



Asymptotic Analysis of Loss Probabilities in *GI/M/m/n* Queueing Systems as *n* Increases to Infinity

Vyacheslav M. Abramov

School of Mathematical Sciences, Monash University, Australia

(Received May 2005, accepted April 2006)

Abstract: The paper studies asymptotic behavior of the loss probability for the *GI/M/m/n* queueing system as *n* increases to infinity. The approach of the paper is based on applications of classic results of Takács (1967) and the Tauberian theorem with remainder of Postnikov (1979-1980) associated with the recurrence relation of convolution type. The main result of the paper is associated with asymptotic behavior of the loss probability. Specifically it is shown that in some cases (precisely described in the paper) where the load of the system approaches 1 from the left and *n* increases to infinity, the loss probability of the *GI/M/m/n* queue becomes asymptotically independent of the parameter *m*.

Keywords: Loss probabilities, *GI/M/m/n* queueing system, asymptotic analysis, Tauberian theorem with remainder.

1. Introduction

It is well known that queueing systems with many servers are well models for communication systems. Study of queues with many servers and especially analysis of the loss probability have a distinguished history going back to the works of Erlang (see [7]), who in 1917 first gave fundamental results for Markovian queueing systems, and to the works of Palm, Pollaczek and other researchers (e.g. [15], [16], [11], [19], [22]), who then developed the Erlang's results to non-Markovian systems. Since then these results have been developed in a large number of investigations, motivated by growing development of modern telecommunication systems. Nowadays the theory of loss queueing theory is very rich. There is a number of different directions of the theory including management and control, redundancy, analysis of retries, impatient customers and so on.

In the present paper we study the *GI/M/m/n* queueing system in which parameter *n*, the number of possible waiting places, is a large value. This assumption is typical for real telecommunication systems. Under this assumption the paper studies asymptotic behavior of the loss probability, where the most significant result seems to be related to the case where the load parameter approaches 1 from the left. The case of a heavy load parameter is the most interesting in practice. In the pre-project study and system design stage an engineer is especially interested to know about the behavior of the loss probability in heavy loaded systems.

We consider the *GI/M/m/n* queue, where $m > 1$ is the number of servers, and $n \geq 0$ is the admissible queue-length. Let $A(x)$ denote the probability distribution function of an interarrival time, and let λ be the reciprocal of the expected interarrival time. For real $s \geq 0$ we denote $\alpha(s) = \int_0^\infty e^{-sx} dA(x)$. The parameter of the service time distribution is

denoted μ , and the load of the system is $\rho = \lambda/(m\mu)$. Note that in Theorem 3.2 the parameter ρ is assumed to depend on n . However we do not write this dependence explicitly, assuming that this dependence is clear from the formulation of the aforementioned theorem.

In the case of $GI/M/m/n$ queueing system for the stationary loss probability p_n we have the representation (Abramov [2])

$$p_n = \frac{1}{\pi_n}, \quad (1)$$

where the generating function $\Pi(z)$ of π_j , $j=0,1,\dots$ is the following:

$$\Pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \frac{\alpha(\mu - \mu z)}{\alpha(\mu - \mu z) - z}, |z| < \sigma, \quad (2)$$

σ is the least in absolute value root of the functional equation $z = \alpha(\mu - \mu z)$ (the variable z is assumed to be real). It is well-known (e.g. Takács [23]), that σ belongs to the open interval $(0, 1)$ if $\rho < 1$, and it is equal to 1 otherwise. Note, that another representation for the loss probability p_n is given in Miyazawa [14].

The value π_n has the following meaning. This is the expected number of arrivals up to the first loss of a customer arriving to the stationary system. π_n satisfies the recurrence relation of convolution type

$$\pi_n = \sum_{i=0}^n r_i \pi_{n-i+1}, n=0,1,\dots, \quad (3)$$

where the initial value is $\pi_0 = 1$. Specifically,

$$r_i = \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^i}{i!} dA(x). \quad (4)$$

As $n \rightarrow \infty$, the asymptotic behavior of the general recurrence relation of convolution type,

$$Q_n = \sum_{i=0}^n f_i Q_{n-i+1}, \quad (5)$$

where $f_0 > 0$, $f_i \geq 0$ ($i \geq 1$) and $f_0 + f_1 + \dots = 1$, has been originally studied by Takács [24], p. 22 and then developed by Postnikov [17], Section 25. For the readers' convenience, some of these results, necessary for the purpose of the paper, are collected in the Appendix.

By exploiting (3), Abramov [2] studied the asymptotic behavior of the loss probability p_n , as $n \rightarrow \infty$. The analysis of Abramov [2] is based on the mentioned results of Takács [24] and Postnikov [17]. For other applications of the mentioned results of Takács [24] and Postnikov [17] see also Abramov [1], [3], [4].

The asymptotic analysis of $GI/M/m/n$ queueing system, as $n \rightarrow \infty$, is a much more difficult problem than the same problem for the $GI/M/1/n$ queue. Recently, Choi *et al.* [9] and Kim and Choi [12] obtained some new results related to the $GI/M/m/n$ and $GI^X/M/m/n$ queues. In Choi *et al.* [9] the exact estimation for the convergence rate of the stationary $GI/M/m/n$ queue-length distribution to the stationary queue-length distribution of the $GI/M/m$ queueing system, as $n \rightarrow \infty$, is obtained. In Kim and Choi [12] detailed analysis of the loss probability of the $GI^X/M/m/n$ is provided. The analysis

of these two aforementioned papers is based on the development of the earlier results of Choi and Kim [8] in a nontrivial fashion.

The analysis of Choi and Kim [8] and Choi *et al.* [10] in turn uses the deepen theory of analytic functions, including an untraditional result of the theory of power series given by a theorem of Wiener.

Ramalhoto and Gomez-Corral [18] discuss retrials in $M/M/r/d$ loss queues and present an appropriate decomposition formulae for losses and delay in those queueing systems. In the case of Markovian system with many servers the results of Ramalhoto and Gomez-Corral [18] are useful for analysis of the effect of retrials in asymptotic analysis of losses.

The present paper provides the asymptotic analysis of the loss probability of the $GI/M/m/n$ queue as $n \rightarrow \infty$, by reduction of the sequence $\pi_{m,0,n}^{(n)}$ to above representation (5), and then estimates the loss probability $p_{m,n} = 1/\pi_{m,0,n}^{(n)}$ by using the results of Takács [24] and Postnikov [17] given in the Appendix. (The precise definition of the sequence $\pi_{m,0,n}^{(n)}$ is given later.)

This is the same idea as in the earlier paper of Abramov [2] related to the $GI/M/1/n$ queue, however it is necessary to underline the following. Whereas in the case of $GI/M/1/n$ the reduction to (3) is straightforward, and the representation of the probabilities r_n and their generating function is very simple, the reduction to the recurrence relation of (5) in the case of $GI/M/m/n$ queue, $m > 1$, is not obvious, and representation for probabilities $r_{k,m-k,n}$ and $r_{0,m,j}$ and associated generating function is more difficult. (The aforementioned probabilities $r_{k,m-k,n}$ and $r_{0,m,j}$ are defined later.) Furthermore, the sequence $\pi_{m,0,j}^{(n)}$, $j \leq n$, is structured as schema with the series classes, and, as $j = n$, the value $p_{m,n} = 1/\pi_{m,0,n}^{(n)}$ coincides with the desired loss probability. For this reason the class of asymptotic results, that we could obtain here for $GI/M/m/n$ queue, is poorer than that for the $GI/M/1/n$ queue in Abramov [2].

Our approach has the following two essential advantages compared to the pure analytical approaches of Choi *et al.* [10], Kim and Choi [12], Choi *et al.* [9], Simonot [20]:

- The problem reduces to the known classic results (Theorem of Takács [24] and Tauberian Theorem of Postnikov [17]), permitting us to substantially diminish the cumbersome algebraic calculations and clearly understand the results.
- Along with standard asymptotic results having a quantitative feature we also prove one interesting property related to the case of n increasing to infinity and the load approaching 1 from the left. (For the more precise assumptions see formulation of Theorem 3.2) Specifically it is proved that the obtained asymptotic representation is the same for all $m \geq 1$, i.e. it coincides with asymptotic representation obtained earlier for the $GI/M/1/n$ queueing system in Abramov [2]. As n increases to infinity, this asymptotic property remains true for all fixed $\rho \geq 1$.

The conditions of Theorem 3.2 and the first two cases of Theorem 3.1 all fall into the domain of heavy traffic theory (e.g. Borovkov [6], Whitt [25, 26]). Although the aforementioned recent results of Whitt [25, 26] are related to more general models, they however do not cover the results of this paper.

Our approach is based on asymptotic analysis of relations (9) and (10) which is based on asymptotic representation for the root of equation $z = \alpha(\mu m - \mu m z)$ (see formulation of Theorem 3) as ρ approaches 1, presented in the book of Subhankulov [21] (Chapter 9 of

this book is devoted to application of Tauberian theorems to a specific moving server problem arising in Operations Research).

The rest of the paper is organized as follows. In Section 2 we give some heuristic arguments preparing the reader to the results of the paper. There are two theorems presenting the main results in Section 3. In Section 4 we derive the recurrence relation of convolution type for the loss probability. These recurrence relations are then used to prove Theorem 3 of the paper. Theorem 3 is proved in Section 5 and Section 6. In Section 7 we study the behavior of the loss probability as the load approaches 1 from the left. In Section 8 numerical example supporting the theory is provided. The Appendix contains auxiliary results necessary for the purpose of the paper: the theorem of Takács [24] and the Tauberian theorem of Postnikov [17].

2. Some Heuristic Arguments

The $GI/M/m/n$ queueing system, $m > 1$, is more complicated than its analog with one server. Letting n be infinity, discuss first the $GI/M/1$ and $GI/M/m$ queueing systems.

It is well-known that the stationary queue-length distribution of $GI/M/1$ queue (immediately before arrival of a customer with large order number) is geometric. The same (customer-stationary) queue-length distribution of $GI/M/m$ queue, provided that immediately before arrival at least $m-1$ servers are occupied, is geometric as well. Thus, a typical behavior of the queue-length processes of $GI/M/1$ and $GI/M/m$ queues is similar, if we assume additionally that a customer arriving into $GI/M/m$ queueing system finds at least $m-1$ servers busy (see Kleinrock [13]).

The similar situation holds in the case of the $GI/M/1/n$ and $GI/M/m/n$ queues. Specifically, in the case of the $GI/M/1/n$ queue the stationary loss probability satisfies (1)-(3). In the case of the $GI/M/m/n$ queue the *conditional* stationary loss probability *provided that upon arrival at least $m-1$ servers are busy* satisfies (3) as well. In the case of the $GI/M/m/n$ queue the only difference is that, the value μ in (2) and (4) should be replaced with μm , and σ should be the least in absolute value root of the equation $z = \alpha(\mu m - \mu m z)$ rather than of the equation $z = \alpha(\mu - \mu z)$. In the sequel, the least root of the functional equation $z = \alpha(\mu m - \mu m z)$ is denoted σ_m .

Let us now discuss the stationary probabilities of the $GI/M/m/n$ system, $m > 1$, without the condition above. It is clear that eliminating the condition above proportionally changes the stationary probabilities $P\{\text{arriving customer meets } m+j-1 \text{ customers in the system}\}$, $j \geq 0$. That is, the loss probability is changed proportionally as well. This enables us to anticipate the behavior of the loss probability as $n \rightarrow \infty$ in some cases.

Specifically, in the case $\rho < 1$ and n large, the loss probability is equal to conditional loss probability, provided that upon arrival at least $m-1$ servers are busy, multiplied by some constant. That is, as n large, the both abovementioned loss probabilities, conditional and unconditional, are of the same order. Following Abramov [2] (see also Choi *et al.* [9]) this order is $O(\sigma_m^n)$. The precise result is given by Theorem 3.1 below.

If n large and $\rho \geq 1$, then the probability that arriving customer meets less than $m-1$ customers in the system is small, and therefore, the loss probability should be approximately the same as the conditional stationary probability that upon arrival of a customer at least $m-1$ servers are busy. That is, following Abramov [2] (see also Choi *et al.* [9]) one can expect, that in the case of $\rho \geq 1$, the limiting stationary loss probability of the

$GI/M/m/n$ queue, as $n \rightarrow \infty$, should be equal to $(\rho - 1)/\rho$ for all m . Can the last property of asymptotic independence of m be extended as n increases to infinity and ρ approaches 1? The paper provides the condition for this asymptotic independence in this case.

3. Formulation of the Main Results

Theorem 3.1. If $\rho > 1$ then for any $m \geq 1$

$$\lim_{n \rightarrow \infty} p_{m,n} = \frac{\rho - 1}{\rho}. \quad (6)$$

If $\rho = 1$ and $\rho_2 = \int_0^\infty (\mu x)^2 dA(x) < \infty$ then for any $m \geq 1$

$$\lim_{n \rightarrow \infty} np_{m,n} = \frac{\rho_2}{2}. \quad (7)$$

If $\rho = 1$ and $\rho_3 = \int_0^\infty (\mu x)^3 dA(x) < \infty$ then for large n and any $m \geq 1$ we have

$$p_{m,n} = \frac{\rho_2}{2n} + O\left(\frac{\log n}{n^2}\right). \quad (8)$$

If $\rho < 1$ then for $p_{m,n}$ we have the limiting relation:

$$\lim_{n \rightarrow \infty} \frac{p_{m,n}}{\sigma_m^n} = K_m [1 + \mu m \alpha'(\mu m - \mu m \sigma_m)], \quad (9)$$

where $\alpha'(\cdot)$ denotes the derivative of $\alpha(\cdot)$,

$$K_m = [1 + (1 - \sigma_m) \sum_{j=1}^m \binom{m}{j} \frac{C_j}{(1 - \varphi_j)} \frac{m(1 - \varphi_j) - j}{m(1 - \sigma_m) - j}]^{-1} \quad (10)$$

$$\varphi_j = \int_0^\infty e^{-\mu_j x} dA(x),$$

$$C_j = \prod_{i=1}^j \frac{1 - \varphi_i}{\varphi_i},$$

and σ_m is the least in absolute value root of functional equation:

$$z = \alpha(\mu m - \mu m z).$$

Theorem 3.1 shows that if $\rho > 1$ then the limiting stationary loss probability is independent of parameter m . If $\rho = 1$ and $\rho_2 < \infty$ then $\lim_{n \rightarrow \infty} np_{m,n}$ is independent of parameter m as well. The proof of (6) seems to be given by simple straightforward arguments (extended version of the heuristic arguments of Section 2). Nevertheless, all results are proved by reduction to the abovementioned theorems of Takács [24] and Postnikov [17] given in the Appendix. The most significant result of Theorem 3.1 is (9). This result is then used to prove the statements of Theorem 3.2 on the behavior of the loss probability as the load approaches 1 from the left.

This behavior of the loss probability is given by the following theorem.

Theorem 3.2. Let $\varrho = 1 - \varepsilon$, where $\varepsilon > 0$, and $\varepsilon n \rightarrow C$ as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$. Assume that $\varrho_3 = \varrho_3(n)$ is a bounded sequence in n , and there exists $\tilde{\varrho}_2 = \lim_{n \rightarrow \infty} \varrho_2(n)$. In the case where $C > 0$ for any $m \geq 1$ we have

$$p_{m,n} = \frac{\varepsilon e^{-2C/\tilde{\varrho}_2}}{1 - e^{-2C/\tilde{\varrho}_2}} [1 + o(1)]. \quad (11)$$

In the case where $C = 0$ for any $m \geq 1$ we have

$$p_{m,n} = \frac{\tilde{\varrho}_2}{2n} + o\left(\frac{1}{n}\right). \quad (12)$$

Theorem 3.2 shows that as ϱ approaches 1 from the left, the loss probability $p_{m,n}$ becomes independent of parameter m when n large, and the asymptotic behavior of the loss probability is exactly the same as for the $GI/M/1/n$ queue.

4. Deviation of the Recurrence Equations for the Loss Probability

For the sake of convenience, in this section we keep in mind that the first $m-1$ states of the $GI/M/m/n$ queue-length process form one special class. If an arriving customer occupies one of servers, then the system is assumed to be in this class, and the states of this class are numbered 1, 2, ..., m . Otherwise, the system is in the other class with states $m+1$, $m+2$, ..., $m+n$, where the last state, $m+n$, is associated with a loss of an arriving customer.

For example, if $n=0$, then the second class of the $GI/M/m/0$ queueing system consists of one state only.

Let us now build the recurrence relation similar to those of the $GI/M/1/n$ queue.

We start from the $GI/M/1/0$ queue. For this queue we have

$$\pi_{1,0} = \frac{1}{r_{0,1}}, \quad (13)$$

where

$$r_{0,1} = \varphi_1 = \int_0^\infty e^{-\mu x} dA(x).$$

Equation (13) formally follows from the recurrence relation $\pi_{0,1} = r_{0,1}\pi_{1,0}$, where $\pi_{0,1} = 1$. The loss probability for the $GI/M/1/0$ queue is equal to

$$p_{1,0} = \frac{1}{\pi_{1,0}} = \varphi_1.$$

Before considering the case of the $GI/M/m/n$ queue, notice that the value $\pi_{m-k,k}$ has the meaning of the expected number of arrivals to the stationary system up to at the first time an arriving customer finds $m-k$ servers busy and k remaining servers free.

In the case of the $GI/M/2/0$ queue, by the total expectation formula we have

$$\pi_{1,1} = r_{0,2}\pi_{2,0} + r_{1,1}\pi_{1,1},$$

where

$$r_{1,1} = 2 \int_0^\infty [1 - e^{-\mu x}] e^{-\mu x} dA(x),$$

$$r_{0,2} = \varphi_2 = \int_0^\infty e^{-2\mu x} dA(x).$$

Then, by the total expectation formula, the recurrence relation for the $GI/M/m/0$ queue looks as follows:

$$\pi_{m-1,1} = \sum_{k=0}^{m-1} r_{k,m-k} \pi_{m-k,k}, \quad (14)$$

where

$$r_{k,m-k} = \binom{m}{k} \int_0^\infty [1 - e^{-\mu x}]^k e^{-(m-k)\mu x} dA(x).$$

It is well-known that

$$\pi_{m,0} = \sum_{i=0}^m \binom{m}{i} \prod_{j=1}^i \frac{1 - r_{0,j}}{r_{0,j}} = \sum_{i=0}^m \binom{m}{i} C_i, \quad (15)$$

and the loss probability is

$$p_{m,0} = \frac{1}{\pi_{m,0}} = \left[\sum_{i=0}^m \binom{m}{i} C_i \right]^{-1} \quad (16)$$

(see Cohen [11], Palm [15], Pollaczek [16], Takács [22] as well as Bharucha-Reid [5]).

A relatively simple proof of (15) and (16) can be found in Takács [22]. It is based on another representation than (14). For our further purposes, representation (14) is preferable. Representation (14) is a recurrence relation of the convolution type (5), and in the following it helps us to reduce the problem to the abovementioned combinatorial results of Takács [24]. Once this is done, we apply then the Tauberian theorem of Postnikov [17].

Let us now consider the $GI/M/m/n$ queueing system. In the case of this system with $n \geq 1$ we add an additional subscript to the notation. Specifically, $r_{k,m-k,0} = r_{k,m-k}$, and for $j \leq n$

$$r_{0,m,j} = \int_0^\infty e^{-m\mu x} \frac{(m\mu x)^j}{j!} dA(x),$$

and

$$r_{k,m-k,n} = \binom{m}{k} \int_0^\infty e^{-(m-k)\mu x} \times \left\{ \int_0^\infty \frac{(m\mu u)^{n-1}}{(n-1)!} (e^{-\mu u} - e^{-\mu x})^k m\mu du \right\} dA(x).$$

In addition, the value $\pi_{m,0,n}^{(n)}$ denotes the expected number of arrivals into the stationary $GI/M/m/n$ queue up to the first loss. (Replacing the indexes n with j has the same meaning for the $GI/M/m/j$ queue.) Also, we will use the notation $\pi_{m-k,k,0}^{(0)}$ instead of the earlier notation $\pi_{m-k,k}$ for the $GI/M/m/0$ queue.

Then the recurrence relation associated with the n th series looks as follows:

$$\pi_{m,0,j}^{(n)} = \sum_{l=0}^j r_{0,m,l} \pi_{m,0,j-l+1}^{(n)} + \sum_{k=1}^{m-1} r_{k,m-k,j} \pi_{m-k,k,0}^{(n)}, \quad (17)$$

$$j = 0, 1, \dots, n-1,$$

where

$$\pi_{m-i,i,0}^{(n)} = \sum_{k=0}^{m-i} r_{k,m-k-i+1,0} \pi_{m-k-i+1,k+i-1}^{(n)} \quad (\pi_{0,m,0}^{(n)} = 1), \quad (18)$$

$$i = 1, 2, \dots, m-1,$$

and the second sum of (17) is equal to zero if $m=1$. Moreover, if $m=1$, then we do not longer need the upper index (n) , showing the series number, and (18). It is not difficult to see, that for the given series n , the recurrence relations (17) and (18) form a recurrence relation of the convolution type given by (5). In the next section we prove relations (6)-(8) of Theorem 3.1.

5. The Proofs of Equations (6)-(8)

First of all note, that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[\sum_{l=0}^n r_{0,m,l} + \sum_{k=1}^{m-1} r_{k,m-k,n} \right] &= \sum_{l=0}^{\infty} r_{0,m,l} \\ &= \sum_{l=0}^{\infty} \int_0^{\infty} e^{-m\mu x} \frac{(m\mu x)^l}{l!} dA(x) \\ &= \int_0^{\infty} \sum_{l=0}^{\infty} e^{-m\mu x} \frac{(m\mu x)^l}{l!} dA(x) = 1. \end{aligned}$$

Therefore, one can apply the theorem of Takács [24] (see Appendix).

Let γ_1 denote

$$\gamma_1 = \sum_{l=1}^{\infty} h_{0,m,l}. \quad (19)$$

Then also

$$\lim_{n \rightarrow \infty} \left[\sum_{l=1}^n h_{0,m,l} + \sum_{k=1}^{m-1} (n+k) r_{k,m-k,n} \right] = \gamma_1. \quad (20)$$

This is because

$$\begin{aligned} (n+k) \binom{m}{k} \int_0^{\infty} e^{-(m-k)\mu x} \left\{ \int_0^{\infty} \frac{(m\mu u)^{n-1}}{(n-1)!} (e^{-\mu u} - e^{-\mu x})^k m\mu du \right\} dA(x) \\ \leq (n+k) \binom{m}{k} \int_0^{\infty} e^{-\mu(m-k)x} \left\{ \int_0^x \frac{(m\mu u)^{n-1}}{(n-1)!} m\mu du \right\} dA(x) \end{aligned}$$

$$= (n+k) \binom{m}{k} \int_0^\infty e^{-\mu(m-k)x} \frac{(m\mu x)^n}{n!} dA(x) \rightarrow 0,$$

as $n \rightarrow \infty$.

According to (19) and (20) we have $\gamma_1 = m\mu/\lambda$, and therefore, $\gamma_1 = 1/\varrho$. Then according to theorem of Takács [24], given in the Appendix, in the case of $\varrho > 1$ we obtain

$$\lim_{n \rightarrow \infty} \pi_{m,0,n}^{(n)} = \frac{1}{1-\gamma_1} = \frac{\varrho}{\varrho-1}.$$

Then, in this case of the limiting loss probability as $n \rightarrow \infty$ we obtain

$$\lim_{n \rightarrow \infty} p_{m,n} = \lim_{n \rightarrow \infty} \frac{1}{\pi_{m,n,0}^{(n)}} = \frac{\varrho-1}{\varrho}.$$

Similarly to (19), Let γ_2 denote

$$\gamma_2 = \sum_{l=2}^{\infty} l(l-1)r_{0,m,l}.$$

Then also

$$\lim_{n \rightarrow \infty} \left[\sum_{l=2}^n l(l-1)r_{0,m,l} + \sum_{k=1}^{m-1} (n+k)(n+k-1)r_{k,m-k,n} \right] = \gamma_2,$$

and $\varrho_2 < \infty$ as $\gamma_2 < \infty$. Indeed, as $n \rightarrow \infty$,

$$\begin{aligned} & (n+k)(n+k-1) \\ & \times \binom{m}{k} \int_0^\infty e^{-(m-k)\mu x} \left\{ \int_0^\infty \frac{(m\mu u)^{n-1}}{(n-1)!} (e^{-\mu u} - e^{-\mu x})^k m\mu du \right\} dA(x) \\ & \leq (n+k)(n+k-1) \binom{m}{k} \int_0^\infty e^{-\mu(m-k)x} \frac{(m\mu x)^n}{n!} dA(x) \rightarrow 0. \end{aligned}$$

Therefore, in the case where $\varrho = 1$ and $\varrho_2 = \int_0^\infty (\mu mx)^2 dA(x) < \infty$ we obtain

$$\lim_{n \rightarrow \infty} np_{m,n} = \frac{\varrho_2}{2}. \quad (21)$$

Limiting relation (21) can be improved with the aid of the Tauberian theorem of Postnikov [17] (see Appendix). In the case where $\varrho = 1$ and $\varrho_3 = \int_0^\infty (\mu mx)^3 dA(x) < \infty$, for large n we obtain

$$p_{m,n} = \frac{\varrho_2}{2n} + O\left(\frac{\log n}{n^2}\right).$$

Indeed, let γ_3 denote

$$\gamma_3 = \sum_{l=3}^{\infty} l(l-1)(l-2)r_{0,m,l}.$$

Then also

$$\lim_{n \rightarrow \infty} \left[\sum_{l=3}^n l(l-1)(l-2)r_{0,m,l} + \sum_{k=1}^{m-1} (n+k)(n+k-1)(n+k-2)r_{k,m-k,n} \right] = \gamma_3,$$

and $\rho_3 < \infty$ as $n \rightarrow \infty$. Indeed, as $n \rightarrow \infty$,

$$\begin{aligned} & (n+k)(n+k-1)(n+k-2) \\ & \times \binom{m}{k} \int_0^{\infty} e^{-(m-k)\mu x} \left\{ \int_0^x \frac{(m\mu u)^{n-1}}{(n-1)!} (e^{-\mu u} - e^{-\mu x})^k m\mu du \right\} dA(x) \\ & \leq (n+k)(n+k-1)(n+k-2) \binom{m}{k} \\ & \times \int_0^{\infty} e^{-\mu(m-k)x} \frac{(m\mu x)^n}{n!} dA(x) \rightarrow 0. \end{aligned}$$

Thus, (7) and (8) follow.

6. The Proof of Equation (9)

Whereas (6)-(8) are proved by immediate reduction to the known results associated with (5), the proof of (9) requires special analysis. In order to simplify the analysis let us concentrate our attention to the constant K_m in relation (9). Multiplying this constant by $(1 - \sigma_m)$ we obtain

$$\begin{aligned} \tilde{K}_m &= (1 - \sigma_m)K_m \\ &= \left[\frac{1}{1 - \sigma_m} + \sum_{j=1}^m \binom{m}{j} \frac{C_j}{(1 - \varphi_j)} \frac{m(1 - \varphi_j) - j}{m(1 - \sigma_m) - j} \right]^{-1}. \end{aligned} \tag{22}$$

The constant \tilde{K}_m , given by (22), is well-known from the theory of $GI/M/m$ queueing system. Specifically, let $\tilde{p}_j, j = 0, 1, \dots$, be the stationary probabilities of the number of customers in this system immediately before arrival of a customer. It is known (e.g. Bharucha-Reid [5], Borovkov [6]) that for all $j \geq m$

$$\tilde{p}_j = \tilde{K}_m \sigma_m^{j-m}. \tag{23}$$

Now, in order to prove (9) let us write a new recurrence relation, alternative to (17). For this purpose, join the first m states of the $GI/M/m/n$ process to a single state and label it 0. Other states will be numbered 1, 2, ..., n . In the new terms we have the following recurrence relations

$$\Pi_j^{(n)} = \sum_{i=0}^j r_{0,m,i} \Pi_{j-i+1}^{(n)}, \quad (24)$$

with some initial value $\Pi_0^{(n)}$ for the given series n . For example, for the series $n=0$ we have

$$\Pi_0^{(0)} = \sum_{i=0}^m \binom{m}{i} C_i$$

(see relation (15)). A formal application of the theorem of Takács [24] (see Appendix), applied to recurrence relation (24), for large n yields:

$$\lim_{n \rightarrow \infty} \frac{\Pi_n^{(n)} \sigma_m^n}{\Pi_0^{(n)}} = \frac{1}{1 + \mu m \alpha' (\mu m - \mu m \sigma_m)}. \quad (25)$$

Let us now find $\lim_{n \rightarrow \infty} \Pi_0^{(n)}$. It should be noted that for $j \geq m$ the probability \tilde{p}_j can be rewritten as follows. From (23) we have:

$$\tilde{p}_j = \tilde{K}_m \sigma_m^{j-m} = K_m \sigma_m^{j-m} (1 - \sigma_m) = \frac{K_m P_j}{\sigma_m^m}, \quad (26)$$

where p_j is the conditional probability for the $GI/M/m$ queue, that an arriving customer finds j customers in the queue provided that upon arrival at least $m-1$ servers are occupied. The conditional probability P_{j-m} coincides with the stationary queue-length distribution immediately before arrival of a customer in the $GI/M/1$ queue given under the expected service time $(\mu m)^{-1}$. K_m is the stationary probability for the $GI/M/m$ queue, that upon arrival at least $m-1$ servers are occupied.

From the theory of Markov chains associated with $GI/M/1/n$ queue it is known (e.g. Choi and Kim [8]) that the j -state probability immediately before arrival of a customer is $(\pi_{n-j} - \pi_{n-j-1})/\pi_n$, where π_n is given by (3), and in turn the loss probability is determined by (1).

Then for the same j -state probability of $GI/M/1$ queue with the expected service time $(\mu m)^{-1}$ we have

$$P_j = \lim_{n \rightarrow \infty} \frac{\Pi_{n-j}^{(n)} - \Pi_{n-j-1}^{(n)}}{\Pi_n^{(n)}}. \quad (27)$$

In turn, by (26) and (27), the $j+m$ -state probability of the $GI/M/m$ queue is determined as follows:

$$\tilde{p}_{j+m} = K_m \lim_{n \rightarrow \infty} \frac{\Pi_{n-j}^{(n)} - \Pi_{n-j-1}^{(n)}}{\Pi_n^{(n)}},$$

and

$$\lim_{n \rightarrow \infty} \Pi_0^{(n)} = \frac{1}{K_m}. \quad (28)$$

In view of (28) and (25) and according to Takács theorem [24] we obtain:

$$\lim_{n \rightarrow \infty} \left[\Pi_n^{(n)} - \frac{1}{K_m \sigma_m^n [1 + \mu m \alpha'(\mu m - \mu m \sigma_m)]} \right] = \frac{\varrho K_m}{1 - \varrho}. \tag{29}$$

Now, taking into consideration that the loss probability

$$p_{m,n} = \frac{1}{\Pi_n^{(n)}},$$

we obtain statement (9) of the theorem.

7. The Proof of Theorem 3.2

It was shown in Subhankulov [21], p. 326, that if $\varrho^{-1} = 1 + \varepsilon$, $\varepsilon > 0$ and $\varepsilon \rightarrow 0$, $\varrho_3(n)$ is a bounded sequence, and there exists $\tilde{\varrho}_2 = \lim_{n \rightarrow \infty} \varrho_2(n)$, then

$$\sigma_m = 1 - \frac{2\varepsilon}{\tilde{\varrho}_2} + O(\varepsilon^2), \tag{30}$$

where $\sigma_m = \sigma_m(n)$ is the minimum in absolute value root of the functional equation $z = \alpha(\mu m - \mu m z)$, $|z| \leq 1$, and where the parameter μ and the function $\alpha(z)$, both or one of them, are assumed to depend on n . (Asymptotic representation (30) can be immediately obtained by expanding the equation $z - \alpha(\mu m - \mu m z) = 0$ for small z)

Then, after some algebra we have

$$[1 + \mu m \alpha'(\mu m - \mu m \sigma_m)] = \varepsilon + o(\varepsilon), \tag{31}$$

and

$$\sigma_m^n = e^{-2C/\varrho_2} [1 + o(1)]. \tag{32}$$

In view of (30), the term

$$(1 - \sigma_m) \sum_{j=1}^m \binom{m}{j} \frac{C_j}{(1 - \varphi_j)} \frac{m(1 - \varphi_j) - j}{m(1 - \sigma_m) - j}$$

has the order $O(\varepsilon)$. Therefore, for term (10) we have

$$K_m = 1 + O(\varepsilon), \tag{33}$$

and in the case where $C > 0$, in view of (31)-(33) and (29), we obtain:

$$p_{m,n} = \frac{\varepsilon e^{-2C/\tilde{\varrho}_2}}{1 - e^{-2C/\tilde{\varrho}_2}} [1 + o(1)]. \tag{34}$$

Equation (11) is proved.

The proof of (12) follows by expanding the main term of the asymptotic expression of (34) for small C .

Theorem 3.2 is proved.

8. Numerical Example

In this section a numerical example supporting the theory is provided. Specifically, we simulate $D/M/1/n$ and $D/M/2/n$ queues and check statements (12) and (11) of Theorem 10 numerically. The results of simulation are reflected in the table below. The value ρ is taken 0.999, so that $\varepsilon=0.001$. The value n varies from 10 to 50, and parameter $C = \varepsilon n$ varies from 0.01 to 0.05. The theoretical values of the loss probability for these n are calculated by (12). There are also the loss probabilities for $n=100$. The theoretical value for the loss probability related to this case is calculated by (11).

Table 1. The comparison table of the loss probability for $D/M/1/n$ and $D/M/2/n$ queues.

n	Loss probability theoretical	Loss probability simulated for $D/M/1/n$ queue	Loss probability simulated for $D/M/2/n$ queue
10	0.0501	0.0426	0.039
15	0.0334	0.0292	0.0275
20	0.0251	0.0221	0.0211
25	0.02	0.018	0.0173
30	0.0167	0.0151	0.0146
35	0.0143	0.0128	0.0124
40	0.0125	0.0111	0.0108
45	0.0111	0.0098	0.0096
50	0.01	0.0087	0.0085
100	0.0045	0.004	0.0039

The table is structured as follows: Column 1 contains the values of parameter n , Column 2 contains the theoretical values for the loss probability given by (12) and (11), Column 3 and 4 contain the loss probabilities obtained by simulation for the $D/M/1/n$ and $D/M/2/n$ queueing systems respectively.

As we can see from this table the difference between the loss probabilities of the single-server and two-server queueing systems obtained by simulation is not large, and difference between these loss probabilities decreases as n increases. As n increases the both simulated loss probabilities approach the theoretical loss probability.

Acknowledgements

The author thanks the referees for useful comments. Especial thank is to the referee calling attention of the author to the results of Whitt [25, 26] having immediate relation to the main result of this paper. The initiation of the paper was due to a question (conjecture) of Professor Henk C. Tijms related to asymptotic behavior of the loss probability in the overloaded $GI/M/m/n$ queue. The question had relation to the talk of the author at the First Madrid Conference on Queueing Theory. The research was supported by the Australian Research Council grant, No. DP0771338.

References

1. Abramov, V. M. (1997). On a property of a refusals stream. *Journal of Applied*

- Probability*, 34, 800-805.
2. Abramov, V. M. (2002). Asymptotic analysis of the $GI/M/1/n$ queueing system as increases to infinity. *Annals of Operations Research*, 112, 35-41.
 3. Abramov, V. M. (2004). Asymptotic behavior of the number of lost messages. *SIAM Journal on Applied Mathematics*, 64, 746-761.
 4. Abramov, V. M. (2005). Optimal control of a large dam. arXiv: math/PR 0512118.
 5. Bharucha-Reid, A. T. (1960). *Elements of the Theory of Markov Processes and Their Application*. McGraw-Hill, New York.
 6. Borovkov, A. A. (1976). *Stochastic Processes in Queueing Theory*. Springer-Verlag, Berlin.
 7. Brockmeyer, E., Halström, H. L. and Jensen, A. (1948). *The Life and the Works of A.K.Erlang*. The Copenhagen Telephone Company, Copenhagen.
 8. Choi, B. D. and Kim, B. (2000). Sharp results on convergence rates for the distribution of the $GI/M/c/K$ queues as K tends to infinity. *Journal of Applied Probability*, 37, 1010-1019.
 9. Choi, B. D., Kim, B., Kim, J. and Wee, I.-S. (2003). Exact convergence rate for the distributions of $GI/M/1/K$ queue as K tends to infinity. *Queueing Systems*, 44: 125-136.
 10. Choi, B. D., Kim, B. and Wee, I.-S. (2000). Asymptotic behavior of loss probability in $GI/M/1/K$ queue as tends to infinity. *Queueing Systems*, 36, 437-442.
 11. Cohen, J. W. (1957). The full availability group of trunks with an arbitrary distribution of interarrival times and negative exponential holding time distribution. *Simon Stevin*, 31, 169-181.
 12. Kim, B. and Choi, B. D. (2003). Asymptotic analysis and simple approximation of the loss probability of the $GI^X/M/c/K$ queue. *Performance Evaluation*, 54, 331-356.
 13. Kleinrock, L. (1975). *Queueing Systems. Volume 1: Theory*. John Wiley, New York.
 14. Miyazawa, M. (1990). Complementary generating functions for the $M^X/GI/1/k$ and $GI/M^Y/1/k$ queues and their application to the comparison for loss probabilities. *Journal of Applied Probability*, 27, 682-692.
 15. Palm, C. (1943). Intensitätsschwankungen im Fernsprechverkehr. *Ericsson Technics*, 44, 1-189.
 16. Pollaczek, F. (1953). Generalisation de la theorie probabiliste des systemes telephoniques sans dispositif d'attente. *Comptes Rendus de l'Academie des Sciences (Paris)*, 236, 1469-1470.
 17. Postnikov, A. G. (1979-1980). *Tauberian Theory and Its Application*. Trudy Matematicheskogo Instituta Steklova 2 (1979) 1-147 (In Russian). Engl. transl. in: Proceedings of the Steklov Mathematical Institute 2 (1980) 1-137.
 18. Ramalhoto, M. F. and Gomez-Corral, A. (1998). Some decomposition formulae for $M/M/r/r+d$ queues with constant retrial rate. *Stochastic Models*, 14, 123-145.
 19. Sevastyanov, B. A. (1957). An ergodic theorem for Markov processes and its application to telephone systems with refusals. *Theory of Probability and Its Applications*, 2, 104-112.
 20. Simonot, F. (1998). A comparison of the stationary distributions of $GI/M/c/n$ and $GI/M/c$. *Journal of Applied Probability*, 35, 510-515.
 21. Subhankulov, M.A. (1976). *Tauberian Theorems with Remainder*. Nauka, Moscow. (In Russian)

22. Takács, L. (1957). On a probability problem concerning telephone traffic. *Acta Mathematica Academia Scientiarum Hungaricae*, 8, 319-324.
23. Takács, L. (1962). *Introduction to the Theory of Queues*. Oxford University Press, New York/London.
24. Takács, L. (1967). *Combinatorial Methods in the Theory of Stochastic Processes*. John Wiley, New York.
25. Whitt, W. (2004). Heavy-traffic limits for loss proportions in single-server queues. *Queueing Systems*, 46, 507-536.
26. Whitt, W. (2005). Heavy-traffic limits for the $G/H_2^*/n/m$ queue. *Mathematics of Operations Research*, 30, 1-27.

Appendix:

In the appendix we recall the main results on asymptotic behavior of the sequence Q_n , as $n \rightarrow \infty$ (see relation (5)).

Denote $f(z) = \sum_{i=0}^{\infty} f_i z^i, |z| \leq 1$, $\gamma_i = \sum_{j=i}^{\infty} (\prod_{k=j-i+1}^j k) f_j$.

Theorem A1. (Takács [24], p. 22, 23.) If $\gamma_1 < 1$ then

$$\lim_{n \rightarrow \infty} Q_n = \frac{Q_0}{1 - \gamma_1}.$$

If $\gamma_1 = 1$ and $\gamma_2 < \infty$, then

$$\lim_{n \rightarrow \infty} \frac{Q_n}{n} = \frac{2Q_0}{\gamma_2}.$$

If $\gamma_1 > 1$ then

$$\lim_{n \rightarrow \infty} \left[Q_n - \frac{Q_0}{\delta^n (1 - f'(\delta))} \right] = \frac{Q_0}{1 - \gamma_1},$$

where δ is the least in absolute value root of equation $z = f(z)$.

Theorem A2. (Postnikov [17], Section 25.) If $\gamma_1 = 1$ and $\gamma_3 < \infty$, then as $n \rightarrow \infty$

$$Q_n = \frac{2Q_0}{\gamma_2} n + O(\log n).$$

Author's Biography:

Vyacheslav M. Abramov graduated from Tadjik State University (Dushanbe, Tadjikistan) in 1977. During the period 1977-1992 he worked at the Research Institute of Economics under the Tadjikistan State Planning Committee (GosPlan). In 1992 he repatriated to Israel and during 1994-2001 worked in software companies of Israel as a software engineer and algorithms developer. In 2002-2005 he was an assistant and lecturer in Judea and Samaria College, Tel Aviv University and Holon Institute of Technology. In 2004 he received a PhD degree from Tel Aviv University, and since 2005 has been working at School of Mathematical Sciences of Monash University (Australia). The scientific interests of him are mainly focused on the theory and application of queueing systems. He is an author of a monograph and various papers published in *Journal of Applied Probability*, *Annals of Operations Research*, *Queueing Systems*, *SIAM Journal on Applied Mathematics* and other journals.