SOME ASPECTS OF GAMBLING WITH THE KELLY CRITERION

Ravi PHATARFOD

School of Mathematical Sciences

Monash University, Clayton, Victoria, Australia 3168

In this paper we consider the problem of gambling with the Kelly Criterion, i.e. gambling so as to maximize the expected exponential rate of growth of capital. We consider gambling on games of chance such as horse races, as well as gambling involved in buying and selling of shares on the stock market. For both the situations we obtain results which in some way are surprising and run counter to intuition.

Keywords : Gambler's ruin, Kelly Criterion, hedging, stock market, exponential growth.

1. Introduction

In Phatarfod (1999) we considered betting strategies in horse races in the framework of the gambler's ruin problem. That is, we have a gambler with a finite capital, who bets for a fixed positive expected gain in each race, and we are interested in the probability of him losing his capital. It was shown that a betting strategy which intuition would suggest as being optimal was, in fact, not so. For example, suppose in a race there are, among others, three horses A, B and C , with odds on offer against them winning the race being 5/1, 3/1, and 3/1 , and the probabilities of their winning the race being 0.2, 0.3 and 0.3 respectively. Betting on A alone with its high odds and expectation of gain of 0.2 per unit stake seems intuitively appealing. However, with a capital of $80, and betting $8 per race, the probability of the gambler's ruin is 0.4662, whereas betting amounts 2, 3, 3, dollars on A, B, C respectively (to give the same expectation of $1.60 of gain on the race) would dramatically reduce this probability to 0.00013. This is because the variance of the net gain in the first case is 368.64, whereas in the second case it is 23.04. An approximate relation between ruin probability and variance is Pr. (ruin) $\cong \exp(-2\mu a/\sigma^2)$, where μ and σ^2 are the mean and variance of the net gain, and a is the gambler's capital. The importance of the reduction of variance is so crucial that, for example, in some situations, betting on a favourable bet (expected gain positive) alone is not as advantageous as having an additional fair bet (expected gain zero) and in some circumstances, having an unfavourable bet (expected gain negative); i.e., in terms of reducing the probability of ruin it is better to dilute the favourableness of the original favourable bet with a fair or an unfavourable bet. This is because, the so diluting reduces the variance of the net gain at each race.

In this paper we consider gambling in the framework of the Kelly Criterion - see e.g.Rotando and Thorp (1992), Breiman (1961). The present paper essentially follows the discussion in the paper by Rotando and Thorp, by considering the importance of the variance of the net gain. We have a gambler with a finite capital who wants to bet in such a way as to maximize the expected exponential rate of growth of his capital – The Kelly Criterion. This is achieved by wagering an optimal fraction of his capital at each stage. As in Rotando and Thorp (1992), we consider games of chance in general, (Binomial games), not necessarily horse races, as well as gambling involved in the buying and selling of shares on the stock market. In the latter case, it is convenient to consider the net gain as a continuous random variable. Another difference from the Binomial games is that the time period per game is much greater, typically a year. The purpose of this paper is to show that here too the results obtained run counter to intuition. There are two types of results which fall in this category. For example, suppose the gambler wins a units for each unit wagered when he wins , and suppose the win probability is $p > 0$, so that the game is favourable with mean gain $\mu = ap - q > 0$, ($q=1- p$). It is known(see Rotando and Thorp (1992)) that the optimal fraction to bet at each stage is $f^* = (ap - q)/a$. Suppose that the gambler's capital is \$1000. Then, for the case p=0.9, q=0.1, a=0.2, we have μ=0.08, and the expectation of capital at the end of 100 games, using the optimal fraction f*, is 23332.95, whereas for the case p =0.54, q =0.46 and a=1 , the value of μ is 0.08 as before , but the expectation of the capital at the end of 100 games is a mere 1892.62. This is so because the optimal fraction of the capital wagered for the former case is 0.4, while for the latter it is 0.08, values which are directly dependent on the value of a , which also influences the value of the variance .

Secondly, consider again the case, p = 0.54, q = 0.46, and a =1. Here the gambler's edge is 0.08, i.e. somewhat substantial one, the fraction wagered is f* = 0.08, a fairly conservative value. So one expects a steady increase of the capital at an exponential rate to the not very spectacular value of 1892.62. However, simulation results obtained differ quite substantially from the 'expected values '. Half of the simulations gave final capitals below $1000. i.e. the gambler lost on the exercise. In fact there is a significant probability of such an event to occur. Of course, we know that eventually the final capital must increase indefinitely. However, in many practical situations one is interested not in the ultimate future, but in a finite time horizon. The reason we have a significant probability of losing in a finite time horizon is that although the expectation of the capital increases exponentially, so does the standard deviation, which in fact increases faster than the expectation.

For both the cases, the Binomial games as well as continuous variable games, we consider a sequence of 100 games . The mumber 100 has been arbitrarily chosen, but is of the same order of magnitude as the numbers chosen by others in this area- see Wong (1981). For games of chance this may well be a small value, but for stock market gambles the value is rather high. For both the cases the problem of having a substantial probability of being on the losing side at the end of the trials can be somewhat alleviated by diversification, i.e. having more than one bet at the same time. Perhaps this may be somewhat difficult to carry out in the horse race situation, unless one has the resources to bet at many venues at the same time.

The results on the stock market case can be extended to the situation concerning a commercial company's operations, and in particular its bankruptcy, an area which was, in fact the main motivation behind the present study of gambling with Kelly criterion. A commercial company's operations can be likened to those of a stock market investor. The latter invests i.e. gambles with a fraction f of the capital available to him, leaving the rest uninvested. Similarly, a commercial company invests a fraction of its capital in ventures which may be somewhat risky, i. e. are effectively gambles. Of course, not to invest in this way may not be the best course of action, as such investments may well bring substantial returns. The important thing is to arrive at the correct value of the fraction of money so invested. There are many reasons why commercial companies fail, but surely one of the reasons is the company overextending itself, i.e. in the context of the present theory, having a large value of f than is desirable. In section 3 we show the relation between the parameters of the underlying distribution and the value of the optimal fraction. It is suggested that a company wishing to minimize the probability of bankruptcy would need to correctly estimate the parameters so as to arrive at the correct value of the optimal fraction.

2 Binomial games

Let Y be the net gain made by a gambler on a game with unit stake and let $E(Y) = \mu > 0$. Let us assume that we have a sequence of games, for each of which the net gain per unit stake has a distribution the same as that of Y. Let the capital at the end of game n be X_n . Taking the simple case when the probability of winning a unit amount is p , and losing the bet is q , $(p + q = 1)$, we have, with X_0 as the initial capital, and with wagering a fraction f of the capital at each stage,

$$X_n = X_0 (1+f)^S (1-f)^F,$$

where S and F are the number of successes and failures respectively in n trials, $(S + F = n)$. To make it possible to wager a fraction at each stage, it is assumed that capital is infinitely divisible. Taking $0 < f < 1$, we have, $\Pr. (X_n = 0) = 0$; thus ruin in the sense of the gambler's ruin problem does not occur. If ruin is interpreted to mean that for any arbitrary small positive ε , $\lim_{n\to\infty}$ [Pr ($X_n \leq \varepsilon$)] = 1 , then ruin is possible if the value of f is large. Now, since

$$\exp\{n \log (X_n / X_0)^{1/n}\} = X_n / X_0$$

the quantity

$$\frac{1}{n}\log(X_n / X_0) = \frac{S}{n}\log(1+f) + \frac{F}{n}\log(1-f)$$

measures the exponential rate of growth per trial . The Kelly criterion maximizes the expected value of this, namely

$G(f) = E (1/n \log(X_n /X_0)) = p\log(1+ f) +q \log(1- f)$.

Setting $G'(f) = 0$, we have the optimal fraction f* as $f^* = p - q$. The behaviour of G(f) can be shown to be as in Figure 1.

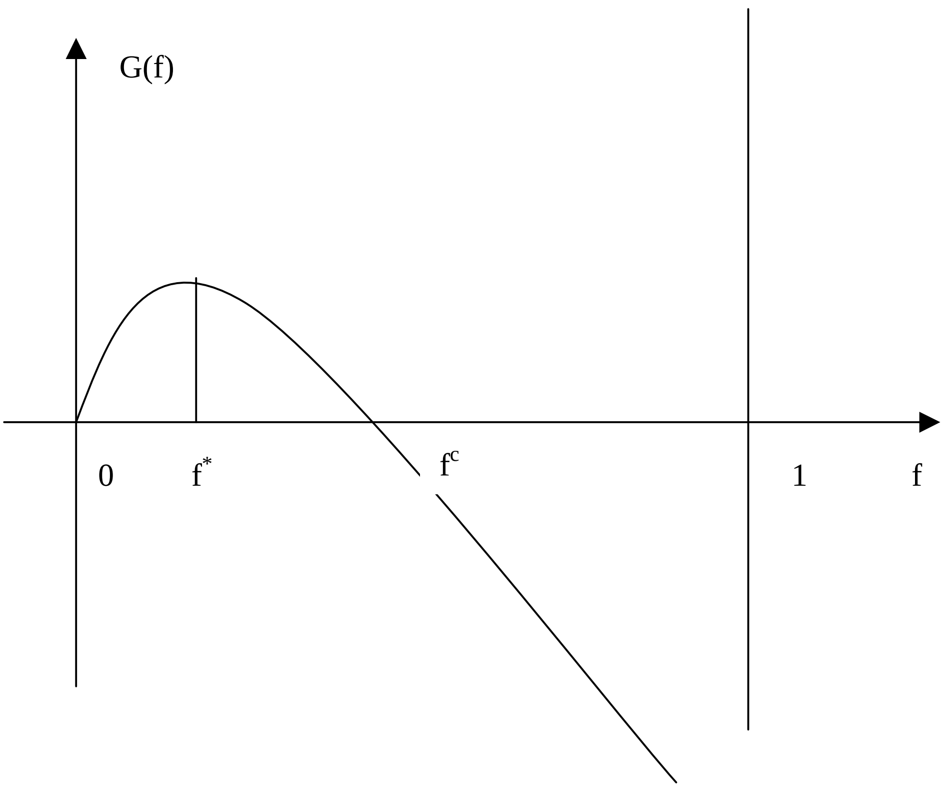

Figure1. The graph of G(f) against f

Table 1 gives the values of f* and f^c for a selection of the values of μ . It is seen that $f^c \cong 2 f^*$. That this is generally true can be shown by showing that G(f) is approximately symmetrical about f*. To show this let $x = f - (p - q)$. Using Taylor's series expansion for $H(x) = G(x+p-q)$, we have,

$$H(x) \cong \log 2 + p\log p + q\log q - x^2 / 8pq + x^3 (q - p)/ 24 p^2q^2 .$$

For example, for p=0.53, q =0.47, we have

$$H(x) \cong 0.6949 - 0.5018 x^2 + 0.0403 x^3 ,$$

a very nearly symmetrical function.

When $f < f^c$, ($\neq f^*$), expected value of capital increases exponentially, but at a slower rate than with f*. For $f > f^c$, ruin is certain. Breiman (1961) showed that

a) If $G(f) > 0$, then $\lim_{n\to\infty} X_n = \infty$ almost surely.

b) If $G(f) < 0$, then $\lim_{n\to\infty} X_n = 0$ almost surely.

It is easy to work out the mean and variance of X_n. We have, when f = f*,

$X_n = X_{n-1} (1+ f^*)$ with probability p

$= X_{n-1} (1 - f^*)$ with probability q.

Thus,

$$E(X_n \mid X_{n-1}) = X_{n-1} [1+ f^* (p - q)],$$

and hence,

$$E(X_n) = X_0 K^n ,$$

where , $K = 1 + f^* (p - q)$.

Similarly , $E (X_n^2) = X_0^2 L^n$,

where, $L = 1 + f^{*2} + 2f^* (p - q)$.

Thus, $\mathrm{Var} (X_n) = X_0^2 (L^n - K^{2n})$.

Table 1 gives the values of $E (X_n)$ and St. Dev. (X_n), for n=100, $X_0 = 1000$. Basically, as μ increases, the fraction wagered f* increases, resulting in an increase in the expected value of the final capital X_{100}. However, the St. Dev. of X_n increases much faster than its expected value .

Table 1. Values of f*, f^c , E(Xn) , St. Dev. (Xn) and P* for different values of the mean net gain μ

p , q	p= 0.53, q=0.47	p=0.54 , q=0.46	p=0.6, q=0.4
μ	0.06	0.08	0.20
f*	0.06	0.08	0.20
f^c	0.1197	0.1593	0.3894
E (Xn)	1432.40	1892.62	50504.95
St. Dev (Xn)	935.89	1765.20	284554
P*	0.3821	0.3446	0.1554

Table 2. Values of E (X_n) , St. Dev. (X_n) and P* for different values of a for μ = 0.08

a, p	a = 0.2, p = 0.9	a = 0.5, p = 0.9	a=1, p= 0.54	a= 1.5, p =0.432	a=2, p=0.36	a = 10 p=0.09818
f*	0.4	0.16	0.08	0.0533	0.04	0.008
E (Xn)	23332.95	3567.54	1892.62	1530.75	1376.42	1066.07
St.Dev (Xn)	56568.4	5148.08	1763.20	1124.35	839.30	283.77
P*	0.1513	0.2796	0.3446	0.3730	0.3897	0.4509

We obtain similar results when the gambler wins a units for every unit wagered.

Here, μ = ap – q, and f* = (ap – q) / a , and,

$E(X_n) = X_0 \, [\, 1 + (ap - q) f^* \,]^n$, and

$Var(X_n) = X_0^2 \{ [\, 1 + f^{*2} (pa^2 + q) + 2f^*(pa - q)]^n - [1 + f^* (pa - q)]^{2n} \}$

Table 3. Simulated Values of X_{100} for the case $X_0 = 1000$, p =0.54, q=0.46

646.32	783.44	872.00	1197.00	1254.92
3427.00	628.49	1060.93	665.00	1802.39
853.05	4656.26	341.93	696.81	986.90
2080.44	1110.93	595.67	1268.01	1302.56

Table 2 gives the values of E (X_n) and St. Dev. (X_n) for different values of a, for the same mean, μ = 0.08. These show what effect the value of a i.e., indirectly the variance of the net gain, has on the expected value and the standard deviation of X_n. For expected values they range from 23332.95 to 1066.07. It can be shown by some straightforward but somewhat tedious algebra that for a fixed μ, as a increases, f* decreases , resulting in decreasing values of E (X_n) and St. Dev (X_n).

The preceding discussion leads one to expect that if the gambler sticks to the optimal fraction f*, a steady increase of his capital at an exponential rate is guaranteed particularly so for a case like , p= 0.54 , q = 0.46. Here, the gambler's edge is 0.08, i.e. somewhat substantial, and the fraction of the capital wagered is f* = 0.08, a fairly conservative value. However, the results are quite different from what one would expect. Table 3 gives the results of 20 simulations for this case with n=100, and X_0 = 1000. Half of the final capital values are smaller than the initial capital of $1000. ! Of course, we know that eventually X_n must increase indefinitely. However, in many practical situations one is interested not in the ultimate future but in a finite time horizon.

To derive the probability that the gambler would be on the losing side at the n^{th} stage, i.e. $P^* = Pr. [X_n < X_0]$, it is easy to derive the equivalent $Pr. [\log X_n < \log X_0]$.

To derive the latter, we have for the case of a general a ,

$\log X_n = \log X_0 + S \log (1 + af^*) + F \log (1- f^*)$

so that $E (\log X_n) = \log X_0 + c$,

where $c = n [p \log \{ (1 + af^*)/ (1- f^*)\}+ \log (1 – f^*)]$,

and St. Dev ($\log X_n$) = d , where

$d = \sqrt{npq} \log [(1+ af^*)/ (1-f^*)]$.

For large n , $\log X_n$ is approximately normal . Hence,

$P^* = Pr. (Z< -c/d)$

where Z is a standardized normal variable.

Tables 1 and 2 give the values of P* for different μ with a= 1, and for different a for $\mu = 0.08$. It can be shown that for a constant value of μ, as a increases, i.e. as the variance of the net gain increases, f*, $E (\log X_n)$ and St. Dev.($\log(X_n)$) decrease. However, P* increases .

As in Phatarfod (1999), here too it is advantageous to dilute a favourable bet by having an additional fair bet, since such a dilution would reduce the variance of the net gain, keeping the mean the same. Suppose we have bets of equal amounts on a horse A with odds of a/1, and probability of win equal to $p_1 = (1+\mu)/ (1 + a)$, and on a fair bet at even money on horse B, whose probability of win is 0.5, the total amount determined by the fraction f of the capital at that stage. When A wins, the net gain for a unit amount is a-1; when B wins , there is no net gain or loss; when neither A nor B wins, there is a loss of 2 units. Let S be the number of times A wins, and F be the number of times neither A nor B wins. Then we have,

$$X_n = X_0 (1 + fa/2)^S (1-f)^F ,$$

from which we obtain, in an obvious manner,

$$f^* = \mu / [(a - 1)(p_1 + q)] \qquad \text{where} \quad p_1 + 0.5 + q = 1.$$

Table 4. Values of f* ,E(X*n*), St. Dev (X*n*), and P* for different values of a for μ = 0.08,when an additional fair bet at even money was taken

a	1.5	2.0	10
f*	0.32	0.16	0.01778
E(Xn)	3567.54	1892.62	1073.65
St. Dev (X*n*)	4412.28	1669.64	300.32
P*	0.2660	0.3228	0.4483

E (X_n), St. Dev. (X_n) , P* etc., can be obtained in the usual manner. Table 4 gives these values for the cases , a = 1.5, 2 , and 10. It is clear, comparing Tables 2 and 4 that the values of E (X_n) and St. Dev. (X_n) are substantially greater than for the case when the fair bet was not taken. However the values of P* are substantially smaller. It is intuitively obvious that similar conclusions would apply if instead of the fair bet we have an unfavourable bet (albeit only marginally unfavourable)

3 Continuous games.

We now consider the case of an investor involved in buying and selling of shares on the stock market. For simplicity we assume that the share price movement is uniformly distributed. Suppose the net gain per unit investment is a continuous random variable U uniformly distributed over the interval (a, b), $-1 < a < 0$, $0 < b < 1$. Translated into the stock market situation, an investor purchases a stock for \$100 per share, and the anticipated price per share in one year's time is uniformly distributed over (100 (1 +a), 100 (1 + b)). For example, for a = -0.7, b=1, the current share price is $100 , and the anticipated price is uniform over (30, 200) . As before, we have the growth coefficient,

$$G(f) = \int_a^b [\log(1+fu)]/(b-a)du .$$

Setting G′(f) equal to zero, gives us f* , the optimal fraction of the capital wagered (invested) at each stage as the positive solution of

(b-a) f = log [(1 + bf)/ (1 + af)].

To determine the expected value and the variance of the capital at the n^{th} stage, using the optimal value of f, we have,

$X_n = X_{n-1} (1 - f^*) + X_{n-1} f^* (1 + U)$.

This is because the capital at the n^{th} stage is the sum of the capital not invested at the $(n-1)^{th}$ stage, namely $X_{n-1} (1- f^*)$ and the return on the investment of $X_{n-1} f^*$ (we assume, for simplicity, that the capital not invested remains idle during the year, without attracting interest). From the above we have

$X_n = X_{n-1} (1 + f U)$.

Working in a manner similar to that in section 2, and using E (U) = (a + b)/2,

Var (U) = $(b – a)^2 /12$, we obtain ,

$E (X_n) = X_0 K^n$,

$Var (X_n) = X_0^2 [\{K^2 + f^{*2} (b-a)^2/12 \}^n - K^{2n}]$

where K = 1 + f* (a + b) /2.

Note, sometimes we have f* >1. For example, when a= - 0.3, and b = 0.5 we obtain f *= 1.95. When this happens, adopting the optimal policy involves borrowing an amount $(f^*-1)X_n$ at the n^{th} stage, to invest the amount f^*X_n. We assume that out of the proceeds of the investment, the amount (f*-1) X_n is returned , with no interest charged. Thus,

$$X_n = X_{n-1} f^* (1+U) - X_{n-1} (f^*-1)$$

$$= X_{n-1} (1 + f^* U)$$

as before.

Table 5 gives the expected values and the St. Dev. of X_{100} when X_0 =$1000.We see that the expected values are not only phenomenally large, but are widely disparate, and so are the standard deviations. Note, case B is obtained from case A by only a scale change, and so the values regarding the final capital remain the same. Also, case C has smaller mean net gain per investment than case A; however the standard deviation is also small. This results in a large value of f* and hence phenomenally large values of E (X_{100}) and St. Dev. (X_{100}) . Case D can be compared to an example of the Binomial case. The equivalent Binomial gamble case, i.e., with μ =0.08, σ^2 = 0.2821 is when p = 0.8052, a = 0.3412. This has f* = 0.2345. For this case, E (X_{100}) = 6414.87, and St.Dev (X_{100}) = 11845.37. The higher vales for these for case D is attributable to the fact that the value of f* is higher—0.28485 as against 0.2345.

We shall now see that in spite of the phenomenal growth of the capital, there is a significant probability that at stage 100 the capital has fallen below the original value.

It is easy to work out the mean and the variance of log (X_n/ X_0). We have,

$$\text{Log} (X_n / X_0) = \Sigma \log (1 + f^* U_i) ,$$

where U_i is uniformly distributed over (a , b) . This gives,

$$E [\log (X_n / X_0)] = n A,$$

$$\text{Var} [\log (X_n / X_0)] = nB$$

where,

$$A = [(1 + b f^*) \log (1 + bf^*) - (1 + af^*) \log (1 + af^*) - (b - a) f^*] / [f^* (b- a)]$$

$B = [(1 + bf^*)\{ \log(1+bf^*)\}^2 - (1+af^*) \{ \log(1+af^*)\}^2]/[f^*(b-a)] - 2A - A^2$.

This gives,

$P^* = \Pr. (X_n < X_0) = \Pr. (\log X_n < \log X_0)$

$= \Pr. (Z < - A\sqrt{n/B})$.

Table 5 summarizes the situation for the various cases.

Table 5. Mean, standard Deviation of X*n* and P* values for various cases

CASE	A	B	C	D
a	- 0.7	- 0.35	- 0.3	- 0.84
b	1.0	0.5	0.5	1.0
μ	0.15	0.075	0.10	0.08
σ^2	0.2408	0.0602	0.0533	0.2821
f*	0.63	1.26	1.95	0.28485
E (X_{100})	8347887	8347887	54549469127.7	9518.37
St. Dev. (X_{100})	387709456	387709456	41722243155.4	26435.60
P*	0.0582	0.0582	0.0122	0.2246

References

Breiman, L. (1961), Optimal gambling systems for favourable games, *Fourth Berkeley Symposium on Probability and Statistics,* I, 65-78

Phatarfod, R. M. (1999), Betting strategies in horse races, *Asia- Pacific Journal of Operations Research* 16, 87-98

Rotando, L.M. and E.O. Thorp (1992), The Kelly Criterion and the stock market, *American Mathematical Monthly 99,* 922-931.

Wong, S. (1981),What proportional betting does to your win rate. *Blackjack World 3,* 162-168.