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On Existence of Limiting Distribution for Time-Nonhomogeneous Countable Markov Process

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Abstract. In this paper, sufficient conditions are given for the existence of limiting distribution of a nonhomogeneous countable Markov chain with time-dependent transition intensity matrix. The method of proof exploits the fact that if the distribution of random process $Q = (Q_t)_{t\geqslant 0}$ is absolutely continuous with respect to the distribution of ergodic random process $Q^\circ = (Q_t^\circ)_{t\geqslant 0}$, then

$$Q_t \xrightarrow[t\to\infty]{\text{law}} \pi,$$

where π is the invariant measure of Q°. We apply this result for asymptotic analysis, as $t \to \infty$, of a nonhomogeneous countable Markov chain which shares limiting distribution with an ergodic birth-and-death process.

Keywords: countable Markov process, existence of the limiting distribution, birth-and-death process

1. Introduction

There is a large number of papers in the queueing literature devoted to analysis of state dependent and time dependent queueing systems as $M_t/M_t/1$ and $M_t/M_t/c$ and associated with their Markovian queueing networks (e.g., see [23–27]). An analysis of such queueing system is motivated by a wide spectra of practical problems well-known in the literature. For instance, a simple example corresponding to the police dispatching problem is given in [26, p. 60] (see also [15] for further discussion). The $M_t/M_t/\infty$ queue, used for a model of emergency ambulances and intensive care units, is considered in [6]. Other applications are known for client/server computer networks, when arrival and service rates of nodes depend on amount of unfinished work and number of available tasks on server (see, e.g., [2,3,19,20]).

Typically, an asymptotic analysis of $M_t/M_t/1$ and $M_t/M_t/c$ uses differential equations for transition probabilities and asymptotic analysis, as $t \to \infty$, for their solutions. This type of analysis is similar to an investigation of stability for Markov chain and is associated with a verification of stationarity (quasi-stationarity) (see [1,4,5, 7–10,21,29] and many others.

We mention now results related to time-nonhomogeneous stochastic models converging, as $t \to \infty$, to time-homogeneous ones. Although first results were published more than 30 years ago by Gnedenko and Soloviev [12] and Gnedenko [11], a remarkable progress was achieved not a long time ago (see [13,14,16,23,30–36]). In particular, Zeifman developed a number of effective tools, permitting investigate successfully ergodicity conditions for special classes of time-nonhomogeneous birth-and-death processes including $M_t/M_t/1$, $M_t/M_t/S$ and $M_t/M_t/S/0$ queues (for further discussion, see [14]).

In the present paper, we give sufficient conditions, under which a time-nonhomogeneous countable Markov chain with the transition intensity matrix $\Lambda(t)$ shares the limiting distribution with time-homogeneous ergodic Markov chain with the transition intensity matrix Λ. Our setting is heavily related to the above-mentioned settings and the result obtained supplements the results from [14,31,32,35,36] (more detailed comparison is given in section 6). The main difference with known approaches to this problem is that the convergence

$$\Lambda(t) \to \Lambda, \quad t \to \infty$$

is not required. In contrast to that we assume the existence of nonnegative λ_{ij}, $i \neq j$ such that (here $\lambda_{ij}(t)$ are entries of $\Lambda(t)$)

$$\begin{gathered}\sum_{j\neq i} \lambda_{ij} < \infty, \qquad \int_0^\infty \left(\sqrt{\lambda_{ij}(t)} - \sqrt{\lambda_{ij}}\right)^2 \mathrm{d}t < \infty, \\ \int_0^t \lambda_{ij}(s) I(\lambda_{ij} > 0)\,\mathrm{d}s = \int_0^t \lambda_{ij}(s)\,\mathrm{d}s, \quad \forall t > 0,\end{gathered} \tag{1.1}$$

and create the matrix Λ with entries λ_{ij}, $i \neq j$ and $\lambda_{ii} = -\sum_{j\neq i} \lambda_{ij}$. We assume that the Markov chain with this transition intensity matrix Λ is ergodic.

To explain our method with more details, notice that (1.1) guarantees the absolute continuity of the distribution for $\Lambda(t)$-Markov chain with respect to the distribution for Λ-Markov chain. It is also assumed that Λ-Markov chain is ergodic but the geometrical ergodicity is not required. We show in theorem 2.1 (section 2) that the above-mentioned absolute continuity of distributions provides the limiting distribution, as $t \to \infty$, for $\Lambda(t)$-Markov chain coinciding with the invariant measure of Λ-Markov chain.

In section 3, we give the proof of theorem 2.1. In section 4, we show that not only the limiting distribution but also other limiting functionals are the same as for Λ-Markov chain. In section 5, an asymptotic equivalence of $\Lambda(t)$-Markov chain to an ergodic birth-and-death process is established.

2. The main result

We consider a nonhomogeneous Markov chain $Q = (Q_t)_{t \geqslant 0}$ with the countable set of states $\mathbb{S} = \{0, 1, \ldots\}$ and the transition intensity matrix $\Lambda(t)$ with entries $\lambda_{ij}(t)$. Suppose that for any pair (i, j) with $i \neq j$ there is a nonnegative constant λ_{ij} such that (1.1) holds true, and introduce Markov chain $Q^\circ = (Q^\circ_t)_{t \geqslant 0}$ with the set of its states $\mathbb{S}$, and $Q^\circ_0 = Q_0$, and the transition intensity matrix Λ (see section 1).

Our main assumption is that Q° is ergodic, i.e. there is the unique probabilistic measure π on $\mathbb{S}$ such that $\pi\Lambda = 0$ and

$$\lim_{t\to\infty} P\big(Q^\circ_t = j \mid Q^\circ_s = i\big) = \pi_j, \quad \forall s, i, j, \tag{2.1}$$

where π_j are entries of π.

Theorem 2.1. Under (1.1) and (2.1),

$$\lim_{t\to\infty} P(Q_t = j \mid Q_s = i) = \pi_j, \quad \forall s, i, j. \tag{2.2}$$

3. The proof of theorem 2.1

3.1. Preliminaries

Without loss of generality one may assume that the Markov chains Q and Q° have paths in the Skorokhod space $\mathbb{D} = \mathbb{D}_{[0,\infty)}$ of right continuous having limits to the left functions $x = (x_t)_{t \geqslant 0}$.

Let ν, ν° be the distributions of Q and Q° respectively, that is ν, ν° are probabilistic measures on $(\mathbb{D}, \mathcal{G})$, where $\mathcal{G}$ is the Borel σ-algebra on $\mathbb{D}$. Without loss of generality we may assume that $\mathcal{G}$ is completed with respect to the measure $(\nu + \nu^\circ)/2$. We shall use in the sequel that $\nu \ll \nu^\circ$. Recall (see, e.g., [28]) that $\nu \ll \nu^\circ$ provides that $\nu(A) = 0$ for any $A \in \mathcal{G}$, if $\nu^\circ(A) = 0$.

For the verification of $\nu \ll \nu^\circ$, we apply [17, theorem 24]. Following this theorem, $\nu \ll \nu^\circ$ if for all $i, j \in \mathbb{S}$

(a) $\nu^\circ(x_0 = i) = 0 \Leftarrow \nu(x_0 = i) = 0$;

(b) $\int_0^t I(x_s = j)\lambda_{ij}(s)\,\mathrm{d}s = \int_0^t I(x_s = j)\lambda_{ij}(s) I(\lambda_{ij} \neq 0)\,\mathrm{d}s$, ν-a.s.;

(c) with $\infty \cdot 0 = 0$

$$P\left(\int_0^\infty \sum_{j\neq i} \left[1 - \sqrt{\frac{\lambda_{ij}(t)}{\lambda_{ij}}}\right]^2 I(\lambda_{ij} \neq 0)\lambda_{ij} I(x_t = i)\,\mathrm{d}t < \infty\right) = 1, \quad \nu\text{-a.s.}$$

Notice that for any $j \neq i$, (c) is provided by the condition

$$\int_0^\infty \left[1 - \sqrt{\frac{\lambda_{ij}(t)}{\lambda_{ij}}}\right]^2 I(\lambda_{ij} \neq 0)\lambda_{ij}\,\mathrm{d}t < \infty$$

equivalent to the first part in (1.1).

Introduce a stochastic basis $(\mathbb{D}, \mathcal{G}, (\mathcal{G}_t)_{t\geqslant 0}, \nu^\circ)$ with the general condition (see, e.g., [22]), where $(\mathcal{G}_t)_{t\geqslant 0}$ is the filtration generated by x. Henceforth, E^{ν} and E^{ν° denote the expectations with respect to ν and ν°, respectively.

Set $Z(x) = \mathrm{d}\nu/\mathrm{d}\nu^\circ(x)$ and $Z_t(x) = E^{\nu^\circ}(Z \mid \mathcal{G}_t)(x)$. We shall use the fact that $(Z_t(x))_{t\geqslant 0}$ is positive uniformly integrable martingale with respect to ν°. Throughout the paper we use the notation "$\wedge$" ("$\vee$") for minimum (maximum) of two numbers.

3.2. Auxiliary lemma

Lemma 3.1. Under the assumptions of theorem 2.1, for any $s \geqslant 0$ and $j \in \mathbb{S}$

$$P(Q_t = j|Q_s) \xrightarrow[t\to\infty]{\text{prob.}} \pi_j.$$

Proof. With $s < s' < t$, using Markov property, write

$$P(Q_t = j \mid Q_s) = \nu(x_t = j \mid \mathcal{G}_s) = E^{\nu}\big(\nu(x_t = j \mid \mathcal{G}_{s'}) \mid x_s\big).$$

According to well known formula for the conditional expectation under absolute continuous change of measure: for any integrable random variable α, $E^{\nu}(\alpha \mid \mathcal{G}_{s'}) = E^{\nu^\circ}((Z/Z_{s'})\alpha \mid \mathcal{G}_{s'})$ we find

$$\begin{aligned}\nu(x_t = j \mid \mathcal{G}_{s'}) &= E^{\nu^\circ}\left(\frac{Z(x)}{Z_{s'}(x)} I(x_t = j) \mid \mathcal{G}_{s'}\right)\\ &= \nu^\circ(x_t = j \mid x_{s'}) + E^{\nu^\circ}\left(\left[\frac{Z(x)}{Z_{s'}(x)} - 1\right] I(x_t = j) \mid \mathcal{G}_{s'}\right).\end{aligned}$$

By (2.1), $\nu^\circ(x_t = j \mid x_{s'}) \xrightarrow[t\to\infty]{\nu^\circ-\text{a.s.}} \pi_j$ and by $\nu \ll \nu^\circ$ the same convergence holds ν-a.s. too well. So, it remains to show that

$$\varlimsup_{t\to\infty} E^{\nu^\circ}\left(\left[\frac{Z(x)}{Z_{s'}(x)} - 1\right] I(x_t = j) \mid \mathcal{G}_{s'}\right) \xrightarrow[s'\to\infty]{\nu^\circ-\text{prob.}} 0. \tag{3.1}$$

Notice that

$$\begin{aligned}&E^{\nu^\circ}\left|\varlimsup_{t\to\infty} E^{\nu^\circ}\left(\left[\frac{Z(x)}{Z_{s'}(x)} - 1\right] I(x_t = j) \mid \mathcal{G}_{s'}\right)\right|\\ &\leqslant E^{\nu^\circ}\left|\frac{Z(x)}{Z_{s'}(x)} - 1\right|\\ &\leqslant E^{\nu^\circ}\left(\left|\frac{Z(x)}{Z_{s'}(x)} - 1\right| \wedge 3\right) + E^{\nu^\circ}\left(\frac{Z(x)}{Z_{s'}(x)} + 1\right) I\left(\frac{Z(x)}{Z_{s'}(x)} > 2\right)\\ &= E^{\nu^\circ}\left(\left|\frac{Z(x)}{Z_{s'}(x)} - 1\right| \wedge 3\right) + E^{\nu^\circ}\left(\frac{Z(x)}{Z_{s'}(x)} + 1\right)\\ &\quad - E^{\nu^\circ}\left(\frac{Z(x)}{Z_{s'}(x)} + 1\right) I\left(\frac{Z(x)}{Z_{s'}(x)} \leqslant 2\right).\end{aligned}$$

As was mentioned above $Z_t(x)$ is the positive uniformly integrable ν°-martingale. Hence $\lim_{s'\to\infty} Z_{s'}(x) = Z(x)$, ν°-a.s. Consequently, by Lebesgue dominated theorem

$$\lim_{s'\to\infty} E^{\nu^\circ}\left(\left|\frac{Z(x)}{Z_{s'}(x)} - 1\right| \wedge 3\right) = 0. \tag{3.2}$$

Now, it remains to show that

$$\lim_{s'\to\infty} E^{\nu^\circ}\left(\frac{Z(x)}{Z_{s'}(x)} + 1\right) = \lim_{s'\to\infty} E^{\nu^\circ}\left(\frac{Z(x)}{Z_{s'}(x)} + 1\right) I\left(\frac{Z(x)}{Z_{s'}(x)} \leqslant 2\right).$$

Since $Z_{s'}(x) \to Z(x), s' \to \infty$, by the Lebesgue dominated theorem the right hand side of the above equality is equal to 2. At the same time for any s' we have $E^\circ((Z(x)/Z_{s'}(x)) \mid \mathcal{G}_{s'}) = 1$ and so for any s' it holds $E^\circ(Z(x)/Z_{s'}(x) + 1) = 2$.

Thus,

$$\overline{\lim_{t\to\infty}} E^{\nu^\circ}\left(\left[\frac{Z(x)}{Z_{s'}(x)} - 1\right] I(x_t = j) \mid \mathcal{G}_{s'}\right) \underset{s'\to\infty}{\overset{\nu^\circ-\text{prob.}}{\longrightarrow}} 0. \qquad \square$$

3.3. Final part of proof

By lemma 3.1 we have

$$\sum_{j=0}^{\infty} P(Q_t = j \mid Q_s = i) I(Q_s = i) \underset{t\to\infty}{\overset{\text{prob.}}{\longrightarrow}} \pi_j.$$

Hence, for any $i \in \mathbb{S}$

$$P(Q_t = j \mid Q_s = i) I(Q_s = i) \underset{t\to\infty}{\overset{\text{prob.}}{\longrightarrow}} \pi_j I(Q_s = i) \tag{3.3}$$

and the statement of theorem 2.1 follows. $\square$

4. Asymptotic equivalence for other functionals

Denote $h(x_t) = I(x_t = j)$. Theorem 2.1 guarantees the asymptotic equivalence

$$\lim_{t\to\infty} E^{\nu}\big(h(x_t) \mid \mathcal{G}_s\big) = \lim_{t\to\infty} E^{\nu^\circ}\big(h(x_t) \mid \mathcal{G}_s\big).$$

An analysis of the proof of theorem 2.1 shows that the same type of asymptotic equivalence holds for any bounded functional $h(x_{[t,\infty)})$ of argument $x_{[t,\infty)} = \{x_u, u \geqslant t\}$ provided that $\lim_{t\to\infty} E^{\nu^\circ}(h(x_{[t,\infty)}) \mid \mathcal{G}_s)$ exists, that is under the assumptions of theorem 2.1

$$\lim_{t\to\infty} E^{\nu}\big(h(x_{[t,\infty)}) \mid \mathcal{G}_s\big) = \lim_{t\to\infty} E^{\nu^\circ}\big(h(x_{[t,\infty)}) \mid \mathcal{G}_s\big). \tag{4.1}$$

5. Asymptotic equivalence to birth-and-death process

Let

$$\Lambda(t)=\begin{pmatrix} -\lambda_0(t) & \lambda_0(t) & 0 & 0 & \dots \\ \mu_1(t) & -(\lambda_1(t)+\mu_1(t)) & \lambda_1(t) & 0 & \dots \\ 0 & \mu_2(t) & -(\lambda_2(t)+\mu_2(t)) & \lambda_2(t) & \dots \\ 0 & 0 & \mu_3(t) & -(\lambda_3(t)+\mu_3(t)) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Assume that there exist positive numbers λ_j, μ_j's such that

$$\sum_{n=1}^{\infty}\prod_{j=1}^{n}\frac{\lambda_{j-1}}{\mu_j}<\infty \tag{5.1}$$

and

$$\int_0^{\infty}\Big[\big(\sqrt{\lambda_j(t)}-\sqrt{\lambda_j}\big)^2+\big(\sqrt{\mu_j(t)}-\sqrt{\mu_j}\big)^2\Big]\,\mathrm{d}t<\infty,\qquad \forall j\geqslant 0. \tag{5.2}$$

We mention here that (5.2) does not provide $\lambda_j(t)\to\lambda_j$, $\mu_j(t)\to\mu_j$, $t\to\infty$.

Introduce the matrix

$$\Lambda=\begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and notice that Markov chain with the transition intensity matrix Λ generates the birth-and-death process.

It is well known (see, e.g., [18, chapter 7, section 5]) that under (5.1) the birth-and-death process is ergodic with the unique stationary distribution on $\mathbb{S}$:

$$\pi_0=\frac{1}{1+\sum_{n=1}^{\infty}\prod_{j=1}^{n}(\lambda_{j-1}/\mu_j)},\qquad \pi_n=\pi_0\prod_{j=1}^{n}\frac{\lambda_{j-1}}{\mu_j},\quad n=1,2,\dots. \tag{5.3}$$

By theorem 2.1, the Markov chain with transition intensity matrix $\Lambda(t)$ possesses the same stationary distribution.

It is known that the sojourn time T_i in state i for the birth-and-death process is exponentially distributed

$$P(T_i\leqslant x)=1-\mathrm{e}^{-(\lambda_i+\mu_i)x}.$$

Under (5.1) and (5.2), for Markov chain with the transition intensity $\Lambda(t)$ we have the following. Let $T_i(t)=v_i(t)-t$, where

$$v_i(t)=\inf\{s\geqslant t\colon\ Q_s\neq i,\ Q_t=i\}.$$

Then, applying the result from section 4, we obtain

$$\lim_{t\to\infty} P\big(T_i(t) \leqslant x\big) = 1 - \mathrm{e}^{-(\lambda_i+\mu_i)x}.$$

6. Discussion

We consider here $M_t/M_t/1$ model. Let A_t and D_t be independent and time-nonhomogeneous Poisson processes with positive rates $\lambda(t)$ and $\mu(t)$ respectively. Let Q_0 be a random variable, independent of A_t and D_t, taking values in $\mathbb{S} = \{0, 1, \ldots\}$. We define the queue-length process $Q = (Q_t)_{t\geqslant 0}$ in the $M_t/M_t/1$ as follows (notice that jumps of A_t and D_t are disjoint and so $Q_t \in \mathbb{S}$)

$$Q_t = Q_0 + A_t - \int_0^t I(Q_{s-} > 0)\,\mathrm{d}D_s.$$

Let A_t° and D_t° be independent and homogeneous Poisson processes with positive rates λ and μ, let $Q_0^\circ = Q_0$ be independent of A_t° and D_t°, and let $Q^\circ = (Q_t^\circ)_{t\geqslant 0}$ be the queue-length process in the $M/M/1$ queue with parameters λ and μ for arrival and service of customers, respectively,

$$\lambda < \mu.$$

The queue-length process Q_t° is defined as

$$Q_t^\circ = Q_0^\circ + A_t^\circ - \int_0^t I\big(Q_{s-}^\circ > 0\big)\,\mathrm{d}D_s^\circ.$$

By theorem 2.1 the existence of the limiting distribution for $M_t/M_t/1$ is provided by

$$\int_0^\infty \Big[\big(\sqrt{\lambda} - \sqrt{\lambda(t)}\big)^2 + \big(\sqrt{\mu} - \sqrt{\mu(t)}\big)^2\Big]\,\mathrm{d}t < \infty. \tag{6.1}$$

On the other hand, it is known from [14,31,32,35] that the existence of the limiting distribution is provided by

$$\lim_{t\to\infty} \big[\big|\lambda - \lambda(t)\big| + \big|\mu - \mu(t)\big|\big] = 0. \tag{6.2}$$

Generally,

$$(6.1) \not\Leftrightarrow (6.2),$$

and so (6.1) and (6.2) supplement each other. If $\lambda(t)$ and $\mu(t)$ are uniformly continuous on $[0, \infty)$ functions, then (6.1) $\Rightarrow$ (6.2), that is (6.2) is weaker than (6.1). Notice also that (6.2) $\Rightarrow$ (6.1), say, under additional condition: for small positive ε and t large enough $|\lambda - \lambda(t)| + |\mu - \mu(t)| = \mathrm{O}(t^{-(1/2+\varepsilon)})$.

It would be noted that in [36], it is studied an ergodicity problem, in the uniform operator topology, of time-nonhomogeneous Markov chains with transition intensity matrices possessing summable perturbations. These results have some connection with our one too well.

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