Rigorous lower bounds for extinction probabilities of the contact process

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Abstract

Positive correlations for extinction probabilities in the contact process are used to obtain rigorous lower bounds

1 The contact process

The contact process is one of the most frequently studied interacting particle systems. Among other things it can be used to model the spatial spread of an infection. In d-dimensions the state-space is $\{0,1\}^{Z^d}$ where $\eta(x) = 0$ is interpreted as the site x being empty (uninfected) and $\eta(x) = 1$ is interpreted as it being occupied (infected). The sites act independently with the changes $1 \to 0$ at rate 1 and $0 \to 1$ at rate $\lambda\times$ the number of occupied neighbours. For a detailed treatment see Liggett(1985), Chapter VI. In Sudbury(2001) rigorous lower bounds for the critical infection rate were found for the diffusive contact process. The method was to use a computer to find a suitable submartingale. In this paper a computer is used to create a set of inequalities.

In general the difficulty in treating interacting particle systems derives from the fact that equations for sets of sites with span n depend on the state of the sites with span $n+1$ in one-dimension, and more than that in higher dimensions. For example, designating the probability of extinction from the set of sites ....0000001010000... as (101) we have

$$(101) = \frac{1}{(2+4\lambda)}\left[\lambda(1101) + 2\lambda(111) + \lambda(1011) + 2(1)\right], \quad (1)$$

since at rate λ the configuration 101 may change to 1101, or at rate 1 to ...00001000...etc. Note that the latter configuration is designated (1) using the convention that we only give the configuration between the leftmost and rightmost occupied sites. There is no systematic way of solving these equations because the system is infinite. Our treatment is very similar to that of Konno(1994), although in this note we emphasize extinction probabilities rather than occupation probabilities for the upper invariant measure. The new ingredient is the use of a computer to find bounds which are more accurate, because it is then possible to treat much longer strings of 0's and 1's.

We aim to find lower bounds on the extinction probabilities by using the positive correlations between extinction probabilities. For example,

$$(1001100110101) > (100110)(0110101) = (10011)(110101). \tag{2}$$

Lemma 1 *If s_1 and s_2 are strings of $0's$ and $1's$ then $(s_1 s_2) \geq (s_1)(s_2)$.*

Proof. We follow the treatment in Konno(1994), Chapter 3. Defining ν_λ as the upper invariant measure, he shows that for any finite subsets of Z, A and B,

$$\overline{\rho}_\lambda(A \cup B) \geq \overline{\rho}_\lambda(A)\overline{\rho}_\lambda(B),$$

where $\overline{\rho}_\lambda(A) = \nu_\lambda(\eta : \eta(x) = 0 \text{ for all } x \in A)$. Because of the well-known duality for the contact process,

$$P(\eta_t^Z \cap A = \phi) = P(\eta_t^A \cap Z = \phi),$$

$\overline{\rho}_\lambda(A)$ is equal to the probability of extinction with initial set A. ■

We initially set the lower bounds on the extinction probabilities to be 0. We then calculate the equations iteratively and use the above Lemma to ensure that all the values we get are lower bounds for the extinction probabilities. The first 4 equations are equation (1) above and

$$(1)_{m+1} = \frac{1}{1+2\lambda}\left(1 + 2\lambda(11)_m\right), (11)_{m+1} = \frac{1}{2+2\lambda}\left(2(1)_m + 2\lambda(111)_m\right), \tag{3}$$

$$(111)_{m+1} = \frac{1}{3+2\lambda}\left(2(11)_m + (101)_m + 2\lambda(1111)_m\right),$$

equivalent to Lemma 3.2.6 of Konno(1994). If we were only going to solve for spans of maximum length 3, the last equation would be replaced by

$$(111)_{m+1} = \frac{1}{3+2\lambda}\left(2(11)_m + (101)_m + 2\lambda(11)_m(11)_m\right).$$

It should be noted that the best bounds are found when s_1, s_2 in the lemma above are as equal as possible. Use of the lemma in equation (3) gives:

Theorem 2

$$(1) \geq \frac{(1+\lambda)}{\lambda(1+2\lambda)}, (11) \geq \frac{1}{2\lambda^2}.$$

These are only reproduced here because they are simple, as better bounds of this type can be found in Lemmas 3.3.5 and 3.3.9 in Konno(1994).

2 Results

The following results are for a variety of initial sets with the maximum span being 14. 20000 simulations were performed starting from a single site to give an indication as to how close the lower bounds are in that case.

		Initial occupied sites							
λ	Simulations	1	11	111	101	11111	10001	11111111	10000001
1.7	0.627	0.546	0.412	0.333	0.367	0.233	0.329	0.141	0.304
1.8	0.504	0.481	0.337	0.257	0.291	0.163	0.256	0.086	0.237
1.9	0.444	0.431	0.281	0.202	0.236	0.116	0.204	0.054	0.189
2.0	0.400	0.391	0.238	0.162	0.195	0.085	0.167	0.035	0.155
2.5	0.268	0.271	0.125	0.066	0.093	0.022	0.077	0.005	0.074
3.0	0.206	0.210	0.078	0.034	0.054	0.008	0.045	0.001	0.044

3 References

KONNO,N. (1994) Phase transitions of interacting particle systems.
World Scientific Publishing Singapore.
LIGGETT,T.M. (1985) Interacting particle systems.Springer-Verlag.New York.
SUDBURY,A.W. (2001) Rigorous lower bounds for the critical infection rate in the diffusive contact process. *J.Appl.Prob.*,**38**,1074-1078.