

Stochastic Inequalities for Single-Server Loss Queueing Systems

Vyacheslav M. Abramov

School of Mathematical Sciences, Monash University, Clayton, Australia

Abstract: The present article provides some new stochastic inequalities for the characteristics of the $M/GI/1/n$ and $GI/M/1/n$ loss queueing systems. These stochastic inequalities are based on substantially deepen up- and down-crossings analysis, and they are stronger than the known stochastic inequalities obtained earlier. Specifically, for a class of $GI/M/1/n$ queueing system, two-side stochastic inequalities are obtained.

Keywords: Loss probability; Probability distribution classes; Single-server queueing systems; Stochastic inequalities.

1991 Mathematics Subject Classification: 60K25; 60E15.

1. INTRODUCTION

The goal of this article is to establish stronger stochastic inequalities for the number of losses during a busy period than those are obtained earlier [1]. The number of losses during a busy period is a significant characteristic for analysis of loss probability and other performance measures of real telecommunication systems, and detailed stochastic analysis of losses in queueing systems seems to be very important.

For the purpose of detailed stochastic analysis of losses we develop the up- and downcrossings approach initiated in a number of earlier

Received April 27, 2006; Accepted July 16, 2006

Address correspondence to Vyacheslav M. Abramov, School of Mathematical Sciences, Monash University, Building 28M, Clayton Campus, Clayton, VIC 3800, Australia; E-mail: vyacheslav.abramov@sci.monash.edu.au

works of the author [1–4]. It is proved in [1] that if the interarrival time distribution of $GI/M/1/n$ queue belong to the class NBU (NWU), then the number of losses during a busy period is stochastically not smaller (respectively not greater) than the number of offspring in the $n + 1$ st generation of the Galton-Watson branching process with given offspring generating function (see below for the more details). The Galton-Watson branching process is a well-known process having relatively simple explicit expressions for its characteristics. At the same time the explicit results for the number of losses in the $M/GI/1/n$ and $GI/M/1/n$ queues are very hard for applications.

In this article we obtain two-side stochastic inequality for the number of losses during a busy period of the $GI/M/1/n$ queueing system, where the left and right sizes are branching processes.

Note that other inequalities related to the number of losses during a busy period in the different loss queueing systems were obtained in [5–7] and others articles.

This article starts from elementary extension of the inequalities obtained in [1] to some special class of $GI/GI/1/n$ queues, which includes $M/GI/1/n$ queueing systems with NBU (NWU) service time and $GI/M/1/n$ queueing systems with NBU (NWU) interarrival time as particular cases. For our further convenience the $GI/GI/1/n$ queueing system will be denoted $A/B/1/n$, where $A(x)$ is the probability distribution function of an interarrival time, and $B(x)$ is the probability distribution function of a service time. Then, for the $M/GI/1/n$ and $GI/M/1/n$ queueing system we often use the notation $M/B/1/n$ and $A/M/1/n$ respectively. For the definition of the classes of distributions such as NBU, NWU and all other, that are used in the article, see [8].

Throughout the article the following notation is used. For $\Re(s) \geq 0$ we denote the Laplace-Stieltjes transforms of the probability distributions $A(x)$ and $B(x)$ by $\widehat{A}(s)$ and $\widehat{B}(s)$, respectively, and the reciprocals of the expected interarrival and service times are denoted by λ and μ , respectively. The aforementioned Laplace-Stieltjes transforms are in fact used for real values of argument, specifically only the values $\widehat{A}(\mu)$ and $\widehat{B}(\lambda)$ are used throughout the article.

The number of losses during a busy period is denoted L_n .

For the $A/M/1/n$ queue we have the inequality $L_n \geq_{st} Z_{n+1}$ in the case where an interarrival time is NBU, and the opposite inequality, $L_n \leq_{st} Z_{n+1}$, in the case where an interarrival time is NWU (see [1]). Z_n denotes the number of offspring in the n th generation of the Galton-Watson branching process with $Z_0 = 1$ and the offspring generating function

$$g_z(z) = \frac{1 - \widehat{A}(\mu)}{1 - z\widehat{A}(\mu)}, \quad |z| \leq 1.$$

The method of [1], adapted to the $M/B/1/n$ queue, provides the following inequality:

$$L_n \leq_{st} Y_{n+1} \quad (L_n \geq_{st} Y_{n+1}) \tag{1.1}$$

in the case where the service time is NBU (NWU). Y_n is the number of offspring in the n th generation of the Galton-Watson branching process with $Y_0 = 1$ and the offspring generating function

$$g_Y(z) = \frac{\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)}, \quad |z| \leq 1.$$

(See Section 2 for details of proof.)

A deepen analysis of these two queueing systems, given in Sections 3 and 4, enables us to obtain the following stronger results than that permits us the method of [1].

For the $M/B/1/n$ queue in the case where $B(x)$ belongs to the class of NBU (NWU) distributions it is shown that

$$L_n \leq_{st} \sum_{i=1}^{Y_n} \tau_i \quad \left(L_n \geq_{st} \sum_{i=1}^{Y_n} \tau_i \right), \tag{1.2}$$

where τ_1, τ_2, \dots is a sequence of independent identically distributed nonnegative integer random variables,

$$P[\tau_i = k] = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} dB(x).$$

Representation (1.2) is preferable than (1.1). For example, it follows from (1.1) that

$$E[L_n] \geq \left[\frac{1 - \widehat{B}(\lambda)}{\widehat{B}(\lambda)} \right]^{n+1} \left(E[L_n] \leq \left[\frac{1 - \widehat{B}(\lambda)}{\widehat{B}(\lambda)} \right]^{n+1} \right). \tag{1.3}$$

In turn, by using the Wald’s equation, from (1.2) we obtain

$$E[L_n] \geq \frac{\lambda}{\mu} \left[\frac{1 - \widehat{B}(\lambda)}{\widehat{B}(\lambda)} \right]^n \left(E[L_n] \leq \frac{\lambda}{\mu} \left[\frac{1 - \widehat{B}(\lambda)}{\widehat{B}(\lambda)} \right]^n \right). \tag{1.4}$$

Clearly that (1.4) is stronger than (1.3) since in the case of the NBU (NWU) service time distribution we have:

$$\frac{1 - \widehat{B}(\lambda)}{\widehat{B}(\lambda)} \geq \frac{\lambda}{\mu} \quad \left(\frac{1 - \widehat{B}(\lambda)}{\widehat{B}(\lambda)} \leq \frac{\lambda}{\mu} \right).$$

For a subcritical $A/M/1/n$ queue ($\rho = \lambda/\mu \leq 1$), in the case where an interarrival time distribution belongs to the IHR (DHR) class of distributions we obtain $L_n \leq_{st} X_{n+1}$ ($L_n \geq_{st} X_{n+1}$). The process $\{X_n\}$ is a branching process, but not classical (the precise definition of this process is given in Section 4). Thus, combining this result with the result of [1] we conclude the following. If $\rho \leq 1$, then in the case when an interarrival time distribution belongs to the IHR (DHR) class of distributions we have $Z_{n+1} \leq_{st} L_n \leq_{st} X_{n+1}$ ($X_{n+1} \leq_{st} L_n \leq_{st} Z_{n+1}$).

The article is organized as follows. It consists of 4 sections. Section 2 introduces the reader to the up- and downcrossings method of [1] and extends the results of [1] to the special class of $A/B/1/n$ queues (described exactly in that Section 2). The results related to $M/B/1/n$ and $A/M/1/n$ queues are then developed in Sections 3 and 4, respectively. In turn, Section 4 is divided into subsections, containing preliminary information on the properties of the $A/M/1/n$ queues. The most significant property is a monotonicity, which is considered in Section 4.1. Section 4.2 introduces and studies a special type of branching process, which is then used for the main result of Section 4, Theorem 4.3.

2. STOCHASTIC INEQUALITIES FOR $GI/GI/1/n$ QUEUES

In this section we establish stochastic inequalities for a class of $A/B/1/n$ queues. Specifically, assuming that the probability distributions $A(x)$ and $B(x)$ belong to the opposite classes of NBU and NWU, i.e., either $A(x)$ belongs to NBU and $B(x)$ belongs to NWU, or $A(x)$ belongs to NWU and $B(x)$ belongs to NBU, we have the following.

Theorem 2.1. *Under the assumption that $A(x)$ belongs to NBU (NWU), and $B(x)$ belongs to NWU (NBU), and a busy period is finite with probability 1, we have*

$$L_n \geq_{st} X_{n+1} (L_n \leq_{st} X_{n+1}). \quad (2.1)$$

X_n in (2.1) is the number of offspring in the n th generation of the Galton-Watson branching process with $X_0 = 1$ and the offspring generating function

$$g(z) = \frac{1-r}{1-zr}, \quad |z| \leq 1, \quad (2.2)$$

where $r = 1 - \int_0^\infty [1 - A(x)] dB(x)$.

Proof. The proof is provided only in the case where $A(x)$ belongs to the NBU class and $B(x)$ belongs to the NWU class. The opposite case is analogous.

Let $f_n(j)$, $0 \leq j \leq n + 1$, denote the number of customers arriving during a busy period who, upon their arrival, meet j customers in the system. Under the assumption that a busy period is finite we have $f_n(0) = 1$ with probability 1. Let $t_{j,1}^n, t_{j,2}^n, \dots, t_{j,f_n(j)}^n$ be the instants of arrival of these $f_n(j)$ customers, and let $s_{j,1}^n, s_{j,2}^n, \dots, s_{j,f_n(j)}^n$ be the instants of service completions (departures) at which there remain only j customers in the system. Notice, that $t_{n+1,k}^n = s_{n+1,k}^n$, $1 \leq k \leq f_n(n + 1) = L_n$.

For $0 \leq j \leq n$ let us consider the following intervals:

$$[t_{j,1}^n, s_{j,1}^n), [t_{j,2}^n, s_{j,2}^n), \dots, [t_{j,f_n(j)}^n, s_{j,f_n(j)}^n). \tag{2.3}$$

It is clear that the intervals

$$[t_{j+1,1}^n, s_{j+1,1}^n), [t_{j+1,2}^n, s_{j+1,2}^n), \dots, [t_{j+1,f_n(j+1)}^n, s_{j+1,f_n(j+1)}^n) \tag{2.4}$$

are contained in intervals (2.3). Let us delete the intervals in (2.4) from those in (2.3) and connect the ends. That is, we connect every point $t_{j+1,k}^n$ with the corresponding point $s_{j+1,k}^n$, $1 \leq k \leq f_n(j + 1)$, if the set of intervals (2.4) is not empty.

We will use the following notation. Take the interval $[t_{j,k}^n, s_{j,k}^n)$. Within this interval there is a number of inserted points, say m . If $m > 0$ then these points are numbered as $i = 1, 2, \dots, m$. Let $A_{j,k}^{(i)}(x)$ denote the probability distribution of the residual time in point i until the next arrival, and let $B_{j,k}^{(i)}(x)$ denote the probability distribution of the residual service time in point i . Then $A_{j,k}^{(0)}(x)$ is the probability distribution of the residual time in the initial point $t_{j,k}^n$ of the interval $[t_{j,k}^n, s_{j,k}^n)$ until the next arrival. Since $t_{j,k}^n$ is the moment of arrival, then $A_{j,k}^{(0)}(x) = A(x)$ for all $j \geq 0$ and $k \geq 1$. $B_{j,k}^{(0)}(x)$ is the probability distribution of the residual service time in the initial point $t_{j,k}^n$.

Let us take the interval $[t_{j,k}^n, s_{j,k}^n)$ and a customer in service in time $t_{j,k}^n$. Let $\tau_{j,k}$ be the time elapsed from the moment of the service begun for that customer until time $t_{j,k}^n$. Then for residual service time $\vartheta_{j,k}$ of the tagged customer we have

$$\begin{aligned} P[\vartheta_{j,k} \leq x] &= P[\chi \leq \tau_{j,k} + x \mid \chi > \tau_{j,k}] \\ &= \int_0^\infty P[\chi \leq y + x \mid \chi > y] dP[\tau_{j,k} \leq y]. \end{aligned} \tag{2.5}$$

According to the above convention, the probability of (2.5) is denoted by $B_{j,k}^{(0)}(x)$. Let $\kappa_{j,k}$ denote the number of inserted points within the interval $[t_{j,k}^n, s_{j,k}^n)$, so

$$\sum_{k=1}^{f_n(j)} \kappa_{j,k} \triangleq f_n(j + 1).$$

Then,

$$P[\kappa_{j,k} = 0] = \int_0^\infty [1 - A_{j,k}^{(0)}(x)] dB_{j,k}^{(0)}(x),$$

and for $m \geq 1$

$$P[\kappa_{j,k} = m] = \prod_{i=0}^{m-1} \left[1 - \int_0^\infty [1 - A_{j,k}^{(i)}(x)] B_{j,k}^{(i)}(x) \right] \times \int_0^\infty [1 - A_{j,k}^{(m)}(x)] dB_{j,k}^{(m)}(x). \tag{2.6}$$

Relationship (2.6) looks cumbersome, but it has a simple explanation. The term

$$\int_0^\infty [1 - A_{j,k}^{(m)}(x)] dB_{j,k}^{(m)}(x)$$

is the probability that during the residual service time corresponding to the m th inserted point there is no arrival, or in other words, the m th inserted point is last. Similarly, the product term

$$1 - \int_0^\infty [1 - A_{j,k}^{(i)}(x)] dB_{j,k}^{(i)}(x)$$

is the probability that during the residual service time corresponding to the i th inserted point there is at least one arrival.

Taking into account that both $A(x) \leq A_{j,k}^{(i)}(x)$ and $B_{j,k}^{(i)}(x) \leq B(x)$ for all j, k , and i , we have the following. Let κ_X be geometrically distributed random variable, $P[\kappa_X = m] = r^m(1 - r)$, $m = 0, 1, \dots$, where the parameters r is determined in the formulation of the theorem. Then, $\kappa_X \geq_{st} \kappa_{j,k}$, for all $j \geq 0$ and $k \geq 1$, and we have the following. Let $\kappa_X^{(j,k)}$ be the sequences of independent identically distributed integer random variables all having the same distribution as the random variable κ_X . We have

$$\sum_{k=1}^{f_n(j)} \kappa_{j,k} \leq_{st} \sum_{k=1}^{f_n(j)} \kappa_X^{(j,k)}.$$

Taking into account that

$$X_{j+1} = \sum_{k=1}^{X_j} \kappa_X^{(j,k)},$$

owing to induction we have

$$X_j \leq_{st} f_n(j),$$

and, therefore, $f_n(n + 1) = L_n \geq_{st} X_{n+1}$. The statement of the theorem is proved. □

From Theorem 2.1 we have the following special cases.

Corollary 2.2. *Under the assumption that $A(x) = 1 - e^{-\lambda x}$, and $B(x)$ belongs to class NWU (NBU), we have (2.1). X_n in (2.1) is the number of offspring in the n th generation of the Galton–Watson branching process with $X_0 = 1$ and the offspring generating function*

$$g(z) = \frac{\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)}, \quad |z| \leq 1, \tag{2.7}$$

Proof. Putting $A(x) = 1 - e^{-\lambda x}$, we have

$$r = 1 - \int_0^\infty e^{-\lambda x} dB(x) = 1 - \widehat{B}(\lambda), \tag{2.8}$$

and the statement follows by substituting (2.8) for (2.2). □

Corollary 2.3 (Abramov [1]). *Under the assumption that $B(x) = 1 - e^{-\mu x}$, and $A(x)$ belongs to class NBU (NWU), we have (2.1). X_n in (2.1) is the number of offspring in the n th generation of the Galton–Watson branching process with $X_0 = 1$ and the offspring generating function*

$$g(z) = \frac{1 - \widehat{A}(\mu)}{1 - z\widehat{A}(\mu)}, \quad |z| \leq 1.$$

Proof. Putting $B(x) = 1 - e^{-\mu x}$, we have

$$\begin{aligned} r &= 1 - \int_0^\infty [1 - A(x)]\mu e^{-\mu x} dx \\ &= \int_0^\infty A(x)\mu e^{-\mu x} dx \\ &= \int_0^\infty e^{-\mu x} dA(x) \\ &= \widehat{A}(\mu). \end{aligned} \tag{2.9}$$

Substituting (2.9) for (2.2) we obtain the desired representation. □

3. STRONGER INEQUALITIES FOR $M/GI/1/n$ QUEUES

In this section we develop the result for the $M/B/1/n$ queue given by Corollary 2.2. The main result of this section is the following.

Theorem 3.1. *Under the assumption that $A(x) = 1 - e^{-\lambda x}$, and $B(x)$ belongs to class NWU (NBU), we have*

$$L_n \geq_{st} \sum_{i=1}^{X_n} \tau_i \left(L_n \leq_{st} \sum_{i=1}^{X_n} \tau_i \right),$$

where the branching process $\{X_n\}$ is the same as in Corollary 2.2, and τ_1, τ_2, \dots is a sequence of independent identically distributed nonnegative integer random variables,

$$P[\tau_i = k] = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} dB(x).$$

Proof. Considering first the $M/B/1/0$ loss queue without waiting places it is not difficult to see that

$$L_0 \underset{st}{=} \text{Number of Poisson arrivals per service time,}$$

that is,

$$P[L_0 = k] = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} dB(x).$$

Let us now consider the $M/B/1/n$ queue, where $f_n(n)$ is the number of cases during a busy period when an arriving customer meets n customers in the system (recall that $L_n \overset{\Delta}{=} f_n(n + 1)$). Then, the number of losses L_n coincides in distribution with

$$\sum_{i=1}^{f_n(n)} \tau_i,$$

where the sequence τ_1, τ_2, \dots is a sequence of independent identically distributed integer random variables, coinciding in distribution with L_0 .

It follows from the proof of Theorem 2.1, that if in the $A/B/1/n$ queue $A(x)$ is NBU (NWU) and $B(x)$ is NWU (NBU), then

$$f_n(n) \underset{st}{\geq} X_n \quad (f_n(n) \underset{st}{\leq} X_n) \tag{3.1}$$

where the branching process $\{X_n\}$ is defined in Theorem 2.1, that is, $X_0 = 1$, and the offspring generating function is determined by 2.2. Therefore, in the case of $A(x) = 1 - e^{-\lambda x}$ we obtain 3.1, where now the offspring generating function of the branching process is defined by (2.7). This enables us to conclude that under the assumptions of the theorem

$$L_n \underset{st}{\geq} \sum_{i=1}^{X_n} \tau_i \quad \left(L_n \underset{st}{\leq} \sum_{i=1}^{X_n} \tau_i \right),$$

and the statement therefore is proved.

Considering now the $A/B/1/n$ queueing system, let T_n, v_n denote the length of a busy period and the number of served customers during a busy period respectively, and let $\chi^{(1)}, \chi^{(2)}, \dots$ be a sequence

of independent identically distributed random variables all having the probability distribution function $B(x)$. We have

$$v_n = \sum_{j=0}^n f_n(j),$$

$$T_n = \sum_{m=1}^{v_n} \chi^{(m)}.$$

Immediately from the above proof, under the assumption that $A(x)$ is NBU (NWU), and $B(x)$ is NWU (NBU), we have

$$v_n \geq_{st} \sum_{i=0}^n X_i, \quad \left(v_n \leq_{st} \sum_{i=0}^n X_i \right), \tag{3.2}$$

where the branching process $\{X_n\}$ is defined in Theorem 2.1. If $A(x) = 1 - e^{-\lambda x}$, then (3.2) holds true. The only difference that the offspring generating function of the process $\{X_n\}$ is given by (2.7).

Whereas the sequence of $\chi^{(1)}, \chi^{(2)}, \dots$ consists of independent identically distributed random variables, the random variable v_n is independent of the future, that is the event $\{v_n = i\}$ is independent of $\chi^{(i+1)}, \chi^{(i+2)}, \dots$ (e.g., [9]). Therefore, $E[T_n]$ is determined by the Wald's identity: $\mu E[T_n] = E[v_n]$. Then under the above assumptions that $A(x)$ is NBU (NWU) and $B(x)$ is NWU (NBU), we have

$$E[T_n] \geq \frac{1}{\mu} E \left[\sum_{i=0}^n X_i \right] \quad \left(E[T_n] \leq \frac{1}{\mu} E \left[\sum_{i=0}^n X_i \right] \right). \tag{3.3}$$

Taking into account that $E[X_n] = r^n / (1 - r)^n$, under the above assumptions from (3.3) we obtain

$$E[T_n] \geq \frac{1}{\mu} \sum_{i=0}^n \frac{r^i}{(1 - r)^i} \quad \left(E[T_n] \leq \frac{1}{\mu} \sum_{i=0}^n \frac{r^i}{(1 - r)^i} \right).$$

Clearly, that in the case where $A(x) = 1 - e^{-\lambda x}$ the parameter r is equal to $1 - \hat{B}(\lambda)$ (see the proof of Corollary 2.2).

4. FURTHER STOCHASTIC INEQUALITIES FOR THE GI/M/1/n LOSS SYSTEM

Being the special case of Theorem 2.1, Corollary 2.3 provides the stochastic inequalities for the $A/M/1/n$ under the assumption that $A(x)$ belongs to the class NBU (NWU). Assuming now that $A(x)$ belongs to the class IHR (DHR), we provide a deepen analysis in order to obtain stronger stochastic inequalities.

4.1. Monotonicity

For the sake of simplicity the $A/M/1/n$ queueing system is denoted \mathcal{Q}_n . Recall that parameter n excludes the position of a customer in service. For n and k different, $\mathcal{Q}_n, \mathcal{Q}_k$ are two queueing systems with the same probability distribution functions of interarrival and service time but different number of waiting places. For example, \mathcal{Q}_0 denotes the $A/M/1/0$ queueing system without waiting places, a busy period of which contains only a single service time.

Consider a busy period of the queueing system \mathcal{Q}_n . Let us consider the interval $[t_{0,1}^n, s_{0,1}^n)$ after the procedure of deleting from it all the intervals $[t_{1,l}^n, s_{1,l}^n)$, $l = 1, 2, \dots, f_n(1)$, and connecting the ends as it is described in the proof of Theorem 2.1. Then, let $i_1^n, i_2^n, \dots, i_{f_n(1)}^n$ denote the inserted points within the interval $[t_{0,1}^n, s_{0,1}^n)$, and let d_l^n denote the distance between the two adjacent points i_l^n and i_{l+1}^n ($l = 1, 2, \dots, f_n(1) - 1$), that is, $d_l^n = i_{l+1}^n - i_l^n$. If $f_n(1) = 0$, i.e., there is no inserted points, then the distance between inserted points is not defined. If $f_n(1) = 1$, then by the value d_1^n we mean the distance between the point i_1^n and the next arrival of a customer at the system after the instant $s_{0,1}^n$.

Lemma 4.1. *Let \mathcal{Q}_k and \mathcal{Q}_n be two queueing systems, and let $A(x)$ belong to the IHR (DHR) class of distributions. If $k \leq n$ then*

$$d_l^n \leq_{st} d_l^k \quad (d_l^n \geq_{st} d_l^k).$$

Proof. Let us consider the queueing system \mathcal{Q}_n , and the interval $[t_{0,1}^n, s_{0,1}^n)$ after the procedure of deleting from it all intervals $[t_{1,l}^n, s_{1,l}^n)$, $l = 1, 2, \dots, f_n(1)$, and connecting the ends. For convenience, we denote the sequence of independent identically and exponentially distributed random variables with parameter μ by $\chi^{(1)}, \chi^{(2)}, \dots$, and a random variable τ , having the probability distribution $A(x)$, is independent of this sequence $\chi^{(1)}, \chi^{(2)}, \dots$.

The probability, that during the interval $[t_{0,1}^n, s_{0,1}^n)$ there is no arrival, is

$$1 - \int_0^\infty \mu e^{-\mu x} A(x) dx = 1 - \widehat{A}(\mu).$$

Obviously, that this probability is independent of parameter n . Let us assume that there is the inserted point i_1^n and, therefore, the instant of arrival $t_{1,1}^n$.

Let q_n denote the stationary number of customers in the queueing system \mathcal{Q}_n immediately after arrival of a customer at the system *during a busy period*, that is, not into the empty system. (An arriving customer, who finds all waiting places busy, leaves the system without incrementing

and decrementing the number of customers in the queue.) Let $\tilde{q}_n = q_n - 1$, and let

$$v = \inf \left\{ m : \sum_{j=1}^{\tilde{q}_n} \chi^{(j+m-1)} \leq \tau \right\}.$$

(The empty sum is assumed to be 0. The case of empty sum arises only by considering of the queueing system \mathcal{Q}_0 .) Then

$$d_l^n \stackrel{d'}{=} \tau - \sum_{j=1}^{\tilde{q}_n} \chi^{(v+j-1)}. \tag{4.1}$$

For example, in the case of the queueing system \mathcal{Q}_0 , we have

$$P[d_l^0 \leq x] = P[\tau \leq x] = A(x),$$

and in the case of the queueing system \mathcal{Q}_1 we have

$$\begin{aligned} P[d_l^1 \leq x] &= P[\tau - \chi^{(1)} \leq x \mid \tau > \chi^{(1)}] \\ &= \int_0^\infty P[\tau \leq x + y \mid \tau > y] \mu e^{-\mu y} dy \\ &= \int_0^\infty \frac{A(x + y) - A(y)}{1 - A(y)} \mu e^{-\mu y} dy. \end{aligned} \tag{4.2}$$

By analysis of sample paths it is clear that for these two queueing systems \mathcal{Q}_n and \mathcal{Q}_{n+1}

$$\tilde{q}_n \leq_{st} \tilde{q}_{n+1}. \tag{4.3}$$

Since $A(x)$ belongs to the IHR (DHR) class of distributions, then (4.3) together with (4.1) yield $d_l^{n+1} \leq_{st} d_l^n$ ($d_l^{n+1} \geq_{st} d_l^n$). The statement of lemma follows. \square

Remark 4.2. Lemma 4.1 establishes a property of external monotonicity. However, from Lemma 4.1 we obtain the property of internal monotonicity as well. Indeed, in the case of the $GI/M/1/n$ queueing system, because of the property of the lack of memory of the exponential distribution of a service time, any interval of (2.3) is distributed as a busy period of the queueing system \mathcal{Q}_{n-j} , $0 \leq j \leq n$. Therefore the distance between two inserted points of each interval (2.3) coincides in distribution with d_1^{n-j} .

4.2. A Branching Process

Let us consider the $A/M/1$ queueing system (with infinite number of waiting places), denoting it by \mathcal{Q} and remaining for this system all the above notation given earlier for the queueing system \mathcal{Q}_n . Assume additionally that the load $\rho = \lambda/\mu \leq 1$.

Analogously to the case of the queueing system \mathcal{Q}_n , for the queueing system \mathcal{Q} let $f(j)$, $j \geq 0$, denote the number of customers, arriving during a busy period, who, upon their arrival meet j customers in the system ($f(0) = 1$). Let $t_{j,1}, t_{j,2}, \dots, t_{j,f(j)}$ be the instants of these arrivals, and let $s_{j,1}, s_{j,2}, \dots, s_{j,f(j)}$ be the instants of corresponding service completions defined analogously to the case of the queueing system \mathcal{Q}_n . Let $\mathcal{F}_j = \sigma\{f(0), f(1), \dots, f(j)\}$.

It is claimed in [4], that the stochastic sequence $\{f(j), \mathcal{F}_j\}$ is a Galton-Watson branching process, and $E[f(1)] = \varphi$, where φ is the least in absolute value root of the functional equation $z = \widehat{A}(\mu - \mu z)$.

According to the standard definition of the Galton-Watson branching process, the number of offspring generated by all particles are mutually independent random variables (e.g., Harris [10]). The Galton-Watson branching process $\{f(j), \mathcal{F}_j\}$, considered in [4] for the case of $GI/M/1$ queues, is not traditional. The number of offspring generated by particles of different generations are not independent random variables. More precisely, the number of offspring of the n th generation is an independent of the future random variable with respect to the numbers of offspring generated by particles of the n th generation.

Notice, that connection between standard branching process and optimal stopping times has been discussed by Assaf et al. [11].

For a more detailed explanation the structure of the abovementioned dependence, related to the above case of the $A/M/1$ queueing system, let us consider the interval $[t_{0,1}, s_{0,1})$, and assume that there is a point $t_{1,1}$. Let $d_1 = t_{1,2} - s_{1,1}$ denote the distance between the begin of the second interval and the end of the first one (provided that the second interval does exist). If there is only a single interval then d_1 also has sense as it is explained in Section 4.1.

If during the time interval $[t_{1,1}, s_{1,1})$ there is no new arrival (denote this event by E_0), then

$$\begin{aligned} P[d_1 \leq x | E_0] &= P[\tau - \chi_1 \leq x | \tau > \chi_1] \\ &= \int_0^\infty P[\tau \leq x + y | \tau > y] \mu e^{-\mu y} dy \\ &= \int_0^\infty \frac{A(x + y) - A(y)}{1 - A(y)} \mu e^{-\mu y} dy. \end{aligned} \tag{4.4}$$

Recall that $P[\tau \leq x] = A(x)$, and $P[\chi_1 \leq x] = 1 - e^{-\mu x}$. Thus (4.4) coincides with (4.2), and $P[d_1 \leq x | E_0] = P[d_1^* \leq x]$. For example, if

$P[\tau = 1] = 1$ and $\mu \geq 1$, then from (4.4) we obtain

$$P[d_1 \leq x | E_0] = \min \left\{ 1, \frac{e^{\mu x - \mu} - e^{-\mu}}{1 - e^{-\mu}} \right\}, \quad x \geq 0.$$

If during the time interval $[t_{1,1}, s_{1,1})$ there is at least one arrival (denote this event by E_1), then we have the following. Let $\{q(i)\}_{i \geq 1}$ be a stationary sequence of the numbers of customers in the system immediately *before* arrival of a customer during a busy period (i.e., not into the empty system). Let us consider the sequence $\{q(i)1_{[q(i) \geq 2]}\}_{i \geq 1}$. Taking only the positive elements of this sequence one can construct a *new stationary sequence* $\{\tilde{q}(i)\}_{i \geq 1}$ all elements of which are not smaller than 2. Then,

$$v = \inf \left\{ m : \sum_{j=1}^{\tilde{q}(1)} \chi^{(j+m-1)} \leq \tau \right\},$$

and

$$P[d_1 \leq x | E_1] = P \left[\tau - \sum_{j=1}^{\tilde{q}(1)} \chi^{(j+v-1)} \leq x \right]. \tag{4.5}$$

Comparing (4.4) and (4.5) it is not difficult to conclude that if $A(x)$ belongs to the IHR (DHR) class of distributions, then

$$P[d_1 \leq x | E_0] \leq P[d_1 \leq x | E_1]$$

$$(P[d_1 \leq x | E_0] \geq P[d_1 \leq x | E_1])$$

For example, if $P[\tau = 1] = 1$, and $\mu \geq 1$, then we have the strong inequality:

$$P[d_1 \leq x | E_0] = \min \left\{ 1, \frac{e^{\mu x - \mu} - e^{-\mu}}{1 - e^{-\mu}} \right\}$$

$$< P[d_1 \leq x | E_1] \quad (x \geq 0).$$

Thus, the random variable $f(1)$ depends on the events E_0 and E_1 . In other words $f(1)$ can have different distributions if a particle of the first generation has or does not have an offspring. Let us call such Galton-Watson branching process by *GI/M/1 type Galton-Watson branching process*.

Notice, that the known property of a Galton-Watson branching process that $E[f(j)] = \varphi^j$ (e.g., Doob [12]; Harris [10]), also remains in force for the *GI/M/1 type Galton-Watson branching process*.

Indeed, according to the total expectation formula, for $E[f(1)]$ we obtain:

$$E[f(1)] = \sum_{n=0}^{\infty} E[f(n)] \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^n}{n!} dA(x) \tag{4.6}$$

By the same arguments for all $j \geq 1$ we have:

$$E[f(j + 1)] = \sum_{n=0}^{\infty} E[f(n + j)] \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^n}{n!} dA(x).$$

Therefore, $Ef(n) = z^n$, and from (4.6) we have:

$$E[f(1)] = z = \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^n}{n!} dA(x) = \widehat{A}(\mu - \mu z).$$

Since $z < 1$, then $z = \varphi$, and $E[f(n)] = \varphi^n$.

4.3. The Number of Losses During a Busy Period

Returning to the queueing system \mathcal{Q}_n once again, assume additionally that the load $\rho \leq 1$. All queueing systems \mathcal{Q}_n with different n and the queueing system \mathcal{Q} are assumed to be given on the same probability space, and the probability distribution function $A(x)$ belongs to the IHR (DHR) class of distributions. According to Lemma 4.1 we have

$$d_l \leq_{st} d_l^n \quad (d_l \geq_{st} d_l^n), \tag{4.7}$$

where d_l is the distance between the l th and $l + 1$ st inserted points of the queueing system \mathcal{Q} , as it is precisely defined in Section 4.2. Stochastic inequality (4.7) is the limiting case, as $k \rightarrow \infty$, of a series of inequalities for the distances $d_l^k \leq_{st} d_l^n$ ($d_l^k \geq_{st} d_l^n$), given for all $k > n$.

Let us now consider the interval $[t_{0,1}^n, s_{0,1}^n)$ after deleting all the intervals $[t_{1,j}^n, s_{1,j}^n)$ and connecting the ends, as it is explained above. Then the remaining interval, because of the property of the lack of memory, is exponentially distributed with parameter μ , and it coincides in distribution with the interval $[t_{0,1}, s_{0,1})$, associated with the queueing system \mathcal{Q} , remaining after deleting of all the intervals $[t_{1,j}, s_{1,j})$ and connecting the ends. Under the assumption that both queueing processes of \mathcal{Q}_n and \mathcal{Q} are defined on the same probability space, one may consider only one of these intervals, comparing then the sample path of relevant processes. Then for the number of losses L_n during a busy period of the queueing system \mathcal{Q}_n we have the following.

Theorem 4.3. *If $A(x)$ belongs to the IHR (DHR) class of distributions, and the load $\rho \leq 1$, then*

$$L_n \leq_{st} Y_{n+1} \quad (L_n \geq_{st} Y_{n+1}),$$

where Y_n denotes the number of offspring in the n th generation of the GI/M/1 type Galton-Watson branching process generated by the queueing system \mathcal{Q} .

Notice, that under the assumptions of Theorem 4.3 we have the inequality

$$E[L_n] \leq \varphi^{n+1} \quad (E[L_n] \geq \varphi^{n+1}). \tag{4.8}$$

On the other hand, taking into account that the class IHR (DHR) is contained in the class NBU (NWU), from Corollary 2.3 we obtain the inequality:

$$\left(\left[\frac{\widehat{A}(\mu)}{1 - \widehat{A}(\mu)} \right]^{n+1} \leq E[L_n] \right. \\ \left. \left(\left[\frac{\widehat{A}(\mu)}{1 - \widehat{A}(\mu)} \right]^{n+1} \geq E[L_n] \right) \right). \tag{4.9}$$

Joining (4.8) and (4.9), under the assumptions of Theorem 4.3 we obtain the two-side inequalities

$$\left[\frac{\widehat{A}(\mu)}{1 - \widehat{A}(\mu)} \right]^{n+1} \leq E[L_n] \leq \varphi^{n+1} \\ \left(\left[\frac{\widehat{A}(\mu)}{1 - \widehat{A}(\mu)} \right]^{n+1} \geq E[L_n] \geq \varphi^{n+1} \right). \tag{4.10}$$

For example, in the case of the $M/M/1/n$ queuing system, when $A(x) = 1 - e^{-\lambda x}$, from (4.10) we obtain $E[L_n] = \rho^{n+1}$.

It is interesting to note the following property. It follows from (4.8) that if $A(x)$ belongs to the IHR (DHR) class of distributions and $\rho \leq 1$ ($\rho \geq 1$), then $E[L_n] \leq 1$ ($E[L_n] \geq 1$) for all $n \geq 0$. This is the special case of the more general result of Wolff [7] for losses in $GI/GI/1/n$ queues under the assumption that interarrival time distribution belongs to the class NBUE (NWUE).

Let us provide inequalities for a busy period T_n and the number of customers served during a busy period of the $A/M/1/n$ queue. Under the assumption that $A(x)$ is IHR (DHR) and $\rho < 1$, we have

$$v_n \leq_{st} \sum_{j=0}^n Y_j \quad \left(v_n \geq_{st} \sum_{j=0}^n Y_j \right), \tag{4.11}$$

where the branching process $\{Y_j\}$ is as in Theorem 4.3.

From (4.11), assuming that $A(x)$ is IHR (DHR) and $\rho < 1$, we obtain

$$E[v_n] \leq \sum_{i=0}^n \varphi^i \quad \left(E[v_n] \geq \sum_{i=0}^n \varphi^i \right). \tag{4.12}$$

On the other hand, taking into account that class IHR (DHR) is contained in class NBU (NWU), from Corollary 2.3 we obtain the following inequality:

$$\begin{aligned} \sum_{i=0}^n \left[\frac{\widehat{A}(\mu)}{1 - \widehat{A}(\mu)} \right]^i &\leq E[v_n] \\ \left(\sum_{i=0}^n \left[\frac{\widehat{A}(\mu)}{1 - \widehat{A}(\mu)} \right]^i \right) &\geq E[v_n] \end{aligned} \quad (4.13)$$

Combining (4.12) and (4.13), under the above assumptions we obtain the two-side inequalities:

$$\begin{aligned} \sum_{i=0}^n \left[\frac{\widehat{A}(\mu)}{1 - \widehat{A}(\mu)} \right]^i &\leq E[v_n] \leq \sum_{i=0}^n \varphi^i \\ \left(\sum_{i=0}^n \left[\frac{\widehat{A}(\mu)}{1 - \widehat{A}(\mu)} \right]^i \right) &\geq E[v_n] \geq \sum_{i=0}^n \varphi^i. \end{aligned}$$

Finally, by Wald's identity we have

$$\begin{aligned} \frac{1}{\mu} \sum_{i=0}^n \left[\frac{\widehat{A}(\mu)}{1 - \widehat{A}(\mu)} \right]^i &\leq E[T_n] \leq \frac{1}{\mu} \sum_{i=0}^n \varphi^i \\ \left(\frac{1}{\mu} \sum_{i=0}^n \left[\frac{\widehat{A}(\mu)}{1 - \widehat{A}(\mu)} \right]^i \right) &\geq E[T_n] \geq \frac{1}{\mu} \sum_{i=0}^n \varphi^i. \end{aligned}$$

ACKNOWLEDGMENT

The author thanks Taleb Samira (USTHB, Algeria) for useful comment.

REFERENCES

1. Abramov, V.M. 2001. Inequalities for the $GI/M/1/n$ loss system. *J. App. Prob.* 38:232–234.
2. Abramov, V.M. 1994. On the asymptotic distribution of the maximum number of infectives in epidemic models with immigration. *J. App. Prob.* 31:606–613.
3. Abramov, V.M. 1991. *Investigation of a Queueing System with Service Depending on Queue Length*. Donish, Dushanbe, Tadjhikistan (in Russian).
4. Abramov, V.M. 2001. Some results for large closed queueing networks with and without bottleneck: Up- and down-crossings approach. *Queueing Systems* 38:149–184.
5. Peköz, E.A., Righter, R., and Xia, C.H. 2003. Characterizing losses in finite buffer systems. *J. App. Prob.* 40:242–249.

6. Richter, R. 1999. A note on losses in the $M/GI/1/n$ queue. *J. App. Prob.* 36:1240–1243.
7. Wolff, R.W. 2002. Losses per cycle in a single-server queue. *J. App. Prob.* 39:905–909.
8. Stoyan, D. 1983. *Comparison Methods for Queues and Other Stochastic Models*. John Wiley, Chichester.
9. Borovkov, A.A. 1986. *Theory of Probability*. Nauka, Moscow (in Russian).
10. Harris, T.E. 1963. *The Theory of Branching Processes*. Springer-Verlag, Berlin.
11. Assaf, D., Goldstein, L., and Samuel-Sahn, E. 2000. An unexpected connection between branching processes and optimal stopping. *J. App. Prob.* 37:613–626.
12. Doob, J.L. 1953. *Stochastic Processes*. John Wiley, New York.