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Dualities for the Domany–Kinzel Model

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We study the Domany–Kinzel model, which is a class of discrete-time Markov processes in one-dimension with two parameters $(p_1, p_2) \in [0, 1]^2$. When $p_1 = \alpha\beta$ and $p_2 = \alpha(2\beta - \beta^2)$ with $(\alpha, \beta) \in [0, 1]^2$, the process can be identified with the mixed site-bond oriented percolation model on a square lattice with probabilities α of a site being open and β of a bond being open. This paper treats dualities for the Domany–Kinzel model ξ_t^A and the DKdual η_t^A starting from A. We prove that (i) $E(x^{|\xi_t^A \cap B|}) = E(x^{|\xi_t^B \cap A|})$ if $x = 1 - (2p_1 - p_2)/p_1^2$, (ii) $E(x^{|\xi_t^A \cap B|}) = E(x^{|\eta_t^B \cap A|})$ if $x = 1 - (2p_1 - p_2)/p_1$, and (iii) $E(x^{|\eta_t^A \cap B|}) = E(x^{|\eta_t^B \cap A|})$ if $x = 1 - (2p_1 - p_2)$, as long as one of A, B is finite and $p_2 \leqslant p_1$.

KEY WORDS: The Domany–Kinzel model; duality; the DKdual.

1. INTRODUCTION

Pascal's triangle (mod 2) can be viewed as a deterministic cellular automaton alternately on the odd and even integers. Site $2x$ is occupied at time $2n$ if exactly one of $2x-1, 2x+1$ is occupied at time $2n-1$, and site $2x+1$ is occupied at time $2n+1$, if exactly one of $2x, 2x+2$ is occupied at time $2n$. Starting from a single occupied site at 0 at time 0, it is easy to see that at epoch $2^n - 1$ there are 2^n particles and at epoch 2^n there are only 2 particles. This is discussed in Durrett,[(2)] Section 5d. The number of particles fluctuates

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unboundedly as time tends to ∞, and the average density of particles tends to 0. What is astonishing is that, if a site is occupied with probability $1-\varepsilon$, far from there being fewer particles, the process appears to converge to a limit with average density close to 1/2. This is the special case $p_1 = 1-\varepsilon$, $p_2 = 0$ of a widely-discussed interacting particle system, the Domany–Kinzel model, which is defined by 3 parameters, p_i, $i = 0, 1, 2$, the probabilities that site x is occupied at epoch n if the number of $x+1, x-1$ occupied at epoch $n-1$ is i. We are not able to demonstrate convergence of the DK-model when the initial occupied set is finite, but using an argument that is given, for example, in Sudbury,(5) duality may be used to demonstrate convergence when the initial measure is "essentially infinite." This may be contrasted with Pascal's triangle where it is known that convergence does not occur from any product measure other than $\nu_{1/2}$. The new duals found in this paper enable this result to be proved in a straighforward manner. It is shown that the crucial difference between the $p_1 = 1-\varepsilon$ situation and the Pascal's triangle, $p_1 = 1$, is that, since models with $p_1 < 1$ have the possibility of dying out, the number of particles in the former models have to tend to either 0 or ∞ instead of fluctuating unboundedly.

We have the following result (see Theorem 6 in Section 7): if the initial measure ν of the Domany–Kinzel model is a.s. infinite then the occupied set at time t

$$\xi_t^{\nu} \to \mu_\eta,$$

the measure uniquely determined by $E((\frac{-(p_1-p_2)}{p_1})^{|\mu_\eta \cap A|}) = P(|\eta_\infty^A| = 0)$ for all finite A and $p_1 > p_2$.

This paper presents a method of finding dualities for a fairly general model of oriented percolation often known as the Domany–Kinzel model (see, Domany and Kinzel,(1) and Chapter 5 of Durrett(2)). This is an interacting particle system that evolves in discrete time. Most usually at time 0, and at subsequent even epochs, the sites are located at even sites, and at odd epochs they are located at odd sites. This ensures that the occupied set has no drift and in certain cases corresponds to oriented percolation rotated through 45 degrees. However, it is possible to locate the sites on the half-integers, so that the state space is $\{0, 1\}^{\mathbf{Z} \cup (\mathbf{Z}+1/2)}$, and only at even times are the sites on $\mathbf{Z}$. The set of occupied sites at time t starting from an initial occupied set A is designated ξ_t^A. The evolution satisfies

$$P(x \in \xi_{t+1}^A \mid \xi_t^A) = f(|\xi_t^A \cap \{x-1/2, x+1/2\}|),$$

and given ξ_t^A, the events $\{x \in \xi_{t+1}^A\}$ are independent with

$$f(0) = 0, \qquad f(1) = p_1, \qquad f(2) = p_2, \qquad 0 \leqslant p_1, p_2 \leqslant 1,$$

where $|A|$ is the number of elements in A. In this paper, however, the sites will be located on $\mathbf{Z}$ at every time. The rule above will be modified to one of two possibilities,

$$P(x \in \xi_{t+1}^A \mid \xi_t^A) = f(|\xi_t^A \cap \{x-1, x\}|),$$
$$P(x \in \xi_{t+1}^A \mid \xi_t^A) = f(|\xi_t^A \cap \{x+1, x\}|),$$

as we need to identify sites at times $t, t+1$. There are obvious couplings between these processes and the process on $\mathbf{Z} \cup \mathbf{Z}+1/2$. At time t the first possibility has occupied set translated $t/2$ to the right and the second possibility $t/2$ to the left.

Sudbury and Lloyd[6] showed how to find duals of interacting particle systems evolving in continuous time. Their method relied on two key properties of such a system, first, that with probability 1 only one event takes place at any time and, secondly, that the infinitesimal operator or Q-matrix is a sum of two-site infinitesimal Q-matrices. In discrete time, many events occur simultaneously and these crucial properties are lost. Nevertheless an analysis having some features in common with theirs can be made. The duals we discuss take the form

$$E(x^{|\xi_t^B \cap A|}) = E(x^{|\eta_t^A \cap B|}),$$

where ξ is a process drifting to the right and η a process drifting to the left. If the equation is true for all finite sets A, B and all times, we would also have at any t

$$E(x^{|\xi_t^B \cap A+t|}) = E(x^{|\eta_t^{A+t} \cap B|}).$$

Using the obvious coupling again, it would follow that

$$E(x^{|\xi_t^B \cap A|}) = E(x^{|\eta_t^A \cap B|})$$

for the equivalent processes without drift on $\mathbf{Z} \cup \mathbf{Z}+1/2$. Thus we only need to establish the duality for the processes with drift.

This paper is organized as follows. Section 2 is devoted to a general theory for discrete-time interacting particle processes in one dimension. In Section 3, we give a duality for the Domany–Kinzel model ξ_t^A and the DKdual η_t^A starting from A: $E(x^{|\xi_t^A \cap B|}) = E(x^{|\eta_t^B \cap A|})$ if $x = 1-(2p_1-p_2)/p_1$ (see Theorem 2). Section 4 treats the self-duality for the Domany–Kinzel model: $E(x^{|\xi_t^A \cap B|}) = E(x^{|\xi_t^B \cap A|})$ if $x = 1-(2p_1-p_2)/p_1^2$ (see Theorem 3). In Section 5, we present a result on the self-duality for the DKdual: $E(x^{|\eta_t^A \cap B|}) = E(x^{|\eta_t^B \cap A|})$ if $x = 1-(2p_1-p_2)$ (see Theorem 4). Section 6 contains the thinning-relationship between the Domany–Kinzel model and the DKdual (see Theorem 5). In the final section, we give an application of Theorem 2 for the non-attractive Domany–Kinzel model.

2. A GENERAL THEORY OF DUALITY

As mentioned in Section 1 we aim to find a relationship between $\xi_t^B \cap A$ and $\eta_t^A \cap B$ where ξ drifts to the left and η to the right. We require either A or B to be finite. Assume without loss of generality it to be B. Define

$$B^t = B \cup \{\text{the } t \text{ sites to the left of } B\}.$$

$\xi_t^B \subset B^t$ and $\eta_t^A \cap B = \eta_t^{A \cap B^t} \cap B$, thus if we fix t and prove the duality equation for the processes confined to B^t, it will also be true for processes on the whole of $\mathbf{Z}$. We note that in the finite case the evolutions of the rightmost position of B^t under ξ and the leftmost position under η are not defined. We shall assume the processes evolve as if the sites to the right and left of B^t were fixed to be unoccupied for all t. This would have been true for ξ^B operating on any set containing B^t, and sites to the left of B^t cannot influence $\eta_t^A \cap B$ in any case.

We locate the processes on $\mathbf{G}$, a finite continuous subset of $\mathbf{Z}$. The state-spaces of both ξ and η will be $\{0, 1\}^{\mathbf{G}}$. Each state is represented by a basis vector in a 2^n-dimensional vector space, where $n = |\mathbf{G}|$, the number of sites in $\mathbf{G}$. Such a basis vector can also be represented as a tensor product of single-site vectors. For example, when $\mathbf{G}$ has just 2 sites, the state, site 1 occupied, site 2 unoccupied can be represented as

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and in general

$$\begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix}.$$

Site 1 is only affected by sites 1 and 2, and site 2 only by sites 2 and 3 and so on. Thus the changes at each time step can be broken down into a sequence of changes beginning at 1 and ending at n. So, the operator, Q, representing the Domany–Kinzel model can be represented as

$$Q = Q_{n,n-1} Q_{n-1,n-2} \cdots Q_{2,1},$$

where the operators $Q_{n,n-1}$ are $2^n \times 2^n$ matrices. Note that we are using the convention by which matrices operate to the left of the states so that the columns add to 1. Each $Q_{n,n-1}$ only changes the distribution at site $n-1$ but depends on the states at sites $n, n-1$. It is the product form above that enables us to create a dual equation in a similar way to that done in Sudbury and Lloyd.[6] We shall assume Q can be represented in this way.

$Q_{x,x-1}$ is the operator which defines the probability that $x-1$ is occupied at time $t+1$ given the state of the process at sites $x-1, x$ at time t. $Q_{x,x-1}$ is a $2^n \times 2^n$ matrix, but we may also represent it as a 4×4 matrix operating on the 4-dimensional vector space defining the state of the process at $x-1, x$ which has as basis the tensor products of the states at $x-1, x$. For the Domany–Kinzel model we have:

$$Q_{x,x-1}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}p_2\\0\\1-p_2\\0\end{pmatrix},\qquad Q_{x,x-1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\p_1\\0\\1-p_1\end{pmatrix},$$

$$Q_{x,x-1}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}p_1\\0\\1-p_1\\0\end{pmatrix},\qquad Q_{x,x-1}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

representing the transitions $11 \to 11$ or 01, $10 \to 10$ or 00, $01 \to 11$ or 01, $00 \to 00$. It can be seen that these transitions only affect the site $x-1$. These equations imply that

$$Q_{x,x-1}=\begin{pmatrix}p_2 & \cdot & p_1 & \cdot\\ \cdot & p_1 & \cdot & \cdot\\ 1-p_2 & \cdot & 1-p_1 & \cdot\\ \cdot & 1-p_1 & \cdot & 1\end{pmatrix}$$

where sometimes we shall represent 0 by "·" in order to make the contents of the matrix clearer.

A single-site operator only affects the distribution at a single-site. For example, suppose the probability that site i is occupied is p. The effect of the single-site operator at i

$$X_i=\begin{pmatrix}x & 1\\ 1 & 1\end{pmatrix},$$

is

$$X_i \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} (x-1)\,p+1 \\ 1 \end{pmatrix},$$

or, for example, using the $2^n \times 2^n$ form of X_i,

$$X_i \left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} a_i \\ b_i \end{pmatrix} \otimes \cdots \right) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} xa_i + b_i \\ a_i + b_i \end{pmatrix} \otimes \cdots.$$

Define $X = X_n X_{n-1} \cdots X_1$, then

$$XQ = X_n X_{n-1} Q_{n,\,n-1} X_{n-2} Q_{n-1,\,n-2} \cdots X_1 Q_{2,\,1}, \tag{2.1}$$

since operators on different sites commute.

We now look for another operator P drifting in the opposite direction to Q, such that $XP = Q^T X$. Then,

$$XP^m = Q^T X P^{m-1} = (Q^T)^2\, X P^{m-2} = \cdots = (Q^T)^m\, X. \tag{2.2}$$

Then (2.1) implies

$$Q^T X = Q^T_{2,\,1} X_1 X_2 (X_2^{-1} Q^T_{3,\,2} X_2 X_3) \cdots (X_{n-1}^{-1} Q^T_{n,\,n-1} X_{n-1} X_n).$$

Since P tends to produce a drift in the opposite direction to Q, we may suppose $P = P_{1,\,2} \cdots P_{n-1,\,n}$, so that

$$XP = X_1 X_2 P_{1,\,2} (X_3 P_{2,\,3}) \cdots X_n P_{n-1,\,n}.$$

A sufficient condition for $XP = Q^T X$ is then

$$X_j X_{j+1} P_{j,\,j+1} = Q^T_{j+1,\,j} X_j X_{j+1}$$

for $j = 1,\ldots, n-1$.

Consider two sets of occupied sites A, B. We represent A by a vector

$$\mathbf{a} = \begin{pmatrix} a_1 \\ 1-a_1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} a_n \\ 1-a_n \end{pmatrix},$$

where $a_i = 1$ or 0 according as site i belongs to A or not. $\mathbf{b}$ is similarly defined. We note that

$$(a_1 \quad 1-a_1) \begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ 1-b_1 \end{pmatrix} = (x-1)\, a_1 b_1 + 1,$$

which is x if $a_1 = b_1 = 1$, and 1 otherwise. So the scalar product is multiplied by x when i belongs to both A and B and

$$\mathbf{a}^T X \mathbf{b} = x^{|A \cap B|}. \tag{2.3}$$

If the initial state in the Q-process is $\mathbf{b}$, then $\xi_t^B = Q^t \mathbf{b}$ and, using (2.2) and (2.3),

$$\mathbf{a}^T X Q^t \mathbf{b} = E(x^{|\xi_t^B \cap A|}) = \mathbf{b}^T (Q^T)^t X \mathbf{a} = \mathbf{b}^T X P^t \mathbf{a} = E(x^{|\eta_t^A \cap B|}),$$

where η is the process represented by P. As we have seen, we may also extend the result to cases in which one of A, B is infinite. Further, the result is true if the sets are random, as can be seen by simply summing the expectations over all possible initial configurations. Thus we have shown:

Theorem 1. If ξ and η are discrete-time interacting particle processes on $\mathbf{Z}$ whose operators Q, P are defined by pairwise operations which satisfy

$$X_j X_{j+1} P_{j,j+1} = Q^T_{j+1,j} X_j X_{j+1},$$

then

$$E(x^{|\xi_t^B \cap A|}) = E(x^{|\eta_t^A \cap B|})$$

as long as one of A, B is finite.

3. DUALS OF THE DOMANY–KINZEL MODEL

Given a value of x we can calculate the pairwise transition probabilities for a dual of the Domany–Kinzel model. If all these probabilities are non-negative then we have a proper dual. $X_j X_{j+1}$ operates on two sites and is thus represented by the 4×4 matrix

$$\begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} x^2 & x & x & 1 \\ x & x & 1 & 1 \\ x & 1 & x & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Define M_1 to be the 4×4 matrix of 1's. Because the columns of P and the rows of Q^T sum to 1, $P\mathrm{M}_1 = \mathrm{M}_1$, $Q^T \mathrm{M}_1 = \mathrm{M}_1$. We may thus subtract M_1 from $X_j X_{j+1}$ when satisfying the above equation to give

$$\begin{pmatrix} x^2 & x & x & 1 \\ x & x & 1 & 1 \\ x & 1 & x & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \mathbf{M}_1 + (x-1)\begin{pmatrix} x+1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now we find the P which satisfies $Q^T_{j+1,j}X_jX_{j+1} = X_jX_{j+1}P_{j,j+1}$ where, however, we use the simplified form of X_jX_{j+1} in the above equation. We note that

$$\begin{pmatrix} p_2 & \cdot & 1-p_2 & \cdot \\ \cdot & p_1 & \cdot & 1-p_1 \\ p_1 & \cdot & 1-p_1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} x+1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x+1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\times \frac{1}{1-x}\begin{pmatrix} p_1+(p_1-p_2)\,x & 2p_1-p_2 & 0 & 0 \\ (p_2-2p_1)\,x & p_2-p_1(1+x) & 0 & 0 \\ (1-p_1)-(1-p_2)\,x-p_1x^2 & p_2-p_1(1+x) & 1-x & 0 \\ (p_1-p_2)\,x+p_1x^2 & (2p_1-1)\,x+1-p_2 & 0 & 1-x \end{pmatrix}.$$

For example, one may check the 1,1 position on the r.h.s. is $p_1[(1+x)^2 - 2x-1-x^2] + p_2[-x(1+x)+x+x] + 1 - x = (1-x)(p_2x+1)$. We now look for values of x which make the last matrix positive.

However, before doing this we should be aware of a problem that will arise. The operator of the Domany–Kinzel model satisfies the equation $Q = Q_{n,n-1}Q_{n-1,n-2}\cdots Q_{2,1}$. That is, the change in distribution can be regarded as a sequential change, first at site 1, then site 2, and so on. The two-site operator $Q_{2,1}$ in fact only affects site 1 and this is reflected in the form of the associated matrix which allows $11 \to 01$ but not $11 \to 10$, and thus has a 0 in row 2, column 1. When the dual matrix above has all its elements non-negative, it certainly defines a process when the occupied set is finite, however, this process may be a sequential change of two sites at a time. If this is the case, it is no longer possible to define the occupation probability at site x at time $t+1$ in terms of the values at sites $x+1, x$ at time t. In fact you need to know all the values from $x+1$ to t. With an infinite occupied set it may not define a process at all.

The above considerations suggest that it will not be possible to define a satisfactory dual model which allows the transition $10 \to 01$. In the dual matrix, this transition has probability $p_2 - p_1(1+x)$. We choose x to make this expression 0.

Assumption. $x = -(p_1 - p_2)/p_1$.

As explained above, the matrix does not appear to be in the correct form until we decompose it:

$$\begin{pmatrix} p_2 & p_1 & \cdot & \cdot \\ p_1 - p_2 & \cdot & \cdot & \cdot \\ 1-p_1 & \cdot & 1 & \cdot \\ \cdot & 1-p_1 & \cdot & 1 \end{pmatrix} = \begin{pmatrix} p_2/p_1 & 1 & \cdot & \cdot \\ 1-p_2/p_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} p_1 & \cdot & \cdot & \cdot \\ \cdot & p_1 & \cdot & \cdot \\ 1-p_1 & \cdot & 1 & \cdot \\ \cdot & 1-p_1 & \cdot & 1 \end{pmatrix}.$$

The second matrix allows the changes $11 \to 01$ and $10 \to 00$ with probability $1 - p_1$. It therefore represents a "p_1-thinning" of the first site which stays 1 with probability p_1. We write this equation as $P_{j,j+1} = R_{j,j+1} T_j$. We then have

$$P_{1,2} \cdots P_{n-1,n} = R_{1,2} T_1 \cdots R_{n-1,n} T_{n-1} = R_{1,2} \cdots R_{n-1,n} T_1 \cdots T_{n-1}. \tag{3.1}$$

Remembering that a sufficient number of the outer sites of $1,..., n$ are always unoccupied, we see that P can be interpreted as a p_1-thinning of all sites followed by the changes $11 \to 11$ with probability p_2/p_1 and $10 \to 11$ with probability 1. R then represents a version of the Domany–Kinzel model, drifting in the opposite direction to Q, and with $p_2' = p_2/p_1$, $p_1' = 1$. We shall call this model the DKdual. We note it is only defined for $p_2 \leqslant p_1$.

Theorem 2. The Domany–Kinzel model with parameters $p_2 \leqslant p_1$ is dual with parameter $x = -(p_1 - p_2)/p_1$ to a model which at each stage represents a p_1-thinning of all sites, followed by an application of the Domany–Kinzel model with $p_2' = p_2/p_1$, $p_1' = 1$.

When $p_2 = p_1$ we have site-percolation. This is known to have a coalescing dual which is equivalent to $x = 0$ when 0^0 is interpreted as 1 (see Sudbury and Lloyd[6]).

When $p_2 = 0$ we have $x = -1$. This is equivalent to an annihilating dual and can be retrieved from graphical representations. In general, this is possible for coalescing and annihilating duals, but not when $x \neq 0, -1$.

When $p_2 = 0$ we can split the transition into 2 phases. First, a deterministic phase in which 11 at sites $j, j-1$ always produces a 0 at site $j-1$, and 01 at sites $j, j-1$ always produces a 1. 00 does not change. Then there is a p_1-thinning of all sites. This is equivalent to the Domany–Kinzel model with $p_1 > 0$, $p_2 = 0$. In the first transition, the value at site $j-1$ at time $t+1$ is the sum of the values at sites $j, j-1$ (mod 2) at time t. The p_1-thinning is equivalent to the site being open with probability p_1. The probability a site x is occupied at time t given initial set A, is then the probability that the number of paths from A to x is odd, where a path cannot pass through a closed site. The probability that a set B has an odd number of sites occupied at time t is then the probability that the number of paths from A to B is odd. But this is, of course, equivalent to the event that the number of paths from B to A is odd. The equality of the probabilities can be written

$$E((-1)^{|\xi_t^A \cap B|}) = E((-1)^{|\eta_t^B \cap A|}).$$

In reverse time the process is a p_1-thinning followed by the Domany–Kinzel model with $p_1 = 1$, $p_2 = 0$. We see that in the mixture of bond and site-percolation it is essential to state in which order they are to be observed.

4. SELF-DUALITY OF A DIFFERENT TYPE

The self-duality for the Domany–Kinzel model can also be derived in a somewhat similar way to that given in Section 2. As before we consider two sets of occupied sites A, B. We represented by vectors $\mathbf{a}$, $\mathbf{b}$. We look at the effect of Q at site i. We write the 2-vector of occupation probabilities at site i as $(Q\mathbf{a})_i$.

$$(Q\mathbf{a})_i = \begin{pmatrix} p(i, i+1) \\ 1-p(i, i+1) \end{pmatrix},$$

where $p(i, i+1) = p_2$ if $a_i + a_{i+1} = 2$, $= p_1$ if $a_i + a_{i+1} = 1$, $= 0$ if $a_i + a_{i+1} = 0$. Defining X as in Sections 2 and 3,

$$(XQ\mathbf{a})_i = \begin{pmatrix} (x-1)\, p(i, i+1) + 1 \\ 1 \end{pmatrix}.$$

If $x = 1 - (2p_1 - p_2)/p_1^2$, then $(x-1)\, p_2 + 1 = ((x-1)\, p_1 + 1)^2$. We will then have

$$\begin{pmatrix} b_i \\ 1-b_i \end{pmatrix}^T (XQ\mathbf{a})_i = ((x-1)\, p_1 + 1)^{b_i(a_i + a_{i+1})},$$

since the r.h.s will be 1 if $b_i = 0$. Now writing Q as Q_L as its operation drifts to the left, and noting that the scalar product over a tensor product of vectors is the product of the individual scalar products, we have

$$\mathbf{b}^T X Q_L \mathbf{a} = ((x-1)\,p_1+1)^{\sum_{i=0}^n b_i(a_i+a_{i+1})}.$$

We now consider Q_R, the Domany–Kinzel model moving backwards in time from $\mathbf{b}$ to $\mathbf{a}$ and with the value at site $i+1$ determined by those at sites $i, i+1$. After a similar analysis to that above, we obtain

$$\mathbf{a}^T X Q_R \mathbf{b} = ((x-1)\,p_1+1)^{\sum_{i=0}^n a_{i+1}(b_i+b_{i+1})} = ((x-1)\,p_1+1)^{\sum_{i=0}^n b_i(a_i+a_{i+1})},$$

since $a_0 = b_0 = a_{n+1} = b_{n+1} = 0$. Because

$$\mathbf{b}^T X Q_L \mathbf{a} = \mathbf{a}^T X Q_R \mathbf{b} = \mathbf{b}^T Q_R^T X \mathbf{a}$$

for all $\mathbf{a}$, $\mathbf{b}$, $XQ_L = Q_R^T X$ and, since Q_L, Q_R are both the same Domany–Kinzel model, this model is seen to be self-dual. Then we have the following result:

Theorem 3. The Domany–Kinzel model is self-dual with parameter $x = 1-(2p_1-p_2)/p_1^2$.

5. SELF-DUALITY OF THE DK-DUAL

We saw in Section 3 how the Domany–Kinzel model was dual to a process which we called the DKdual. This process could be split into two phases, a p_1-thinning of all sites, followed by an application of the Domany–Kinzel model with $p_2' = p_2/p_1$, $p_1' = 1$. (3.1) in Section 3 can be written as $P_R = R_R T$, where the subscript R indicates that the operation is to the right. The Domany–Kinzel model can be viewed as these two phases in reverse order, so we may put $Q_L = TR_L$. The duality stems from the equation $Q_L^T X = XP_R$, which is equivalent to

$$(TR_L)^T X = XR_R T \Rightarrow (R_L T)^T\, T^T X = T^T X (R_R T),$$

implying that the DKdual is self-dual. At a single-site the effect of $T^T X$ is

$$\begin{pmatrix} p_1 & 1-p_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} p_1(x-1)+1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} x' & 1 \\ 1 & 1 \end{pmatrix},$$

when $x = -(p_1-p_2)/p_1$, and $x' = 1-(2p_1-p_2)$.

Theorem 4. The DKdual is self-dual with parameter $1-(2p_1-p_2)$.

6. THINNINGS

Theorem 11 of Sudbury and Lloyd,[7] is:
If

$$E(x^{|\xi_t^B \cap A|}) = E(x^{|\eta_t^A \cap B|}), \qquad E(y^{|\xi_t^B \cap A|}) = E(y^{|\zeta_t^A \cap B|}),$$

and, $y < x$, then ζ is a p-thinning of η, where $p = (1-x)/(1-y)$.

In other words, if the initial distribution of ζ is a p-thinning of the initial distribution of η, then the distribution remains a p-thinning at all subsequent times. In an obvious notation

$$\zeta_t^{\mu_p} = (\eta_t^{\mu})_p.$$

We have shown above that the Domany–Kinzel model is self-dual with parameter $1-(2p_1-p_2)/p_1^2$, and dual to a model we called the DKdual with parameter $-(p_1-p_2)/p_1$. However, we cannot use the same argument as in Sudbury and Lloyd,[7] Section 7, since the self-duality does not have the form $X_j X_{j+1} Q_{j,j+1} = Q^T_{j+1,j} X_j X_{j+1}$. But this is the form of the self-duality with parameter $1-(2p_1-p_2)$ of the DKdual. This gives us a thinning with prameter

$$\frac{1-(1-2p_1+p_2)}{1-(-(p_1-p_2)/p_1)} = p_1.$$

This thinning is obvious in another way since $(TR)^m T\mathbf{b} = T(RT)^m \mathbf{b}$.

Theorem 5. The Domany–Kinzel model is a p_1-thinning of the DKdual.

We can see this thinning-relationship by coupling the two processes. We split both the Domany–Kinzel model and the DKdual into two phases rather as in Theorem 2, and we shall allow the first phase to occur at times $k+1/2$ where k is an integer.

1. Let μ be the distribution of the DKdual at time 0.
2. At time $t = 1/2$, it undergoes a p_1-thinning.
3. Start the Domany–Kinzel model at time $t = 0$ with the same configuration as the DKmodel at time $t = 1/2$.
4. Couple the processes together until time $t_0 - 1/2$ for the Domany–Kinzel, t_0 for the DKmodel. This can be done because the transitions for the Domany–Kinzel model are the same as those for the DKmodel lagged by time unit $1/2$.
5. Perform a p_1-thinning for the Domany–Kinzel model at time t_0.

The distribution of the Domany–Kinzel model started and ended as a p_1-thinning of the DK dual.

7. CONVERGENCE IN DISTRIBUTION IN THE NON-ATTRACTIVE CASE

In this section we return to the problem of the convergence of Pascal's triangle, sometimes known a Wolfram's rule 90. We have shown that when $p_2 < p_1$, $E(x^{|\xi_t^B \cap A|}) = E(x^{|\eta_t^A \cap B|})$ where ξ represents the Domany–Kinzel model, η the DKdual and $x = -(p_1 - p_2)/p_1$. Starting ξ with product measure ν_θ and, conditioning on non-extinction of the DKdual, we obtain

$$\begin{aligned} E(x^{|\xi_t^{\nu_\theta} \cap A|}) &= P(|\eta_t^A| = 0) + E(x^{|\eta_t^A \cap \nu_\theta|} \mid |\eta_t^A| > 0)\, P(|\eta_t^A| > 0), \\ &= P(|\eta_t^A| = 0) + E((1-\theta+\theta x)^{|\eta_t^A|} \mid |\eta_t^A| > 0)\, P(|\eta_t^A| > 0). \end{aligned} \tag{7.1}$$

But, since η can die out, $|\eta_t^A| \to \{0, \infty\}$ a.s. and, since $|1-\theta+x\theta| < 1$, we may conclude that

$$E(x^{|\xi_t^{\nu_\theta} \cap A|}) \to P(|\eta_\infty^A| = 0).$$

In fact using the arguments in Sudbury[5] this convergence can be seen to be true for any initial measure ν that is *essentially infinite*, that is, the intersection of which with any infinite set is a.s. infinite. This is because then $E(x^{|\eta_t^A \cap \nu|} \mid |\eta_t^A| > 0) \to 0$ a.s. since $|x| < 1$.

As the convergence is true for all finite sets A, we have

Theorem 6. If the initial measure ν of the Domany–Kinzel model is a.s. infinite then

$$\xi_t^\nu \to \mu_\eta,$$

the measure uniquely determined by $E((\frac{-(p_1-p_2)}{p_1})^{|\mu_\eta \cap A|}) = P(|\eta_\infty^A| = 0)$ for all finite A and $p_1 > p_2$.

In Wolfram's rule 90, because $p_1 = 1$, the thinning phase does not occur and the process is self-dual. $P(|\xi_t^A| = 0) = 0$ and $x = -1$. Then (7.1) becomes

$$E(x^{|\xi_t^{\nu_\theta} \cap A|}) = E((1-2\theta)^{|\xi_t^A|}).$$

Because of the unbounded fluctuations in the number of occupied sites, it is clear this can only converge for $0 < \theta < 1$ when $\theta = 1/2$. Convergence from $\theta = 0, 1$ is trivial. This reproduces a result shown by Miyamoto,[4] Lind,[3] and by Durrett[2] who uses the annihilating duality in a similar way to that given here for the more general duals.

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