it no longer exists once the option expires. With the use of the risk-neutral probability measure and a general change of numéraire, Geman *et al.* (1995) show that the arbitrage-free price of a European call option on the asset S at time t, maturing at time $T > t$ with exercise price K, is of the form

$$L(t) = S(t)\mathbb{Q}_1(S(T) > K) - Ke^{-b(T-t)}\mathbb{Q}_2(S(T) > K) \quad (13)$$

where one interprets this formula as follows. $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are any conditional probabilities (functions) of S, K and T such that at time $t = T$ if $S \geq K$ one has both $\mathbb{Q}_1 = \mathbb{Q}_2 = 1$ and if $S < K$, $\mathbb{Q}_1 = \mathbb{Q}_2 = 0$. In addition, Margrabe (1978), Klebaner (2002) and Le (2003) show that the value of a call option is given by

$$L = S\frac{\partial L}{\partial S} + K\frac{\partial L}{\partial K} \quad (14)$$

where S is the current asset price. Using these formulae, one can construct such a replicating portfolio as in Equation 2 in the form

$$L(u) = S\mathbb{Q}_1(S, v, u) - Ke^{-b(T-u)}\mathbb{Q}_2(S, v, u) \quad (15)$$

A plausible candidate which has such functions $\mathbb{Q}_i$'s is the Black–Scholes formula.

In the previous section, the dynamics of SMV is derived from a given replicating portfolio of the underlying option. This is a monotonic relation between the dynamics of SMV and that of the option. In general, the dynamics of SMV may be the same for different options on the same asset. Next, a number of formulations of SMV are given for a vanilla call option.

IV. First Formulation of SMV

The Black–Scholes formula and stochastic implied volatility

Let the dynamics Equation 5 of L be of the form

$$dL(u) = \left\{\frac{1}{2}I^2(S,u)S^2\frac{\partial^2 L}{\partial S^2} + bS\frac{\partial L}{\partial S} + \frac{\partial L}{\partial u}\right\}du + I(S,u)S\frac{\partial L}{\partial S}\,dW(u) \quad (16)$$

where $I := I(S, u)$ is a well-behaved and bounded function of asset S and time u. I can be the observed *volatility smile* or *volatility surface*. Let the replicating portfolio L in Equation 2 for a vanilla call option be known as a smooth function of S, u and v

$$L(u) = SN(h(u)) - Ke^{-b\tau}N(k(u)) \quad (17)$$

where

$$h(u) = \frac{x + (b + v/2)\tau}{\sqrt{v\tau}}, \quad k(u) = h(u) - \sqrt{v\tau},$$
$$x(t) = \ln(S/K), \quad \tau = T - u$$

N is the standard normal distribution function and v is SMV. Since the proposed option model is the Black–Scholes formula, SMV is also known as *stochastic implied volatility* (SIV), see Carr (2002) and Brace *et al.* (2002). When $v = I^2(S, t) =$ a constant for all t, Equation 16 becomes the well-known Black–Scholes equation for L.

The following are derivative formulae of Equation 17:

$$\frac{\partial L}{\partial S} = N(h), \quad \frac{\partial^2 L}{\partial^2 S} = \frac{n(h)}{S\sqrt{v\tau}}$$
$$\frac{\partial L}{\partial \tau} = Sn(h)\frac{\partial h}{\partial \tau} - e^{-b\tau}Kn(k)\frac{\partial k}{\partial \tau} + bKe^{-b\tau}N(k) = \frac{Sn(h)\sqrt{v}}{2\sqrt{\tau}} + bKe^{-b\tau}N(k)$$
$$\frac{\partial L}{\partial v} = \frac{Sn(h)\sqrt{\tau}}{2\sqrt{v}}, \quad \frac{\partial L}{\partial K} = -e^{-b\tau}N(k)$$
$$\frac{\partial^2 L}{\partial v^2} = \frac{\sqrt{\tau}Sn(h)(hk-1)}{4v\sqrt{v}}, \quad \frac{\partial^2 L}{\partial S\,\partial v} = -\frac{kn(h)}{2v}$$

Equation 14 has been used and the well-known identity

$$Sn(h) = Ke^{-b\tau}n(k) \quad (18)$$

where $n(h) = \partial N(h)/\partial h$.

Using the derived formulae, Equation 9 becomes

$$f = \frac{I^2 - v}{\tau} + \frac{gIk}{\sqrt{v\tau}} - \frac{g^2(hk-1)}{4v} \tag{19}$$

Let the asset price $S(t)$ under Q follow the dynamics

$$dS(t) = bS(t)\,dt + \sigma(S,t)S(t)\,dW^S \tag{20}$$

where σ is a well-behaved and bounded function of asset price S and time t for all $0 \le t \le T$. The backward SDE (Equation 10) of $v(u)$ becomes

$$\begin{aligned} dv(u) &= \left\{ \frac{I^2(S(u),u) - v(u)}{T-u} + \frac{g(v(u),S(u),u)I(S(u),u)k(u)}{\sqrt{v(u)(T-u)}} \right. \\ &\quad \left. - \frac{g(v(u),S(u),u)^2(h(u)k(u)-1)}{4v(u)} \right\} du \\ &\quad + g(v(u),S(u),u)\,dW(u). \end{aligned} \tag{21}$$

In integral form,

$$\begin{aligned} v(S(t),t) &+ \int_t^T \left\{ \frac{I^2(S(u),u) - v(u)}{T-u} + \frac{g(v(u),S(u),u)I(S(u),u)k(u)}{\sqrt{v(u)(T-u)}} \right. \\ &\quad \left. - \frac{g(v(u),S(u),u)^2(h(u)k(u)-1)}{4v(u)} \right\} du \\ &+ \int_t^T g(v(u),S(u),u)\,dW(u) \\ &= v(S(T),T) = I^2(S(T),T) \end{aligned}$$

As a trivial example, let $g=0$ then the backward SDE (Equation 21) becomes:

$$dv(t) = \frac{1}{T-t}(I^2(S,t) - v)\,dt \quad \text{or}$$
$$v(S(t),t) + \int_t^T \frac{1}{T-u}(I^2(S,u) - v)\,du = I^2(S(T),T) \tag{22}$$

It is observed that I^2 behaves as the long-run mean of the market volatility v. When it is close to maturity, v converges rapidly to its long-run mean I^2.

$$\lim_{t \to T} v(t) = I^2(S,T) \tag{23}$$

If g is linear of v and is replaced by $4\beta(y(t),t)v(T-t)$ and let $y(t) = x(t) + b(T-t)$, where β is a bounded function of y and $t < T$.

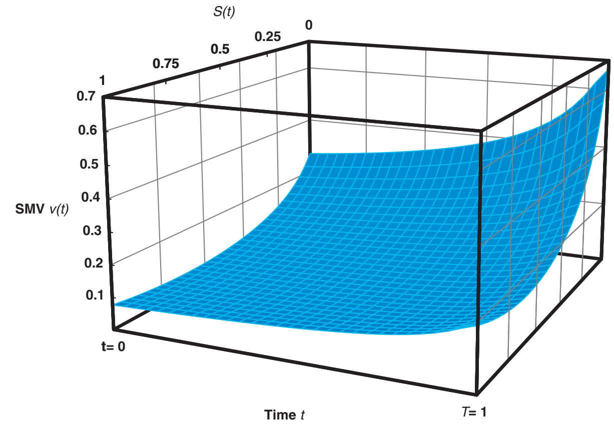

Fig. 1. When $g=0$ and fix asset price $S(t)=S$ for each $t \in [0,T)$, SIV in Equation 2 is plotted as a deterministic function of asset price $S>0$ and time $0 \le t T$, where the volatility smile $I(S,t) = a_0 + a_1 \exp\{-a_2 S\} + a_3 S$ for some nonnegative constants a_0, a_1, a_2 and a_3

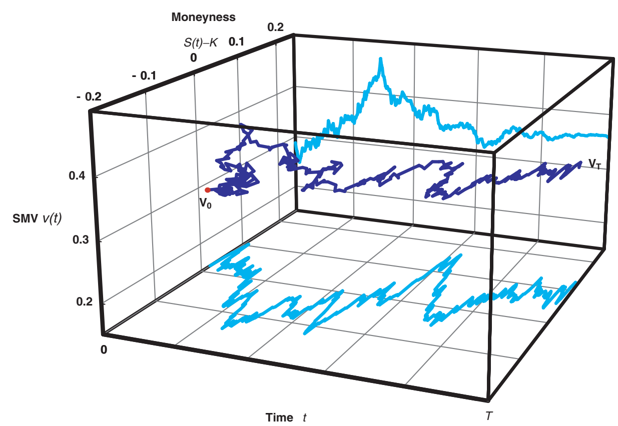

Fig. 2. A sample path of SIV in 3D plotted against moneyness $S(t)-K$ and time $t \in [0,T)$. Its shadows $v(t)$ and $S(t)-K$ are on SMV-Time plane (back wall) and Moneyness-Time plane (floor) respectively. $I(S,t) = 0.25 + 0.4\ \exp\{2S\} + 0.1S$ and $\beta(S,t) = 0.4S$

Equation 21 becomes

$$\begin{aligned} dv(t) &= \left\{ \frac{1}{\tau}\left(I^2(y,t) - v\right) + 2\beta(y,t)I(y,t)(2y - v\tau) \right. \\ &\quad \left. - \beta^2(y,t)\left[4y^2 - v^2\tau^2 - 4v\tau\right]\tau \right\} dt \\ &\quad + 4\beta(y,t)v\tau\,dW(t) \end{aligned} \tag{24}$$

According to $v(t)$, when t tends to T the second and third terms in the drift become insignificant, therefore again $v(t)$ converges to $I^2(y,T)$.

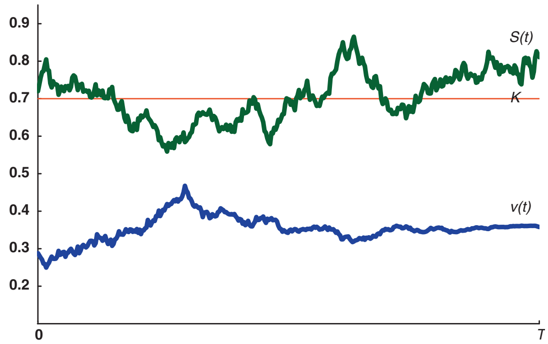

Fig. 3. Sample paths of asset price $S(t)$, SIV $\sqrt{v(t)}$ are given, negative correlation between asset price and SIV is clearly seen

STOCHASTIC MARKET VOLATILITY

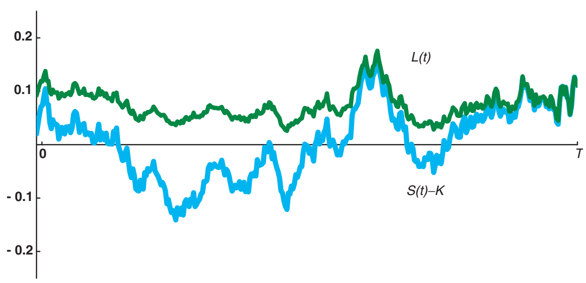

Fig. 4. Sample paths of replicating portfolio $L(t)$ and payoff function $S(t)-K$

Implementation

This section implements the option market model (Equation 17) with SIV (Equation 21). The following simple functional forms for I and β are assumed:

$$\sigma(S,t) = \sigma = \text{positive constant}$$
$$I(S,t) = a_0 + a_1 \exp\{-a_2 S\} + a_3 S, \quad \beta(S,t) = a_4 S$$

where a_0, a_1, a_2, a_3 and a_4 are nonnegative constants. The following system is simulated:

$$dS(t) = bS(t)\,dt + \sigma S(t)\,dW^S(t) \tag{25}$$

$$dv(t) = \left\{ \frac{1}{\tau}\left(I^2(S,t) - v\right) + 2\beta(y,t)I(S,t)(2y - v\tau) + \beta^2(y,t)\left[4y^2 - v^2\tau^2 - 4v\tau\right]\tau \right\} dt + 4\beta(y,t)v\tau\,dW(t) \tag{26}$$

As an example, consider a European call option to be hedged with a market portfolio L at time $t=0$ on asset $S(0)=0.72$, strike $K=0.7$, risk-free rate $r=0$, initial SIV $v(0)=0.3^2$, asset volatility $\sigma=0.4$, time-to-maturity $T=1$ and correlation coefficient $\rho=-0.75$.

Remark. When implementing the system (Equation 25), it is reasonable to assume that $\sigma(S,t) = I(S,t)$ so that at maturity T, the market volatility matches to asset volatility, that is, following Equation 23 one has $v(T) = \sigma(S,T)$.

V. Second Formulation of SMV

This section considers a case similar to the first formulation of SMV except that the function I is now a surface of the functional form

$$I(S,t)^2 = \exp\left\{ \sum_{i=0}^{N} a_i x^i e^{-b_i(T-t)} \right\} \tag{27}$$

Figure 5 gives an example of such initial volatility surface where moneyness is defined as S/K.

Let $f(S,T)$ be the marginal distribution, and the price C of a vanilla call option with strike K and maturity T be written as

$$C(K,T) = \int_0^\infty (S-K)^+ f(S,T)\,dS$$

Breeden and Litzenberger (1978) show that the marginal distribution can be recovered by

$$C(K,T) = \int_K^\infty (S-K) f(S,T)\,dS$$
$$\frac{\partial C(K,T)}{\partial K} = -\int_K^\infty f(S,T)\,dS$$
$$\frac{\partial^2 C(K,T)}{\partial K^2} = f(K,T).$$

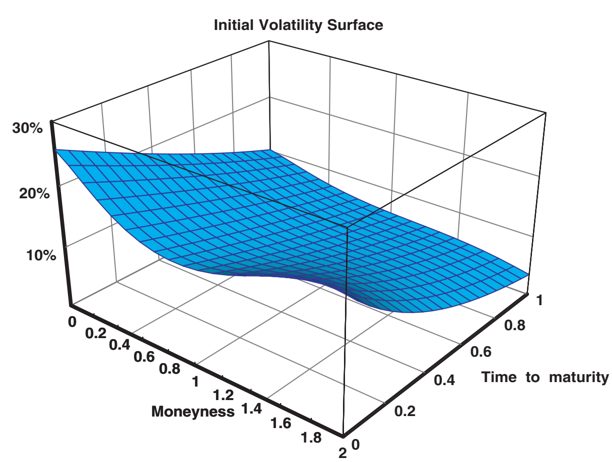

Fig. 5. An initial volatility surface with $a_5 = 0.1158$, $b_5 = 0.4$, $a_4 = -0.9152$, $b_4 = 0.35$, $a_3 = 2.1242$, $b_3 = 0.4$, $a_2 = -1.2976$, $b_2 = 0.35$, $a_1 = -0.5345$, $b_1 = 0.09$, $a_0 = -1.3565$ and $b_0 = -0.52$

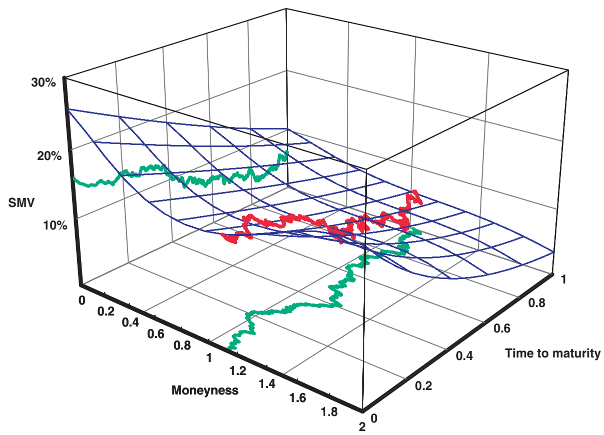

Fig. 6. A sample of SMV is plotted along with its initial volatility surface. Its shadows are also given in moneyness and in time-to-maturity

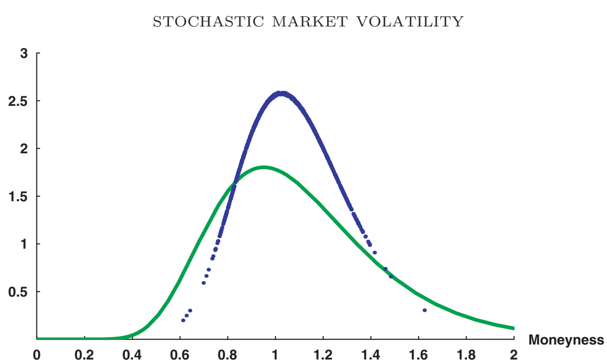

Fig. 7. Marginal distributions from the Black–Scholes model (green) and SMV of second formulation (blue). Asset dynamics is $dS(t)=rS(t)\,dt+\sigma S(t)\,dB(t)$ with $r=0$, $\sigma=0.2$, $S(0)=0.72$, $K=0.7$ and time to maturity $T=0.5$. SMV initial value $v(0)=I(S(0),0)^2$. The correlation between B and W is $\rho=-0.5$

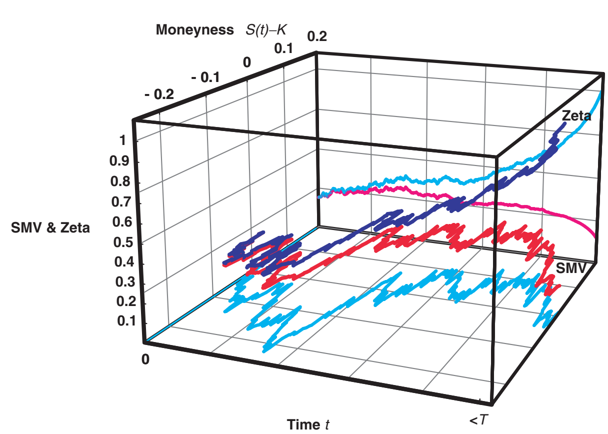

Fig. 8. A sample path of SMV and Zeta in 3D plotted against moneyness $S(t)-K$ and time $t\in[0,T)$ and their shadows $v(t)$, $\zeta(t)$ and $S(t)-K$ are on SMV & Zeta-Time plane (back wall) and Moneyness-Time plane (floor) respectively

To illustrate the effects on option prices, the default initial volatility surface in Fig. 5 is used. For comparison, the Black–Scholes model is used, where the asset volatility σ is a constant. SMV effects the skewness and shape of the marginal distribution. Figure 5 shows the empirical marginal distribution of the SMV model with high peak thin tails (using 100 000 simulations) and the analytical marginal distribution of Black–Scholes model with lower peak and fatter tails, respectively. These simulations show that SMV models can produce a great variety of effects on option prices compared with the Black–Scholes model.

VI. Third Formulation of SMV

This section considers the case where SMV vanishes at maturity. Let the dynamics (Equation 5) of L be of the form

$$dL(u)=\left\{\frac{I(S,u)-v}{T-u}S^2\frac{\partial^2 L}{\partial S^2}+bS\frac{\partial L}{\partial S}+\frac{\partial L}{\partial u}\right\}du + S\frac{\partial L}{\partial S}dW(u) \quad (28)$$

and the asset price $S(t)$ under Q follow a dynamics

$$dS(t)=bS(t)\,dt+\sigma(S(t),t)SdW^S(t) \quad (29)$$

where σ is a bounded function of asset price S and time t for all $0\le t\le T$ and I again is volatility smile. Let the portfolio in (2.2) be known and given by

$$L(u)=S(t)N(h)-Ke^{-b\tau}N(k) \quad (30)$$

where

$$h=\frac{x(t)+v}{\sqrt{v}},\quad k=h-2\sqrt{v} \quad (31)$$

$$x(t)=\ln(S(t)/K),\quad \tau=T-u \quad (32)$$

The following are derivative formulae of Equation 30

$$\frac{\partial L}{\partial S}=N(h),\quad \frac{\partial^2 L}{\partial S^2}=\frac{n(h)}{S\sqrt{v}}$$
$$\frac{\partial L}{\partial \tau}=bKe^{-b\tau}N(k),\quad \frac{\partial L}{\partial v}=\frac{Sn(h)}{\sqrt{v}}$$
$$\frac{\partial^2 L}{\partial v}=\frac{Sn(h)(hk-1)}{2v\sqrt{v}},\quad \frac{\partial^2 L}{\partial S\,\partial v}=-\frac{kn(h)}{2v}.$$

Equation 9 becomes

$$f=\frac{I-v}{\tau}+\frac{gk}{2\sqrt{v}}-\frac{g^2(hk-1)}{4v} \quad (33)$$

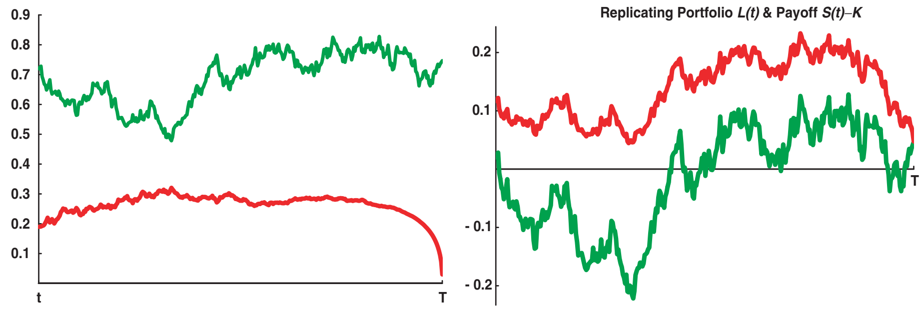

Fig. 9. On the left, sample paths of asset price $S(t)$, SMV $\sqrt{v(t)}$ are given, and negative correlation between asset price and SMV can be seen. On the right are sample paths of replicating portfolio $L(t)$ and payoff function $S(t) - K$

Therefore, the backward SDE (2.10) of $v(t)$ can be written as

$$dv(t) = \left\{ \frac{I(S(t),t) - v(t)}{T-t} + \frac{g(v(t),S(t),t)k}{2\sqrt{v(t)}} - \frac{g(v(t),S(t),t)^2(hk-1)}{4v(t)} \right\} dt + g(v(t),S(t),t)dW(t). \quad (34)$$

To implement the dynamics of Equation 34, one considers simple choices for $g(v,\tau) = 2v\tau$, $\sigma(S,t) = \sigma = constant$ and $I(S,t) = 0$ and simulates the system

$$dv(t) = \left\{ \frac{-v}{T-t} + (x-v)(T-t) - (x^2 - v^2 - v)(T-t)^2 \right\} dt + 2v(T-t)\, dW(t) \quad (35)$$

$$dS(t) = bS(t)\, dt + \sigma S(t)\, dW^S(t) \quad (36)$$

Define the implied volatility process

$$\zeta(t) = \frac{v(t)}{T-t} \quad (37)$$

Figure 9 presents sample paths of asset price $S(t)$, SMV $v(t)$, the implied volatility $\zeta(t)$, portfolio $L(t)$ and the payoff $S(t) - K$.

VII. Existence and Uniqueness of Solution

This section studies the form of stochastic equations in which SMV exists, namely the backward stochastic differential equations (BSDE) of the form

$$v(t) + \int_t^T f(v(s),x(s),s)\, ds + \int_t^T g(v(s),x(s),s)\, dW(s) = v(T) \quad (38)$$

on the interval $I = [0,T]$. In the situation presented in Section III the target $v(T) = V$ is given and one has freedom to formulate a pair f, g under some conditions and to choose $x(t)$ in such a way that $v(t)$ evolves through the time interval $[0,T]$ and reaches $v(T) = V$. Writing Equation 38 in differential form gives

$$dv(t) = f(v(t),x(t),t)\, dt + g(v(t),x(t),t)\, dW(t) \quad (39)$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions. Let $W = \{W(t)\}_{t\geq 0}$ be a one-dimensional Brownian motion defined on the probability space adapted to the filtration.

Definition 6. Let $0 \leq a < b < \infty$. Let $\mathcal{L}^2([a,b],\mathbb{R})$ be the space of all real-valued measurable $\{\mathcal{F}_t\}$-adapted processes $f = \{f(t)\}_{a\leq t\leq b}$ such that

$$\|f\|^2 = E\int_a^b |f(t)|^2\, dt < \infty \quad (40)$$

If there exists $\bar{f} \in \mathcal{L}^2$ such that $\|f - \bar{f}\|^2 = 0$, f and $\bar{f}$ are said to be equivalent and we write $f = \bar{f}$. Let ε be the space of all real-valued stochastic processes $g = \{g(t)\}_{a\leq t\leq b} \cdot g$ is called a simple process if there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$ and bounded random variables ξ_i, $0 \leq i \leq n-1$ such that ξ_i is $\mathcal{F}_{t_i}$-measurable and

$$g(t) = \xi_0 I_{[t_0,t_1]}(t) + \sum_{i=1}^{n-1} \xi_i I_{(t_i,t_{i+1}]}(t) \quad (41)$$

Let us give a precise definition of a solution to the BSDE (Equation 38).

Definition 7. A pair of stochastic processes

$$(v,x) = \{v(t),x(t)\}_{0\leq t\leq T} \in \mathcal{L}^2 \times \mathcal{L}^2$$

is a solution of the BSDE (Equation 38) if the following properties are satisfied:

(i) $f(v(t),x(t),(t))$ and $g(v(t),x(t),(t)) \in \mathcal{L}^2$;
(ii) and Equation 38 holds for all $t \in [0,T]$ with probability 1.

The solution (v, x) is unique if for any other solution $(\bar{v}, \bar{x})$ one has

$$P[v(t) = \bar{v}(t) \quad \text{for all } 0 \le t \le T] = 1$$

and

$$E\int_0^T \left|x(s) - \bar{x}(s)\right|^2 ds = 0$$

Theorem Pardoux and Peng (1990) was proven with a small class of functions g. Using an analogous technique we extend the theorem to a larger class of functions g. Let us proceed to the proof of existence and uniqueness of SMV.

Theorem. *Suppose that $f(v, x, t)$ and $g(v, x, t) \in \mathcal{L}^2$, there exist positive constants K and small δ such that*

$$\left|f(v, x, t) - f(\bar{v}, \bar{x}, t)\right|^2 \le K\left(|v - \bar{v}|^2 + |x - \bar{x}|^2\right) \quad a.s. \tag{42}$$

$$\delta K\left(|v - \bar{v}|^2 + |x - \bar{x}|^2\right) \le \left|g(v, x, t) - g(\bar{v}, \bar{x}, t)\right|^2 \le K\left(|v - \bar{v}|^2 + |x - \bar{x}|^2\right) \quad a.s. \tag{43}$$

for all $v, \bar{v}, x, \bar{x} \in \mathbb{R}$ and $t \in [0, T]$. Then there exists a unique solution $(v(t), x(t))$ to Equation 38 in $\mathcal{L}^2 \times \mathcal{L}^2$.

Proof. Let us first prove the uniqueness. Let (v, x) and $(\bar{v}, \bar{x})$ be the two solutions. From Equation 39, one can see that

$$d[v(t) - \bar{v}(t)] = [f(v(t), x(t), t) - f(\bar{v}(t), \bar{x}(t), t)]\,dt + [(g(v(t), x(t), t) - g(\bar{v}(t), \bar{x}(t), t)]\,dW(t)$$

Applying Itô's formula one has

$$\begin{aligned} d[v(t) - \bar{v}(t)]^2 &= 2[v(t) - \bar{v}(t)][f(v(t), x(t), t) - f(\bar{v}(t), \bar{x}(t), t)]\,dt \\ &\quad + [(g(v(t), x(t), t) - g(\bar{v}(t), \bar{x}(t), t)]^2 dt \\ &\quad + 2[v(t) - \bar{v}(t)][(g(v(t), x(t), t) - g(\bar{v}(t), \bar{x}(t), t)]\,dW(t) \end{aligned}$$

Hence one has in integral form

$$\begin{aligned} -[v(t) - \bar{v}(t)]^2 &= 2\int_t^T [v(s) - \bar{v}(s)][f(v(s), x(s), s) - f(\bar{v}(s), \bar{x}(s), s)]\,ds \\ &\quad + \int_t^T [(g(v(s), x(s), s) - g(\bar{v}(s), \bar{x}(s), s)]^2\,ds \\ &\quad + 2\int_t^T [v(s) - \bar{v}(s)][(g(v(s), x(s), s) \\ &\quad - g(\bar{v}(s), \bar{x}(s), s)]\,dW(s). \end{aligned}$$

Now taking expectation both sides gives one

$$\begin{aligned} &E[v(t) - \bar{v}(t)]^2 + E\int_t^T [(g(v(s), x(s), s) - g(\bar{v}(s), \bar{x}(s), s)]^2\,ds \\ &= 2E\int_t^T [v(s) - \bar{v}(s)][f(v(s), x(s), s) - f(\bar{v}(s), \bar{x}(s), s)]\,ds \end{aligned}$$

Making use of the Lipschitz condition (Equation 43), it follows that

$$\begin{aligned} &E[v(t) - \bar{v}(t)]^2 + \delta KE\int_t^T [v(s) - \bar{v}(s)]^2 ds + \delta KE\int_t^T [x(s) - \bar{x}(s)]^2\,ds \\ &\le -2E\int_t^T [v(s) - \bar{v}(s)][f(v(s), x(s), s) - f(\bar{v}(s), \bar{x}(s), s)]\,ds. \end{aligned}$$

Applying the Lipschitz condition (Equation 42) and an inequality from the elementary calculus $2ab \le a^2/\epsilon + \epsilon b^2$, $\epsilon > 0$, one has:

$$\begin{aligned} &E[v(t) - \bar{v}(t)]^2 + \delta KE\int_t^T [v(s) - \bar{v}(s)]^2\,ds \\ &\quad + \delta KE\int_t^T [x(s) - \bar{x}(s)]^2\,ds \le \frac{1}{\epsilon}E\int_t^T [v(s) - \bar{v}(s)]^2\,ds \\ &\quad + \epsilon KE\int_t^T [v(s) - \bar{v}(s)]^2\,ds + \epsilon KE\int_t^T [x(s) - \bar{x}(s)]^2\,ds \end{aligned}$$

which, by setting $\epsilon = 1/K^2 < \delta = 1/K$, implies

$$E[v(t) - \bar{v}(t)]^2 \le K^2 E\int_t^T [v(s) - \bar{v}(s)]^2\,ds$$

Applying Gronwall's lemma gives us

$$E[v(t) - \bar{v}(t)]^2 = 0 \quad \text{for all } 0 \le t \le T$$

we write $v(t) = \bar{v}(t)$ for all $0 \le t \le T$ a.s. Therefore, one can easily see from (Equation 44) that

$$E\int_0^T [x(s) - \bar{x}(s)]^2\,ds = 0 \tag{44}$$

This completes the proof of uniqueness.

The existence is now proved. First let us prove that there exists a unique $v(t)$ in $\mathcal{L}^2$ such that

$$v(t) + \int_t^T f(s)\,ds + \int_t^T g(x(s), s)\,dW(s) = v(T) \tag{45}$$

for all $t \in [0, T]$. Define

$$M(T) = v(T) - \int_0^T f(s)\, ds$$

and

$$M(t) = E\left[v(T) - \int_0^T f(s)\, ds \middle| \mathcal{F}_t \right]$$

Then $M(t)$ is a square integrable martingale. By martingale representation theorem, there exists a unique $g \in \mathcal{L}^2$ such that

$$M(t) = M(0) + \int_0^t g(x(s), s)\, dW(s)$$

Noting that

$$\begin{aligned} M(T) - M(t) &= \int_0^T g(x(s), s)\, dW(s) - \int_0^t g(x(s), s)\, dW(s) \\ &= \int_t^T g(x(s), s)\, dW(s) \end{aligned}$$

then

$$v(T) - \int_0^T f(s)\, ds - M(t) = \int_t^T g(x(s), s)\, dW(s)$$

Define

$$v(t) = \int_0^t f(s)\, ds + M(t)$$

Clearly, $v(t)$ in $\mathcal{L}^2$ therefore

$$v(T) - \int_t^T f(s)\, ds - v(t) = \int_t^T g(x(s), s)\, dW(s)$$

which is Equation 45. The proof of uniqueness above also proves that Equation 45 is unique. For every $n = 0, 1, 2, \ldots,$ one writes recursively

$$\begin{aligned} v_n(t) &+ \int_t^T f(v_{n-1}(s), x_{n-1}(s), s)\, ds \\ &+ \int_t^T g(v_{n-1}(s), x(s), s)\, dW(s) = v(T) \end{aligned} \quad (46)$$

In the same way the uniqueness was proved, one can show that

$$\begin{aligned} E[v_{n+1}(t) - v_n(t)]^2 &+ \delta K E \int_t^T [v_n - v_{n-1}]^2\, ds \\ &+ \delta K E \int_t^T [x_{n+1} - x_n]^2\, ds \\ &\le \frac{1}{\epsilon} E \int_t^T [v_{n+1} - v_n]^2\, ds + \epsilon K E \int_t^T [v_n - v_{n-1}]^2\, ds \\ &+ \epsilon K E \int_t^T [x_n - x_{n-1}]^2\, ds \end{aligned} \quad (47)$$

which, by setting $\epsilon = 1/K^2,\ \delta = 1/K$, implies that

$$\begin{aligned} E[v_{n+1}(t) - v_n(t)]^2 &+ E \int_t^T [x_{n+1} - x_n]^2\, ds \\ &\le K^2 E \int_t^T [v_{n+1} - v_n]^2\, ds + \frac{1}{K} E \int_t^T [x_n - x_{n-1}]^2\, ds \end{aligned} \quad (48)$$

Denote

$$a_n(t) = E \int_t^T [v_n - v_{n-1}]^2\, ds$$

$$b_n(t) = E \int_t^T [x_n - x_{n-1}]^2\, ds$$

then from the last inequality it follows that

$$-\frac{d}{dt}\left(e^{K^2 t} a_{n+1}(t)\right) + e^{K^2 t} b_{n+1}(t) \le \frac{1}{K} e^{K^2 t} b_n(t)$$

Integrating both sides of this inequality from t to T gives

$$a_{n+1}(t) + \int_t^T e^{K^2(s-t)} b_{n+1}(s)\, ds \le \frac{1}{K} \int_t^T e^{K^2(s-t)} b_n(s)\, ds \quad (49)$$

This in particular, implies that

$$\begin{aligned} \int_0^T e^{K^2 s} b_{n+1}(s)\, ds &\le \frac{1}{K} \int_0^T e^{K^2 s} b_n(s)\, ds \cdots \\ &\le \frac{1}{K^n} b_1(0) \int_0^T e^{K^2 s}\, ds \le \frac{C e^{K^2 T}}{K^{n+2}} \end{aligned}$$

where $C = b_1(0) = E \int_0^T [x_1(s)]^2\, ds$. It then follows from Equations 48 and 49 that

$$a_{n+1}(0) \le \frac{C e^{K^2 T}}{K^{n+2}}$$

and

$$\begin{aligned} b_{n+1}(0) &\le K^2 a_{n+1}(0) + \frac{1}{K} b_n(0) \\ &\le \frac{C e^{K^2 T}}{K^n} + \frac{1}{K} b_n(0) \le \cdots \le \frac{1}{K^n}\left[n C e^{K^2 T} + b_1(0) \right] \end{aligned}$$

The last two inequalities show that $\{v_n\}$ and $\{x_n\}$ are Cauchy sequences in $\mathcal{L}^2$. Let v and x be their limits respectively and let $n \to \infty$, and one obtains Equation 38 which has a solution (v, x). This completes the proof of existence. Thus, the proof of the theorem is complete.

VIII. Conclusion

This paper has proven that formulation of SMV is possible under certain conditions. We have provided the proof of existence and uniqueness of solutions. A fine formulation of SMV is a sound pricing tool for path-dependent options. Implementation can be easily carried out.

Appendix: Itô–Venttsel Formula

Suppose that one is given a probability space $(\Omega, \mathcal{F}, P)$; $\{\mathcal{F}_t\}_{t\in[0,1]}$ is a family of expanding σ-algebras, $\mathcal{F}_t \in \mathcal{F}$, and $W_t : t \in [0, 1]$ is the standard Wiener process compatible with $\{\mathcal{F}_t\}_{t\in[0,1]}$.

Theorem 1. Suppose that $f(t, v)$ is a function which is P-almost surely continuous and twice continuously differentiable with respect to v. $f(t, v)$ is measurable with respect to all variables and compatible with $\{\mathcal{F}_t\}_{t\in[0,1]}$ and admits of the SDE

$$df(t, v) = A(t, v)\,dt + B(t, v)\,dW$$

where $B(t, v)$ is P-almost surely continuous and continuously differentiable with respect to v, and $B(t, v)$ is P-almost surely continuous.

Suppose that $v = \{v_t : t \in [0, 1]\}$ is P-almost surely continuous with respect to t and is a solution of the equation

$$dv_t = C(t, v_t)\,dt + D(t, v_t)\,dW$$

Then the following formula holds

$$\begin{aligned} df(t, v_t) &= A(t, v_t)\,dt + B(t, v_t)\,dW + \left\{ \frac{\partial f(t, v)}{\partial v}\bigg|_{v=v_t} \right\} dv_t \\ &\quad + \frac{1}{2}D^2(t, v_t)\left\{ \frac{\partial^2 f(t, v)}{\partial v^2}\bigg|_{v=v_t} \right\} dt \\ &\quad + D(t, v_t)\left\{ \frac{\partial B(t, v)}{\partial v}\bigg|_{v=v_t} \right\} dt \end{aligned}$$

Rearranging the above equation one has

$$\begin{aligned} df(t, v_t) &= \left[A(t, v_t) + C(t, v_t)\frac{\partial f(t, v)}{\partial v} + \frac{1}{2}D^2(t, v_t)\frac{\partial^2 f(t, v)}{\partial v^2} \right. \\ &\quad \left. + D(t, v_t)\frac{\partial B(t, v)}{\partial v} \right] dt \\ &\quad + \left[B(t, v_t) + D(t, v_t)\frac{\partial f(t, v)}{\partial v} \right] dW. \end{aligned}$$

Gronwall Lemma

The following theorem can be found, for example, in Dieudonné (1960).

Theorem 2. Let $f(t)$, $g(t)$ and $h(t)$ be nonnegative on $[0, T]$, and for all $0 \le t \le T$

$$f(t) \le g(t) + \int_0^t h(s)f(s)\,ds \tag{A1}$$

Then for $0 \le t \le T$

$$f(t) \le g(t) + \int_0^t h(s)g(s)\exp\left\{ \int_s^t h(u)du \right\} ds \tag{A2}$$

Acknowledgement

I wish to thank Prof. Fima Klebaner and Prof. Robert Liptser for their reading the first version of the paper, their helpful comments and discussions.

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