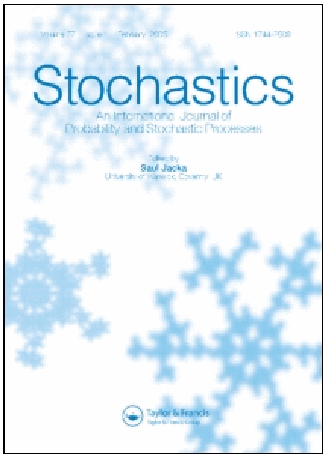

Stochastics An International Journal of Probability and Stochastic Processes
formerly Stochastics and Stochastics Reports
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713652481

On the implicit Black-Scholes formula

Kais Hamza [a]; Fima C. Klebaner [a]
[a] School of Mathematical Sciences, Monash University, VIC, Australia

Online Publication Date: 01 February 2008
To cite this Article: Hamza, Kais and Klebaner, Fima C. (2008) 'On the implicit Black-Scholes formula ', Stochastics An International Journal of Probability and Stochastic Processes, 80:1, 97 - 102
To link to this article: DOI: 10.1080/17442500701607706
URL: http://dx.doi.org/10.1080/17442500701607706

Stochastics: An International Journal of Probability and Stochastics Processes,
Vol. 80, No. 1, February 2008, 97–102

Taylor & Francis
Taylor & Francis Group

On the implicit Black–Scholes formula

KAIS HAMZA* and FIMA C. KLEBANER†

School of Mathematical Sciences, Monash University, VIC 3800, Australia

(Received 11 November 2006; in final form 31 July 2007)

We prove that, if the Black–Scholes formula holds implicitly then the model is the Black–Scholes model. In a recent paper, [Hamza K. and Klebaner F. C. *On nonexistence of non-constant volatility in the Black–Scholes formula*. Discrete and Continuous Dynamical Systems, **6** (2006), 829–834] it is shown that, models with non-constant implied volatility θ_t, assumed to be a function (possibly random) of time t, are not compatible with the Black–Scholes formula, unless θ_t is a constant. Here, we give generalizations by allowing θ_t to also depend on the maturity T, $\theta(t, T)$.

Keywords: Black–Scholes; Semimartingale; Volatility; Merton model; Implied volatility; Stochastic volatility

1991 Mathematics Subject Classification: Primary: 60G44, 60H30; Secondary: 91B28, 91B70

1. Introduction

It is hard to overestimate the importance and impact of the Black–Scholes formula in financial mathematics as well as in practical applications to financial markets and risk management. This formula gives the price of an option when the stock price is modelled by the constant volatility Black–Scholes–Merton model.

It is widely believed and experimentally verified that, stocks do not have a constant volatility, rather this parameter varies with time [4,5,7,15]. In spite of this fact, the Black–Scholes options pricing formula (2) remains the yardstick of option pricing. Practitioners justify its use by incorporating adjustments to the volatility. The question of existence of a model, in which options prices are given by the Black–Scholes formula is of great interest in financial mathematics as well as in practical applications [1–3,13].

In this paper, we show that, if the Black–Scholes formula holds implicitly then the model must be the Black–Scholes–Merton model. The precise definition of "holds implicitly" is given later, but it essentially means that the option price can be calculated by the Black–Scholes formula with *some* "volatility" parameter θ, not necessarily related to the model of

*Corresponding author. Email: kais.hamza@sci.monash.edu.au
†Email: fima.klebaner@sci.monash.edu.au
This work was supported by the Australian Research Council (Grant DP0451657).

Stochastics: An International Journal of Probability and Stochastics Processes
ISSN 1744-2508 print/ISSN 1744-2516 online © 2008 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/17442500701607706

stock. In a recent paper [6], the authors have shown that, when the parameter θ is only a random function of time t, then the assumption that the Black–Scholes formula holds implicitly implies that θ is constant. Here, we extend this conclusion to the case when θ is a function (possibly random) of time t as well as maturity T, $\theta(t, T)$ (Theorem 3). The case when θ also depends on the strike K is still open.

To make the above discussion precise, we introduce notations and recall the basic facts on options pricing [11,14]. Let S_t denote the price of a stock at time t. We can assume without loss of generality, that the riskless interest rate $r = 0$, otherwise work with discounted prices $S_t\,\mathrm{e}^{-rt}$. The first fundamental theorem of asset pricing states that a model does not admit arbitrage if and only if there exists an equivalent probability measure Q such that S_t is a Q-martingale. The price at time t of a call option that pays $(S_T - K)^+$ at time T is given by

$$C_t = \mathbb{E}_Q[(S_T - K)^+ | \mathcal{F}_t^S], \tag{1}$$

where $\mathbb{E}_Q$ is the expectation under Q and $\mathcal{F}_t^S$ is the σ-field generated by the process S_u, $u \leq t$. To introduce the Black–Scholes formula, we let B_t be a Brownian motion on a filtered probability space satisfying the usual conditions, $(\Omega, \mathbb{F}, \mathcal{F}, Q)$. Let Z_t evolve according to the Black–Scholes–Merton model

$$\mathrm{d}Z_t = \sigma Z_t\,\mathrm{d}B_t, \quad Z_0 = z_0.$$

The parameter σ is known as the volatility of the process Z_t.

The price of an option on stock Z_t is given by the famous Black–Scholes formula denoted here by $C(T, t, K, \sigma, z)$,

$$\begin{aligned} C(T, t, K, \sigma, z) &= \mathbb{E}[(Z_T - K)^+ | Z_t = z] \\ &= S_t \Phi\left(\frac{\log \frac{S_t}{K} + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) - K\Phi\left(\frac{\log \frac{S_t}{K} - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned} \tag{2}$$

where Φ denotes the standard normal distribution function.

The following result was proved in [6].

Theorem 1. Let S_t and θ_t be two adapted processes such that, $\theta_0 = \sigma$ and $S_0 = z_0$. Assume that, S_t is strictly positive, that $\mathcal{F}_0$ is trivial and that there exist three terminal times, $T_1 < T_2 < T_3$ such that, for all K and all $t \leq T_i$, $i = 1,2,3$,

$$\mathbb{E}[(S_{T_i} - K)^+ | \mathcal{F}_t] = C(T_i, t, K, \theta_t, S_t).$$

Then $\theta_t^2 = \sigma^2$ for all $t \leq T_1$.

As stated earlier, this paper considers the case of a maturity-dependent θ. It is well known [15] that the Black–Scholes formula holds in the Black–Scholes–Merton model with constant volatility (as mentioned above), but also in the more general case of a volatility that is a non-random function of time: $\mathrm{d}S_t = \vartheta(t) S_t\,\mathrm{d}B_t$, where $\vartheta(t)$ is non-random function of t. In this model, the distribution of the stock S_t is also log-normal, and the Black–Scholes formula holds with averaged future volatility

$$\theta^2(t, T) = \frac{1}{T-t}\int_t^T \vartheta^2(u)\mathrm{d}u.$$

In this paper, we prove that, under certain mild technical conditions, non-random volatility is the only case compatible with the Black–Scholes formula. In the proof, we use the concept of the stochastic logarithm of S, herein denoted as $\mathcal{L}(S)$. It is defined as the unique semimartingale X such that $S = S_0\mathcal{E}(X)$; that is $\mathrm{d}S = S_-\mathrm{d}X$. Its existence is guaranteed by the following result [10,11].

LEMMA 1. Let X be a positive semimartingale such that X and X_- remain strictly positive. Then there exists a unique semimartingale, herein denoted $\mathcal{L}(X)$, such that $X = X_0\mathcal{E}(\mathcal{L}(X))$. It is given by $\mathrm{d}\mathcal{L}(X)_t = (\mathrm{d}X_t/X_{t-})$.

2. Results

Throughout this section S is a non-negative process adapted to a filtration $\mathcal{F}$ on a probability space $(\Omega, \mathbb{F}, Q)$.

DEFINITION. We say that, the Black–Scholes formula holds implicitly at T if there exists an adapted process, $(\theta(t,T))_{t\le T}$, such that for all $t \le T$ and for all $K \ge 0$,

$$\mathbb{E}[(S_T - K)^+|\mathcal{F}_t] = C(T,t,K,\theta(t,T),S_t). \tag{3}$$

THEOREM 2. Suppose that the Black–Scholes formula holds implicitly at T, that S is continuous, that $S_T > 0$, and that $(\theta(t,T))_{t\le T}$ is predictable and of finite variation. Then

$$\theta^2(t,T) = \frac{1}{T-t}\int_t^T \frac{\mathrm{d}\langle S,S\rangle_u}{S_u^2}. \tag{4}$$

In particular, if $\mathcal{F}_0$ is trivial then $\int_0^T \mathrm{d}\langle S,S\rangle_u/S_u^2$ is non-random.

THEOREM 3. Suppose that the Black–Scholes formula holds implicitly at all $T \le T^*$, that S is continuous, that $S_{T^*} > 0$, that $\mathcal{F}_0$ is trivial, and that, for any $T \le T^*$, $(\theta(t,T))_{t\le T}$ is predictable and of finite variation. Then there exists a Gaussian martingale L with non-random quadratic variation $U(t) = \langle L,L\rangle_t$, such that $\mathrm{d}S_t = S_t\mathrm{d}L_t$ and for all $t \le T \le T^*$,

$$\theta^2(t,T) = \frac{1}{T-t}(U(T) - U(t)). \tag{5}$$

Moreover, if $\theta^2(0,T)$, as a function of T, is differentiable, non-decreasing and satisfies $\theta^2(0,0) > 0$, then there exists a non-random function $\vartheta(t)$ and a Brownian motion W_t, such that $\mathrm{d}S_t = \vartheta(t)S_t\,\mathrm{d}W_t$, and

$$\theta^2(t,T) = \frac{1}{T-t}\int_t^T \vartheta^2(u)\mathrm{d}u. \tag{6}$$

Remark. Before we attend to the proofs let us discuss the requirement that the Black–Scholes formula holds implicitly for all $T \le T^*$ rather than a single fixed T. The question can be reformulated as whether non-randomness of the integral $\int_0^T H_t^2\,\mathrm{d}t$ (for a single fixed T) of a predictable process H_t^2 implies non-randomness of the process H_t^2 itself, $0 \le t \le T$. Note that, the standard example of $H_t = \mathrm{sign}(B_t)$ does not meet our requirements since in this case $H_t^2 \equiv 1$ is deterministic.

The following example exhibits a (random, non-constant) predictable H such that $\int_0^1 H_t^2\,\mathrm{d}t = 1$. Let τ be the first time a Brownian motion started at zero hits ± 1, $\tau = \inf\{t : |B_t| = 1\}$,

$$H_t = B_{t\wedge\tau},\ \ t \le 1/2 \ \text{ and } \ H_t = \sqrt{2}\sqrt{1 - \int_0^{1/2} B_{s\wedge\tau}^2\,\mathrm{d}s}, t > 1/2.$$

H is clearly predictable (adapted and left continuous) and satisfies

$$\int_0^1 H_t^2\,\mathrm{d}t = \int_0^{1/2} H_t^2\,\mathrm{d}t + \int_{1/2}^1 H_t^2\,\mathrm{d}t = \int_0^{1/2} B_{t\wedge\tau}^2\,\mathrm{d}t + 2\left(1 - \int_0^{1/2} B_{t\wedge\tau}^2\,\mathrm{d}t\right) \times \frac{1}{2} = 1.$$

The proofs of theorems 2 and 3 are broken into a number of propositions. Note that the assumptions of these are more general, in particular continuity is not always assumed.

Proposition 1. Suppose that the Black–Scholes formula holds implicitly at T. Then S_t and $S_t^2\,\mathrm{e}^{(T-t)\theta^2(t,T)}$, $t \le T$, are martingales.

We give a sketch of the proof only since it follows the steps of Proposition 2 in [6].

Proof. The proof uses the fact that the function $G(K) = \mathbf{E}[(X-K)^+|\mathcal{G}]$ defines the conditional distribution of X given $\mathcal{G}$. The fact that $S_t, t \le T$, is a martingale then immediately follows from the application of (3) to the case $K = 0$,

$$\mathbf{E}[S_T|\mathcal{F}_t] = C(T, t, 0, \theta_t, S_t) = \mathbf{E}[Z_T|Z_t = z]_{\sigma=\theta_t, z=S_t} = z|_{\sigma=\theta_t, z=S_t} = S_t.$$

The second statement uses the fact that, for a positive X,

$$\mathbf{E}[X^2|\mathcal{G}] = 2\int_0^\infty \mathbf{E}[(X-K)^+|\mathcal{G}]\mathrm{d}K.$$

Indeed,

$$\mathbf{E}[S_T^2|\mathcal{F}_t] = \mathbf{E}[Z_T^2|Z_t = z]|_{\sigma=\theta(t,T), z=S_t} = S_t^2\,\mathrm{e}^{(T-t)\theta^2(t,T)},$$

which establishes that $S_t^2\,\mathrm{e}^{(T-t)\theta^2(t,T)}$, $t \le T$, is a martingale. □

Since, S_t^2 is a strict submartingale such that $S_t^2\,\mathrm{e}^{(T-t)\theta^2(t,T)}$ is a true martingale, and $\theta(t,T)$ is predictable and of finite variation, it follows that $\mathrm{e}^{-(T-t)\theta^2(t,T)}$ is the multiplicative compensator of S_t^2. In the terminology of [10], $-(T-t)\theta^2(t,T)$ is the exponential compensator of $2\ln S_t$, i.e. it is the (unique) predictable process of finite variation (in fact, increasing process) in the multiplicative decomposition of the special semimartingale (in fact, submartingale), S_t^2 [8,10].

The following result can be extracted from [8,9,10], but for convenience we include a simple proof.

Lemma 2. Let X be a positive submartingale such that X and X_- remain strictly positive. If $\mathcal{L}(X)$ is a special semimartingale and V denotes the (unique) predictable process of finite variation in its canonical decomposition, then $X\mathcal{E}(V)^{-1}$ is a local martingale. Furthermore, $\mathcal{E}(V)$ is the unique predictable process of finite variation D satisfying $D_0 = 1$ such that $X = LD$ where L is a local martingale.

Proof. As a submartingale, X is special, and the uniqueness of the decomposition $X = LD$ is given in [8], Corollary 3. Since $\mathcal{E}(V)^{-1}$ is a predictable process of finite variation, the integration by parts formula ([9], Theorem 2.53) yields,

$$\mathrm{d}(X\mathcal{E}(V)^{-1})_t = \mathcal{E}(V)_t^{-1}\,\mathrm{d}X_t + X_{t-}\,\mathrm{d}(\mathcal{E}(V)^{-1})_t.$$

Let $M = \mathcal{L}(X) - V$. Then $\mathrm{d}X_t = X_{t-}\,\mathrm{d}\mathcal{L}(X)_t = X_{t-}\,\mathrm{d}M_t + X_{t-}\,\mathrm{d}V_t$ is the canonical decomposition of X, and, writing C_t for $\sum_{s\le t}(\Delta V_s)^2/\mathcal{E}(V)_s$,

$$\begin{aligned}\mathrm{d}(X\mathcal{E}(V)^{-1})_t &= \mathcal{E}(V)_t^{-1}X_{t-}\,\mathrm{d}M_t + \mathcal{E}(V)_t^{-1}X_{t-}\,\mathrm{d}V_t + X_{t-}\left[-\frac{\mathrm{d}V_t}{\mathcal{E}(V)_{t-}} + \mathrm{d}C_t\right]\\ &= \mathcal{E}(V)_t^{-1}X_{t-}\,\mathrm{d}M_t + X_{t-}\left[\Delta(\mathcal{E}(V)^{-1})_t\,\mathrm{d}V_t + \mathrm{d}C_t\right]\\ &= \mathcal{E}(V)_t^{-1}X_{t-}\,\mathrm{d}M_t + X_{t-}\left[-\frac{\Delta V_t}{\mathcal{E}(V)_t}\mathrm{d}V_t + \mathrm{d}C_t\right] = \mathcal{E}(V)_t^{-1}X_{t-}\,\mathrm{d}M_t.\end{aligned}$$

□

Proposition 2. Suppose that the Black–Scholes formula holds implicitly at T, that $S_T > 0$, and that $\theta(t, T)$ is predictable and of finite variation. Then,

$$\theta^2(t,T) = \frac{\ln(\mathcal{E}(V)_T) - \ln(\mathcal{E}(V)_t)}{T-t} \quad = \frac{1}{T-t}\left(V_T - V_t + \sum_{t<s\le T}[\ln(1+\Delta V_s) - \Delta V_s]\right) \tag{7}$$

where V is the (additive) predictable compensator of the stochastic logarithm $\mathcal{L}(S^2)$.

In particular, if $\mathcal{F}_0$ is trivial, then $\ln(\mathcal{E}(V)_T)$ is non-random.

Proof. Since S is a non-negative martingale, $S \equiv 0$ on the stochastic interval $[\tau, +\infty)$, where $\tau = \inf\{t : S_t = 0\}$ [9]. The positivity of S_T then guarantees that $T < \tau$, that S and S_- are strictly positive on $[0,T]$, and that $1/S_-$ is locally bounded [8,10]. This in turn implies that $1/S_-^2$ is also locally bounded and that $\mathcal{L}(S^2)$, as a stochastic integral of a locally bounded predictable process with respect to a special semimartingale, is itself special ([9], Proposition 2.51). Applying the above lemma to $X = S^2$ and using Proposition 1, we get the desired result,

$$\mathcal{E}(V)_t^{-1} = \mathcal{E}(V)_T^{-1}\,\mathrm{e}^{(T-t)\theta^2(t,T)}.$$

□

Proof of Theorem 2. In the special case of a continuous S, Proposition 2 simplifies as follows: $\mathrm{d}V_t = \mathrm{d}\langle S,S\rangle_t/S_t^2$, $\mathcal{E}(V) = \mathrm{e}^V$ and $\theta^2(t,T) = \frac{1}{T-t}\int_t^T \mathrm{d}\langle S,S\rangle_u/S_u^2$. Finally, if $\mathcal{F}_0$ is trivial, then all $\mathcal{F}_0$-measurable functions are constants and $T\theta^2(0,T) = \int_0^T \mathrm{d}\langle S,S\rangle_u/S_u^2$ is non-random. □

Proof of Theorem 3. Applying Theorem 2 to each $T \le T^*$, we get that the process $V_T = \int_0^T \mathrm{d}\langle S,S\rangle_t/S_t^2$, $T \le T^*$, is deterministic. But it is the quadratic variation $U(t)$ of the continuous local martingale $L = \mathcal{L}(S)$. It follows ([12], Exercise V.1.14) that L is a Gaussian martingale; and since it is a stochastic logarithm $\mathrm{d}S_t = S_t\mathrm{d}L_t$, and (5) follows from (4).

Finally, if $\theta^2(0,T)$ is differentiable and non-decreasing, then $U(T) = T\theta^2(0,T)$ is differentiable and strictly increasing (recall that $\theta^2(0,0) > 0$) and the process

$$W_t := \int_0^t \frac{1}{U'(s)} \mathrm{d}L_s$$

is a continuous local martingale with $\langle W, W \rangle_t = \int_0^t 1/(U'(s))\mathrm{d}U(s) = t$. By Lévy's characterization, W is a Brownian motion. Hence, $\mathrm{d}L_t = \vartheta(t)\mathrm{d}W_t$ where $\vartheta(t) = \sqrt{U'(t)}$. This in turn implies that, $\mathrm{d}S_t = \vartheta(t)S_t\mathrm{d}W_t$, $\mathrm{d}U(t) = \vartheta^2(t)\mathrm{d}t$, and (6) follows. □

References

[1] Brace, A., Goldys, B., Klebaner, F. and Wormersley, R., 2001, *Market Model of Stochastic Implied Volatility with Application to BGM Model*, Working paper S01–1, Department of Statistics, University of NSW.
[2] Brace, A., Goldys, B., van der Hoek, J. and Wormersley, R., 2002, *Markovian Models in Implied Volatility Framework* (University of NSW: Statistics preprints).
[3] Carr, P., 2000, A survey of preference free option valuation with stochastic volatility, April Risk Conference, Paris.
[4] Danilo, M. and Spokoinyi, V., 2004, Estimation of time dependent volatility via local change point analysis, *Annals of Statistics*, **32**(2), 577–602.
[5] Goldentyer, L., Klebaner, F.C. and Liptser, R., 2005, Tracking volatility, *Problems of Information Transmition*, **41**, 32–50.
[6] Hamza, K. and Klebaner, F.C., 2006, On nonexistence of non-constant volatility in the Black–Scholes formula, *Discrete and Continuous Dynamical Systems*, **6**, 829–834.
[7] Hull, J. and White, A., 1987, The pricing of options on assets with stochastic volatilities, *Journal of Finance*, **42**, 281–300.
[8] Jacod, J., 1978, Projection prévisible et décomposition multiplicative d'une semi-martingale positive, *Séminaire de Probabilités*, **XII**, 22–34.
[9] Jacod, J., 1979, Calcul stochastique et problèmes de martingales, *Lecture Notes in Mathematics*, 714.
[10] Kallsen, J. and Shiryaev, A.N., 2002, The cumulant process and Esscher's change of measure, *Finance Stochast*, **6**, 397–428.
[11] Klebaner, F.C., 2005, *Introduction to Stochastic Calculus with Applications*, 2nd ed. (London: Imperial College Press).
[12] Revuz, D. and Yor, M., 1999, *Continuous Martingales and Brownian Motion* (Berlin, Heidelberg, New York: Springer Verlag).
[13] Schonbucher, P.J., 1999, A market model for stochastic implied volatility, *Philosophical Transactions of the Royal Society Series A*, **357**, 2071–2092.
[14] Shiryaev, A.N., 1999, *Essential of Stochastic Finance* (Singapore: World Scientific).
[15] Stein, E.M. and Stein, J.C., 1991, Stock price distributions with stochastic volatility—an analytic approach, *Review of Financial Studies*, **4**, 727–752.