A large closed queueing network with autonomous service and bottleneck

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This paper studies the queue-length process in a closed Jackson-type queueing network with the large number N of homogeneous customers by methods of the theory of martingales and by the up- and down-crossing method. The network considered here consists of a central node (hub), being an infinite-server queueing system with exponentially distributed service times, and k single-server satellite stations (nodes) with generally distributed service times with rates depending on the value N. The service mechanism of these k satellite stations is autonomous, i.e., every satellite server j serves the customers only at random instants that form a strictly stationary and ergodic sequence of random variables. Assuming that the first $k-1$ satellite stations operate in light usage regime the paper considers the cases where the kth satellite station is a bottleneck node. The approach of the paper is based both on development of the method from the paper by Kogan and Liptser [16], where a Markovian version of this model has been studied, and on development of the up- and down-crossing method.

Keywords: closed queueing network, autonomous service, bottleneck, martingales and semimartingales, diffusion and fluid approximation, up- and down-crossings

1. Introduction

Consider a closed Jackson-type Markovian network consisting of N homogeneous customers with the following structure. There are a central node (hub), being an infinite server queueing system with identical servers numbered by 0, and k satellite nodes, being single-server stations numbered by $1, 2, \ldots, k$. At the initial time moment all customers are assumed to be at the hub. After service completion at the hub, a customer visits satellite node j with probability p_j ($\sum_{j=1}^{k} p_j = 1$). Being served at the satellite node customers return to the hub. A service time at the hub is exponentially distributed random variable with the expectation λ^{-1} for each customer and a service time at satellite node j is exponentially distributed with the expectation $(\mu_j N)^{-1}$.

Let $Q_j^N(t)$, $j = 1, 2, \ldots, k$, denote a queue-length at time t at satellile node j, and $Q_0^N(t)$ denote a queue-length at the hub. According to convention, $Q_j^N(0) = 0$, $1 \leqslant j \leqslant k$, and $Q_0^N(0) = N$.

We say that a satellite node j is a *bottleneck node* if $\mu_j \leqslant \lambda_j$. Otherwise, that node j is called nonbottleneck node. As $N \to \infty$, Kogan and Liptser [16] showed that the nonstationary queue-length distribution for every nonbottleneck node is the same as some stationary $M/M/1$ queue-length distribution whose parameters explicitly depend on t, i.e., the queue-length process is described by the Yule process (see Bharucha-Reid [6]), and in the case where there is no bottleneck node in the network, the nonstationary queue-length processes for every node is described by geometrical distribution for all $t > 0$.

More accurately, denote $\lambda_j = \lambda p_j$, $1 \leqslant j \leqslant k$, and suppose that nodes $j = 1, 2, \ldots, k-1$ are nonbottleneck nodes and node k is a bottleneck node. It was shown that the distribution of the queue-length $Q_j^N(t)$, $j = 1, 2, \ldots, k-1$, as $N \to \infty$, for any fixed $t > 0$ tends to the stationary distribution of the $M/M/1$ queue-length process with constant arrival rate $\lambda_j[\pi_k + (1-\pi_k)\exp(-\lambda_k t)]$, where $\pi_k = \lambda_k^{-1}\mu_k$ and service rate μ_j.

Under the same assumption on bottlenecks Abramov [4] studied a more general model where service times at the single-server hub are generally distributed and depending on the number of customers residing here. It was shown that the limiting as $N \to \infty$ nonstationary distribution of the queue-length $Q_j^N(t)$, $j = 1, 2, \ldots, k-1$, coincides with the stationary distribution of the $GI/M/1$ queue-length process with parameters explicitly depending on t due to the same effect of bottleneck station k.

In this paper we develop the model of Kogan and Liptser [16] in the following direction.

(*) The service mechanism of k satellite stations is autonomous, i.e., every server $j = 1, 2, \ldots, k$ serves customers only in random instants $\xi_{j,1}^N, \xi_{j,1}^N + \xi_{j,2}^N, \ldots$, and server can accept only one of waiting customers for a service (provided that the correspondent node is not empty). The sequences $\{\xi_{j,1}^N, \xi_{j,2}^N, \ldots\}$ are assumed to be strictly stationary and ergodic sequences of random variables for each j, $\mathbf{E}\xi_{j,1}^N = (\mu_j N)^{-1}$.

Assumption (*) corresponds to models of computer networks in practice, where a message can be processed only in a given random instant. For example, token ring network with a star-ring topology is a network where assumption (*) is appropriate and corresponds to our concerns.

Let $\lambda_j = \lambda p_j$ and $\rho_j = \lambda_j/\mu_j$. It is assumed that for large N, $\rho_j < 1$, $j = 1, 2, \ldots, k-1$, while $\rho_k \geqslant 1$. Node j is said to operate in *light, moderate, heavy* usage regime if respectively $\rho_j < 1$, $\rho_j = 1$, $\rho_j > 1$. As $N \to \infty$ these regimes are respectively called *asymptotically* light, moderate and heavy usage regimes. In the further account the word "asymptotically" will be omitted, and taking the limit as $N \to \infty$ we assume that the series of random sequences indexed by N are given.

As in the Markovian case, a node in moderate or heavy usage regime is also called a bottleneck node, and a node in light usage regime is called a nonbottleneck node. Thus, according to assumption above, the first $k-1$ satellite nodes are nonbottleneck nodes, while kth satellite node is a bottleneck node.

Queueing systems with autonomous service was introduced and studied by Borovkov [8]. He proved the theorems on stability for a wide class of those queueing systems and suggested the methods to calculate the stationary probabilities for certain autonomous queueing systems in which interarrival and departure epochs are independent identically distributed random variables. For a wide class of queuing systems with autonomous service, under assumption that the sequences of interarrival and departure epochs form strictly stationary sequences of random variables, Abramov [3] derived relationships between stationary probabilities immediately before arrival and departure of customers with large number. To obtain appropriate relationship he used an idea of up- and down-crossings. This idea is also used in the present paper. For other papers using an up- and down-crossing approach see also Cohen [11] and Abramov [1,2,4].

As in a number of earlier papers closely related to this one (see already mentioned papers of Kogan and Liptser [16], Abramov [4] as well as Kogan et al. [17]), the present paper studies the cases of moderate and heavy usage regime for the kth satellite node. The methods of analysis for the kth satellite node are based on the development of the methods of the mentioned paper by Kogan and Liptser [16]. Nevertheless, those results related to the queue-length process of the kth satellite node remain the same as the similar results from the respective cases of the paper by Kogan and Liptser [16]. At the same time, the analysis of the queue-length process at the first $k-1$ satellite nodes essentially differs from that part of the paper by Kogan and Liptser [16] and leads to new specific effects.

To obtain limiting nonstationary queue-length distributions Kogan and Liptser [16] used the Doob–Meyer semimartingale decomposition for the indicators of the queue-length processes and then used the standard techniques of stochastic analysis and the theory of martingales. Note, that the Doob–Meyer semimartingale decomposition for indicators of the queue length is the martingale analogue of the Chapman–Kolmogorov equations. The present paper studies the limiting nonstationary queue-length probabilities with the aid of two approaches: up- and down-crossing approach, which is the development of the method of the paper by Abramov [3], and martingale approach, which is in turn the development of the method of the paper by Kogan and Liptser [16].

Whereas both for the Markovian version of the network of Kogan and Liptser [16] and for non-Markovian model from the paper by Abramov [4], the limiting nonstationary queue-length distribution at the first $k-1$ satellite nodes was obtained in explicit form, in the case of the present version of network the general assumption of (*) does not permit us to obtain such kind of the results. The result discussed here connects the nonstationary queue-length probability at time t with nonstationary queue-length probability immediately before the last departure of a customer before time t, and enables us only to judge on the qualitative behavior of the limiting nonstationary queue-length process because of effect of the bottleneck node.

In the case of Markovian network, where all sequences $\{\xi_{j,i}^N\}$ consist of independent identically and exponentially distributed random variables, we obtain the probabilities in explicit form, while the mentioned paper by Kogan and Liptser [16] suggested us more complicated representation in the form of generalized functions.

Unfortunately, approach of the present paper does not permit us to obtain the results giving us something in terms of habital for us transformations, Laplace–Stieltjes transform, say. This is because that approach does not permit us to separate the case of independent identically distributed random variables from considered in the paper case of strictly stationary and ergodic sequence of random variables $\{\xi_{j,i}^N\}_{i\geqslant 1}$. Such kind of result was obtain in Abramov [4] in the case where a hub is a single-server whose service times are conditionally independent random variables depending upon the number of customers residing there.

For different papers closely related to diffusion and fluid approximation as well as to similar models of queueing networks see also Kogan [15], Krichagina et al. [21], Kogan et al. [17,18], Konstantopoulos et al. [19,20], Chen and Mandelbaum [9,10], Mandelbaum and Pats [25], Whitt [30], etc.

The paper is structured as follows. It consists of 10 sections. Section 1 is introduction. Section 2 describes the queue-length process at satellite nodes in terms of arrival and departure point processes to introduce us to the Skorohod reflection principle, one of the main methods of the paper. Section 3 reduces the problem to the Skorohod reflection principle and describes it in terms of stochastic calculus by using the Doob–Meyer semimartingale decomposition. Section 4 studies the normalized queue-length process by dividing the characteristics of that process to large parameter N, the number of customers in the system. The purpose of this section is to study the asymptotic behavior of normalized queue-length process both for the bottleneck node and for satellite nonbottleneck nodes. Section 5 studies another properties of normalized queue-length to prove then the collapse theorem (in the same section) and the theorems on diffusion and fluid approximation (in section 8). Section 6 proves the stability result, which is the existence of limiting nonstationary queue-length probabilities for the first $k-1$ satellite nodes, by using the Skorohod reflection principle and expressing those limiting distributions of the queue-length processes via limiting distributions of supremum of appropriate processes with strictly stationary and ergodic increments. Section 7 proves a main result of this paper on limiting nonstationary queue-length probabilities at the first $k-1$ satellite nodes, which are nonbottleneck nodes. The proof of the theorem uses the up- and down-crossings approach. This section contains also the special case of the Markovian version of network related to results of the earlier paper by Kogan and Liptser [16]. Section 8 studies the same results with the aid of martingale approach by developing the method by Kogan and Liptser [16]. One subsection derives the Doob–Meyer semimartingale decompositions for the indicators of the queue-length processes. The other subsection focuses on the proof of the main result of this section with the aid of asymptotic analysis of those stochastic equations, as the number of customers in network, N, increases to infinity. Section 9 proves the theorems on diffusion and fluid approximations at the bottleneck node. The main results of this section are the diffusion approximation with reflection at zero in the case of moderate usage regime and the fluid and Gaussian diffusion approximation in the case of heavy usage regime. Section 10 consisting of two subsections discusses the future work and the possible extension of the model. The first subsection discusses the situation where arrival process to satellite node j is generated by the nonhomogeneous Poisson process depending on time. For such kind of networks see also the paper by Mandelbaum and Pats [25]. The second subsection discusses a more general situation where arrival process to satellite node j is generated by a process with strictly stationary and ergodic increments.

2. Queue-length processes at satellite nodes

In this section we consider a description of queue-length process for satellite node j. The queue-length process for that node can be written as follows:

$$Q_j^N(t) = A_{j,N}(t) - D_{j,N}(t), \tag{2.1}$$

where $A_{j,N}(t)$ is the *arrival* process to node j, and $D_{j,N}(t)$ is the *departure* process from node j.

To define a departure process let us consider the point process

$$S_{j,N}(t) = \sum_{l=1}^{\infty} \mathbf{I}\{x_{j,l}^N \leqslant t\}, \quad x_{j,l}^N = \sum_{i=1}^{l} \xi_{j,i}^N, \quad l \geqslant 1.$$

Then,

$$\begin{aligned} D_{j,N}(t) &= \int_0^t \mathbf{I}\{Q_j^N(s-) > 0\} \mathrm{d}S_{j,N}(s) \\ &= S_{j,N}(t) - \int_0^t \mathbf{I}\{Q_j^N(s-) = 0\} \mathrm{d}S_{j,N}(s). \end{aligned} \tag{2.2}$$

To define an arrival process, let $\{\pi_{j,i}(t)\}$, $i = 1, 2, \ldots, N$, be a collection of independent Poisson processes with rate λ_j. Then,

$$A_{j,N}(t) = \int_0^t \sum_{i=1}^{N} \mathbf{I}\left\{N - \sum_{l=1}^{k} Q_l^N(s-) \geqslant i\right\} \mathrm{d}\pi_{j,i}(s) \tag{2.3}$$

(for details see Kogan and Liptser [16] and Krichagina et al. [21]).

3. Reduction to the Skorohod problem

In view of (2.1) and (2.2) the queue-length process $Q_j^N(t)$ can be rewritten as

$$Q_j^N(t) = A_{j,N}(t) - S_{j,N}(t) + \int_0^t \mathbf{I}\{Q_j^N(s-) = 0\} \mathrm{d}S_{j,N}(s). \tag{3.1}$$

Equation (3.1) implies that $Q_j^N(t)$ is the normal reflection of the process

$$X_{j,N}(t) = A_{j,N}(t) - S_{j,N}(t), \qquad X_{j,N}(0) = 0 \tag{3.2}$$

at zero. More accurately, $Q_j^N(t)$ is the non-negative solution of the Skorohod problem (see Skorohod [27] as well as Tanaka [29] and Anulova and Liptser [5]) of the normal reflection of the process $X_{j,N}(t)$ at zero (see also Kogan and Liptser [16]), since the function

$$\varphi_j(t) = \int_0^t \mathbf{I}\big\{Q_j^N(s-) = 0\big\}\mathrm{d}S_{j,N}(s) \tag{3.3}$$

satisfies the following two properties:

(1) $\int_0^t h[Q_j^N(s)]\,\mathrm{d}\varphi_j(s) = 0$ for any continuous non-negative function $h(y)$ with $h(0) = 0$;

(2) the function $\int_0^t [Y(s) - Q_j^N(s)]\,\mathrm{d}\varphi_j(s)$ is nondecreasing for any non-negative function $Y(s)$ from the Skorohod space $\mathbf{D}$.

(Detailed proof of those properties is the same as in Kogan and Liptser [16]).

Therefore, the function in (3.3) can be represented as

$$\varphi_j(t) = -\inf_{s\leqslant t} X_{j,N}(s). \tag{3.4}$$

For our further purposes and for the sake of simplicity we shall use the notation

$$\Psi_t(X) = -\inf_{s\leqslant t} X(s) \tag{3.5}$$

for any function $X(t)$, $t \geqslant 0$, from the Skorohod space $\mathbf{D}$ with $X(0) = 0$. From (3.1) and (3.3)–(3.5) we have

$$Q_{j,N}(t) = X_{j,N}(t) + \Psi_t(X_{j,N}). \tag{3.6}$$

Next, let us take into account that the processes $A_{j,N}(t)$ and $S_{j,N}(t)$ are the semimartingales adapted with respect to the filtration $\mathcal{F}_t$ given on stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geqslant 0}, \mathbf{P})$. Then the process $X_{j,N}(t)$ in (3.6) can be rewritten in the form of the Doob–Meyer semimartingale decomposition (see, e.g., Liptser and Shiryayev [24], Jacod and Shiryayev [14]). Denoting $A_{j,N}^p(t)$ and $S_{j,N}^p(t)$ the compensators of the processes $A_{j,N}(t)$ and $S_{j,N}(t)$ respectively, then the processes $A_{j,N}(t) - A_{j,N}^p(t)$ and $S_{j,N}(t) - S_{j,N}^p(t)$ are the local square integrable martingales (see Liptser and Shiryayev [23, chapter 18]). We, therefore, obtain

$$X_{j,N}(t) = A_{j,N}^p(t) - S_{j,N}^p(t) + M_{j,N}(t), \tag{3.7}$$

where

$$M_{j,N}(t) = \big[A_{j,N}(t) - A_{j,N}^p(t)\big] - \big[S_{j,N}(t) - S_{j,N}^p(t)\big]$$

is a local square integrable martingale.

Note, that a compensator of the process $A_{j,N}(t)$ defined in (2.3) has the representation

$$A_{j,N}^p(t) = \int_0^t \lambda_j \left\{N - \sum_{l=1}^k Q_l^N(s)\right\}\mathrm{d}s \tag{3.8}$$

(for details see Dellacherie [13], Liptser and Shiryayev [23; 24, theorem 1.6.1]).

Since the jumps of $A_{j,N}(t)$ and $S_{j,N}(t)$ are disjoint, keeping in mind (3.8), for the predictable quadratic characteristic of $M_{j,N}(t)$ we obtain

$$\begin{aligned}\langle M_{j,N}\rangle_t &= \big\langle A_{j,N} - A^p_{j,N}\big\rangle_t + \big\langle S_{j,N} - S^p_{j,N}\big\rangle_t \\ &= A^p_{j,N}(t) + \int_0^t \big[1 - \Delta S^p_{j,N}(s)\big]\,\mathrm{d}S^p_{j,N}(s) \leqslant A^p_{j,N}(t) + S^p_{j,N}(t), \qquad (3.9)\end{aligned}$$

where $\Delta S^p_{j,N}(t)$ denotes the jump of $S^p_{j,N}(t)$ in point t (see, e.g., Liptser and Shiryayev [23, chapter 18]).

4. Asymptotic properties of normalized queue-length process

In this section we study the queue-length process under appropriate normalization, dividing the characteristics of the process by N.

The process

$$q_j^N(t) = \frac{1}{N}\,Q_j^N(t) \qquad (4.1)$$

is called *normalized* queue-length process. The aim of this section is to calculate the limit of $q_{j,N}(t)$, $j = 1, 2, \ldots, k$, as $N \to \infty$. Along with the process $q_j^N(t)$ we shall consider the following normalized process:

$$x_{j,N}(t) = \frac{1}{N}\,X_{j,N}(t) \qquad (4.2)$$

which has similar properties as the process $X_{j,N}(t)$ being introduced in section 3.

Let us write the semimartingale decomposition for $x_{j,N}(t)$. From (3.7) we have

$$x_{j,N}(t) = \frac{1}{N}\,A^p_{j,N}(t) - \frac{1}{N}\,S^p_{j,N}(t) + m_{j,N}(t) \qquad (4.3)$$

with the local square integrable martingale

$$m_{j,N}(t) = \frac{1}{N}\,M_{j,N}(t).$$

From (3.9) for the predictable quadratic characteristic of that martingale we have

$$\langle m_{j,N}\rangle_t = \frac{1}{N^2}\,\langle M_{j,N}\rangle_t \leqslant \frac{1}{N^2}\,A^p_{j,N}(t) + \frac{1}{N^2}\,S^p_{j,N}(t). \qquad (4.4)$$

Equations (4.1)–(4.3) permit us to write

$$q_j^N(t) = \frac{1}{N}\,A^p_{j,N}(t) - \frac{1}{N}\,S^p_{j,N}(t) + m_{j,N}(t) + \Psi_t(x_{j,N}). \qquad (4.5)$$

Note that

$$q_j^N(t) = x_{j,N}(t) + \Psi_t(x_{j,N}) = \Phi_t(x_{j,N}), \qquad (4.6)$$

where $\Phi_t(X) = X(t) + \Psi_t(X)$, $X(t)$ is a function from Skorokhod space $\mathbf{D}$.

In view of (4.6) and (3.8), the relation (4.3) can be rewritten as follows:

$$x_{j,N}(t) = \int_0^t \lambda_j \left[1 - \sum_{l=1}^{k} \Phi_s(x_{l,N})\right] \mathrm{d}s - \frac{1}{N} S_{j,N}^p(t) + m_{j,N}(t). \tag{4.7}$$

Let us prove that

$$\mathbf{P}\text{-}\lim_{N\to\infty} \frac{1}{N} S_{j,N}^p(t) = \mu_j t, \tag{4.8}$$

where **P**-lim denotes a limit in probability. For this purpose represent $N^{-1}S_{j,N}^p(t)$ in the form

$$\frac{1}{N} S_{j,N}^p(t) = \frac{1}{N}\left[S_{j,N}^p(t) - S_{j,N}(t)\right] + \frac{1}{N} S_{j,N}(t). \tag{4.9}$$

It is clear that the term

$$\frac{1}{N}\left[S_{j,N}^p(t) - S_{j,N}(t)\right] \tag{4.10}$$

is a local square integrable martingale. Its quadratic characteristic is

$$\left\langle \frac{1}{N}\left[S_{j,N}^p - S_{j,N}\right] \right\rangle_t = \frac{1}{N^2}\int_0^t \left[1 - \Delta S_{j,N}^p(s)\right] \mathrm{d}S_{j,N}^p(s) \leqslant \frac{1}{N^2} S_{j,N}^p(t). \tag{4.11}$$

According to Lenglart–Rebolledo inequality (see, e.g., Liptser and Shiryayev [24, p. 66]) we have

$$\mathbf{P}\Big\{\sup_{0\leqslant s\leqslant t} S_{j,N}^p(s) > \varepsilon N^2\Big\} \leqslant \frac{\varepsilon N^{3/2} + 1}{\varepsilon N^2} + \mathbf{P}\big\{S_{j,N}(t) > \varepsilon N^{3/2}\big\} \tag{4.12}$$

and, therefore, the quadratic characteristic (4.11) vanishes as $N \to \infty$. Next, applying the Lenglart–Rebolledo inequality once again we obtain:

$$\mathbf{P}\Big\{\sup_{0\leqslant s\leqslant t} \big|S_{j,N}^p(s) - S_{j,N}(s)\big| > \delta N\Big\} \leqslant \frac{\varepsilon}{\delta^2} + \mathbf{P}\big\{S_{j,N}^p(t) > \varepsilon N^2\big\}. \tag{4.13}$$

It is readily seen that both terms of right hand side of (4.12) vanish as $N \to \infty$ whenever $\varepsilon > 0$, and hence, the left side of (4.13) also vanishes as $N \to \infty$ because of arbitrariness of ε. Therefore, the term (4.10) vanishes in probability as $N \to \infty$. This means that the both terms $N^{-1}S_{j,N}^p(t)$ and $N^{-1}S_{j,N}(t)$ have the same limit in probability as $N \to \infty$. Note by the way that similar result holds for the point process $A_{j,N}(t)$:

$$\frac{1}{N}\left[A_{j,N}(t) - A_{j,N}^p(t)\right]$$

vanishes in probability as $N \to \infty$.

Let us find the limit in probability for the term $N^{-1}S_{j,N}(t)$. For this purpose take into consideration the following result by Krichagina et al. [21].

Lemma 1. Let $\mathcal{A}^N = (\mathcal{A}_t^N)_{t\geqslant 0}$, $N \geqslant 1$, be a sequence of increasing right continuous random processes with $\mathcal{A}_0^N = 0$. Let

$$\mathcal{B}_t^N = \inf\{s\colon\ \mathcal{A}_s^N > t\}, \quad t \geqslant 0,$$

where $\inf(\phi) = \infty$.

If for every t taken from dense in $\mathbf{R}_+$ set S, $\mathcal{B}_t^N \to at$ as $N \to \infty$ ($a > 0$), then, as $N \to \infty$,

$$\sup_{t\leqslant T}\left|\mathcal{A}_t^N - \frac{t}{a}\right| \to 0 \tag{4.14}$$

in probability for each $T > 0$.

Let us apply this lemma. Take $\mathcal{A}_t^N = N^{-1}S_{j,N}(t)$. Then

$$\mathcal{B}_t^N = \sum_{i=1}^{[Nt]+1} \xi_{j,i}^N,$$

where $[Nt]$ denotes the integer part of Nt. It is clear from the weak law of large numbers that, as $N \to \infty$, $\mathcal{B}_t^N \to \mu_j^{-1}t$ in probability. According to lemma, as $N \to \infty$, $N^{-1}S_{j,N}(t) \to \mu_j t$ in probability and, therefore, $N^{-1}S_{j,N}^p(t) \to \mu_j t$ in probability as well.

Next, because of the Lenglart–Rebolledo inequality and (3.8)

$$\begin{aligned}
&\mathbf{P}\Big\{\sup_{0\leqslant s\leqslant t}\big|m_{j,N}(t)\big| > \delta\Big\}\\
&\quad\leqslant \mathbf{P}\Big\{\sup_{0\leqslant s\leqslant t}\big|\big[A_{j,N}(s) + S_{j,N}(s)\big] - \big[A_{j,N}^p(s) + S_{j,N}^p(s)\big]\big| > \delta N\Big\}\\
&\quad\leqslant \frac{\varepsilon}{\delta^2} + \mathbf{P}\big\{A_{j,N}^p(t) + S_{j,N}^p(t) > \varepsilon N^2\big\} \leqslant \frac{\varepsilon}{\delta^2} + \mathbf{P}\big\{S_{j,N}^p(t) > \varepsilon N^2 - \lambda_j N t\big\}
\end{aligned}$$

and taking into account (4.12) and arbitrariness of $\varepsilon > 0$ it is readily seen that $m_{j,N}(t)$ vanishes in probability as $N \to \infty$.

Next, let

$$\mathbf{P}\text{-}\lim_{N\to\infty} x_{j,N}(t) = x_j(t).$$

It follows from (4.7) and (4.8) that $x_j(t)$ should be a solution of the set of equations

$$x_j(t) = \int_0^t \left\{\lambda_j\left[1 - \sum_{l=1}^k \Phi_s(x_l)\right] - \mu_j\right\}\mathrm{d}s, \quad j = 1, 2, \ldots, k. \tag{4.15}$$

This solution is unique because of the Lipshitz condition:

$$\sup_{t\leqslant T}\big|\Phi_t(X) - \Phi_t(Y)\big| \leqslant 2\sup_{t\leqslant T}|X_t - Y_t| \tag{4.16}$$

(for details see Kogan and Liptser [16]). Moreover, following that paper by Kogan and Liptser [16],

$$x_k(t) = q(t) = \left(1 - \rho_k^{-1}\right)\left(1 - \mathrm{e}^{-\lambda_k t}\right), \tag{4.17}$$

$$x_j(t) = (\lambda_j - \mu_j)t - \int_0^t \lambda_j q(s)\,\mathrm{d}s, \quad j = 1, 2, \ldots, k-1. \tag{4.18}$$

5. Further study of the asymptotic properties of normalized queue-length and the collapse theorem

Lemma 2. For any fixed $t > 0$ and $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbf{P}\left\{\sup_{s\leqslant t} \left|x_{j,N}(s) - x_j(s)\right| \geqslant \varepsilon\right\} = 0, \quad j = 1, 2, \ldots, k.$$

Proof. The proof of this lemma is the same as the respective proof by Kogan and Liptser [16, lemma 6.1]. The only difference in the proof is that for every $\delta > 0$,

$$\lim_{N\to\infty} \mathbf{P}\left\{\langle m_{j,N}\rangle_t \geqslant \delta\right\} = 0, \quad j = 1, 2, \ldots, k. \tag{5.1}$$

Indeed,

$$\langle m_{j,N}\rangle_t \leqslant \frac{1}{N}\lambda_j t + \frac{1}{N^2} S^p_{j,N}(t),$$

and (5.1) follows from estimation (4.12). The lemma is proved. □

From proved lemma we have the following properties for normalized queue-length:

$$\lim_{N\to\infty} \mathbf{P}\left\{\sup_{s\leqslant t} q_j^N(s) \geqslant \varepsilon\right\} = 0, \quad j = 1, 2, \ldots, k-1, \tag{5.2}$$

$$\lim_{N\to\infty} \mathbf{P}\left\{\sup_{s\leqslant t}\left|q_k^N(s) - x_k(s)\right| \geqslant \varepsilon\right\} = 0. \tag{5.3}$$

Proof of (5.2), (5.3) is based on the Lipshitz condition (4.16). For details see analogous proof in Kogan and Liptser [16, p. 46]. For our further purposes we need stronger ergodic properties than above given by lemma 2.

Now, let us prove the following theorem.

Theorem 1 (Collapse). For any $t > 0$ and $d > 0$

$$\lim_{N\to\infty} \mathbf{P}\left\{\sup_{s\leqslant t} Q_j^N(s) \geqslant d\sqrt{N}\right\} = 0, \quad j = 1, 2, \ldots, k-1.$$

Proof of this theorem is based on the following result.

Lemma 3. For any fixed $t > 0$ and $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbf{P}\Big\{\sup_{s\leqslant t} \sqrt{N} q_j^N(s) \geqslant \varepsilon\Big\} = 0, \quad j = 1, 2, \dots, k-1, \tag{5.4}$$

and

$$\lim_{N\to\infty} \mathbf{P}\Big\{\sup_{s\leqslant t} \sqrt{N} \big|q_k^N(s) - x_{k,N}(s)\big| \geqslant \varepsilon\Big\} = 0. \tag{5.5}$$

The proof of this lemma is, in turn, based on the following result by Kogan and Liptser [16].

Lemma 4. Let $X^N(t)$, $N \geqslant 1$, be a sequence of random processes such that

$$X^N(t) = -c_N \int_0^t b^N(s)\,\mathrm{d}s + G^N(t), \tag{5.6}$$

where $b^N(t) \geqslant 0$ and $G^N(t)$, $N \geqslant 1$, are the random processes with paths in **D** and $G^N(0) = 0$, $b^N(t) \geqslant 0$, $n \geqslant 1$. Let the following assumptions be fulfilled:

(1) $\{c_N\}$ is a nondecreasing sequence of non-negative numbers tending to infinity;

(2) for any fixed $t > 0$ there exists a positive constant b (possibly depending on t) such that

$$\lim_{N\to\infty} \mathbf{P}\Big\{\inf_{s\leqslant t} b^N(s) \leqslant b\Big\} = 0;$$

(3) the sequence $G^N(t)$, $N \geqslant 1$, converges weakly (in Skorohod–Lindvall topology of space **D**) to the continuous random process $G(t)$.

Then for any fixed $t > 0$ and $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbf{P}\Big\{\sup_{s\leqslant t} \Phi_s\big(X^N\big) \geqslant \varepsilon\Big\} = 0.$$

For the definition of the Skorohod–Lindvall topology of space **D** see, e.g., Stone [28], Bilingsley [7], Lindvall [22] as well as Jacod and Shiryayev [14] and Liptser and Shiryayev [24].

Proof of theorem 1. Denoting

$$X_j^N(t) = \sqrt{N}\, x_{j,N}(t), \quad j = 1, 2, \dots, k-1; \qquad X_k^N(t) = -\sqrt{N}\, x_{k,N}(t),$$

let us apply lemma 4. Each of the processes $X_j^N(t)$, $j = 1, 2, \dots, k$, can be represented in the form (5.6):

$$X_j^N(t) = -\sqrt{N} \int_0^t b_j^N(s)\,\mathrm{d}s + G_j^N(t),$$

where

$$\int_0^t b_j^N(s)\,\mathrm{d}s = -\int_0^t \lambda_j\left[1-\sum_{l=1}^{k} q_l^N(s)\right]\mathrm{d}s + \frac{1}{N}\,S_{j,N}^p(t), \quad j = 1,2,\dots,k-1, \tag{5.7}$$

$$\int_0^t b_k^N(s)\,\mathrm{d}s = \int_0^t \lambda_k\left[1-\sum_{l=1}^{k} q_l^N(s)\right]\mathrm{d}s - \frac{1}{N}\,S_{k,N}^p(t), \tag{5.8}$$

$$G_j^N(t) = \sqrt{N}\,m_{j,N}(t), \quad j = 1,2,\dots,k-1, \qquad G_k^N(t) = -\sqrt{N}\,m_{k,N}(t). \tag{5.9}$$

Taking into consideration (4.8) one can write

$$\lim_{N\to\infty}\mathbf{P}\Big\{\inf_{s\leqslant t} b_j^N(s) \geqslant \mu_j-\lambda_j\Big\} = 1, \quad j=1,2,\dots,k-1,$$

and

$$\lim_{N\to\infty}\mathbf{P}\left\{\inf_{s\leqslant t} b_k^N(s) \geqslant \lambda_k\Big[1-\sup_{s\leqslant t} x_{k,N}(s)\Big] - \lambda_k\sum_{l=1}^{k-1}\sup_{s\leqslant t} q_l^N(s) - \mu_k\right\} = 1. \tag{5.10}$$

By (5.2) and (5.3) the expression

$$\lambda_k\Big[1-\sup_{s\leqslant t} x_{k,N}(s)\Big] - \lambda_k\sum_{l=1}^{k-1}\sup_{s\leqslant t} q_l^N(s) - \mu_k$$

has a non-negative limit in probability as $N\to\infty$. If $\lambda_k < \mu_k$ this limit is equal to $(\lambda_k-\mu_k)\,\mathrm{e}^{-\lambda_k t}$. Hence, the assumptions (1) and (2) of lemma 4 are fulfilled.

Let us verify the assumption (3). Indeed, $\sqrt{N}\,m_{j,N}(t)$, $j=1,2,\dots,k-1$, and $-\sqrt{N}\,m_{k,N}(t)$ are the square integrable martingales with jumps no more than $1/\sqrt{N}$ and with predictable quadratic characteristics

$$\big\langle\sqrt{N}\,m_{j,N}\big\rangle_t = N\langle m_{j,N}\rangle_t \leqslant \int_0^t \lambda_j\left[1-\sum_{l=1}^{k} q_l^N(s)\right]\mathrm{d}s + \frac{1}{N}\,S_{j,N}^p(t).$$

By (4.8), as $N\to\infty$, the term $N^{-1}S_{j,N}^p(t)$ converges in probability to $\mu_j t$, and by (5.2) and (5.3), as $N\to\infty$,

$$\int_0^t \lambda_j\left[1-\sum_{l=1}^{k} q_l^N(s)\right]\mathrm{d}s$$

converges in probability to

$$\int_0^t \lambda_j\big[1-x_k(s)\big]\,\mathrm{d}s.$$

Therefore, $\langle\sqrt{N}\, m_{j,N}\rangle_t = N\langle m_{j,N}\rangle_t$ tends in probability to a limit which is not greater than

$$\int_0^t \{\lambda_j[1 - x_k(s)] + \mu_j\}\mathrm{d}s$$

as $N \to \infty$ for any fixed $t > 0$. According to Liptser and Shiryayev [24, theorem 5.5.4] as $N \to \infty$, $G_j^N(t)$ converges in Skorohod–Lindvall topology to a continuous Gaussian martingale $G_j(t)$, $j = 1, 2, \dots, k$. The relation (5.4) follows immediately from lemma 4 since

$$\sqrt{N}\, q_j^N(t) = \Phi_t\big(X_j^N\big), \quad j = 1, 2, \dots, k-1.$$

The relation (5.5) is also implied by lemma 4 by virtue of

$$\begin{aligned}\sup_{s\leqslant t}\sqrt{N}\,\big|q_k^N(s) - x_{k,N}(s)\big| &= \sup_{s\leqslant t}\sqrt{N}\,\Big|\Big[x_{k,N}(s) - \inf_{s\leqslant t} x_{k,N}(s)\Big] - x_{k,N}(t)\Big| \\ &= \sup_{s\leqslant t} X_k^N(t)\end{aligned}$$

and the obvious estimation

$$\sup_{s\leqslant t} X_k^N(t) \leqslant \sup_{s\leqslant t}\Phi_s\big(X_k^N\big).$$

The proof of lemma is completed.

The statement of theorem 1 follows immediately from relation (5.4). □

6. A stability problem

In this section we prove a statement on stability, that is existence of the limiting nonstationary probability at satellite nodes. That proof of stability extends the similar proof of Borovkov [8]. Below we prove the existence of the limiting probabilities $\lim_{N\to\infty}\mathbf{P}\{Q_j^N(t) = l\}$, $l \geqslant 1$, only. Proof for existence of limiting probabilities for the random time moments such as the last arrival or departure immediatelly before time t is analogous and can be provided by developing the similar proof of Borovkov [8] and using the same idea of proof of theorem 2 given below.

Theorem 2 (Statement on stability). Under assumption that $\{\xi_{j,i}^N\}_{i\geqslant 1}$ is a strictly stationary and ergodic sequence of random variables, there exist $\lim_{N\to\infty}\mathbf{P}\{Q_j^N(t) = l\}$, $l = 0, 1, \dots,$ for all $t > 0$.

Proof. It follows from (4.17) and (5.2), (5.3) that

$$\lim_{N\to\infty}\mathbf{P}\Big\{\inf_{s\leqslant t}\big|q_0^N(s) + 1 - x_k(s)\big| \geqslant \varepsilon\Big\} = 0, \tag{6.1}$$

where

$$q_0^N(t) = \frac{1}{N} Q_0^N(t)$$

is a normalized queue-length at the hub in time t, and

$$\lim_{N\to\infty} \mathbf{P}\Big\{\sup_{s\leqslant t} q_0^N(s) \leqslant 1\Big\} = 1. \tag{6.2}$$

Let us first prove that

$$\mathbf{P}\Big\{\lim_{N\to\infty} \sup_{s\leqslant t}\big[A_{j,N}(s) - S_{j,N}(s)\big] < \infty\Big\} = 1. \tag{6.3}$$

It follows from (6.2) that $A_{j,N}(t) \leqslant \pi_j^N(t)$ with probability 1 for all t, where π_j^N is the Poisson process with rate $\lambda_j N$ given on the same probability space as the point process $A_{j,N}(t)$. Therefore, it is sufficient to prove that

$$\mathbf{P}\Big\{\lim_{N\to\infty} \sup_{s\leqslant t}\big[\pi_j^N(s) - S_{j,N}(s)\big] < \infty\Big\} = 1. \tag{6.4}$$

According to assumption of ergodicity and stationarity of the sequence $\{\xi_{j,i}^N\}_{i\geqslant 1}$ we have

$$\lim_{i\to\infty} \frac{x_{j,i}^N}{i} = \frac{1}{\mu_j N} \quad (\mathbf{P}\text{-a.s.}), \quad x_{j,n}^N = \sum_{i=1}^n \xi_{j,i}^N.$$

Therefore,

$$\lim_{N\to\infty} \frac{S_{j,N}(t)}{Nt} = \mu_j \quad (\mathbf{P}\text{-a.s.}).$$

It is also well known that

$$\lim_{N\to\infty} \frac{\pi_j^N(t)}{Nt} = \lambda_j \quad (\mathbf{P}\text{-a.s.}).$$

Thereby,

$$\lim_{N\to\infty} \frac{\pi_j^N(t) - S_{j,N}(t)}{Nt} = \lambda_j - \mu_j \quad (\mathbf{P}\text{-a.s.}).$$

Since $\rho_j < 1$ implies $\lambda_j - \mu_j < 0$, it holds that

$$\lim_{N\to\infty} \big[\pi_j^N(t) - S_{j,N}(t)\big] = -\infty.$$

Therefore, (6.4) as well as (6.3) follows by the fact that the process $\pi_j^N(t) - S_{j,N}(t)$ has path from Skorohod space **D**.

Next, let

$$\widetilde{Q}_j^N(t) = \pi_j^N(t) - D_{j,N}(t),$$

where $D_{j,N}(t)$ is defined in (2.1). Then, according to Skorohod reflection principle (see section 3)

$$\widetilde{Q}_j^N(t) = \pi_j^N(t) - S_{j,N}(t) - \inf_{s \leqslant t}\big[\pi_j^N(t) - S_{j,N}(t)\big]. \tag{6.5}$$

Under assumptions of this statement, $\pi_j^N(t) - S_{j,N}(t)$ is the process with strictly stationary and ergodic increments, and it is obvious follows from (6.5) that

$$\widetilde{Q}_j^N(t) = \sup_{s \leqslant t}\big[\big(\pi_j^N(t) - S_{j,N}(t)\big) - \big(\pi_j^N(s) - S_{j,N}(s)\big)\big] \tag{6.6}$$

and because of stationarity of increments $\pi_j^N(t) - S_{j,N}(t)$ one can conclude that

$$\sup_{s \leqslant t}\big[\big(\pi_j^N(t) - S_{j,N}(t)\big) - \big(\pi_j^N(s) - S_{j,N}(s)\big)\big] \stackrel{\mathrm{d}}{=} \sup_{s \leqslant t}\big[\pi_j^N(s) - S_{j,N}(s)\big]. \tag{6.7}$$

Hence, in view of (6.6) and (6.7) we have

$$\widetilde{Q}_j^N(t) \stackrel{\mathrm{d}}{=} \sup_{s \leqslant t}\big[\pi_j^N(s) - S_{j,N}(s)\big]. \tag{6.8}$$

Therefore, because of $A_{j,N}(t) \leqslant \pi_j^N(t)$ we obtain

$$\begin{aligned}\lim_{N\to\infty} \mathbf{P}\big\{Q_j^N(t) = l\big\} &\geqslant \lim_{N\to\infty} \mathbf{P}\big\{\widetilde{Q}_j^N(t) = l\big\} \\ &= \lim_{N\to\infty} \mathbf{P}\Big\{\sup_{s \leqslant t}\big[\pi_j^N(s) - S_{j,N}(s)\big] = l\Big\}.\end{aligned} \tag{6.9}$$

Further, keeping in mind (6.1) let us introduce the Poisson process $\Pi_j^N(z)$ with rate $\lambda_j N\,\mathrm{e}^{-\lambda_k t}$. (We use the argument z for the process instead of usual t because the parameter of this process depends on variable t introduced earlier.) Assuming that the both processes $A_{j,N}(z)$ and $\Pi_j^N(z)$ are given on the same probability space, one can write $A_{j,N}(z) \geqslant \Pi_j^N(z)$ for all $z > 0$, and since $\lambda_j\,\mathrm{e}^{-\lambda_k t} < \mu_j$, analogously to (6.9) of the previous case one can prove that

$$\lim_{N\to\infty} \mathbf{P}\big\{Q_j^N(t) = l\big\} \leqslant \lim_{N\to\infty} \mathbf{P}\Big\{\sup_{s \leqslant t}\big[\Pi_j^N(s) - S_{j,N}(s)\big] = l\Big\}. \tag{6.10}$$

Inequalities (6.9) and (6.10) allow us to write:

$$\begin{aligned}\lim_{N\to\infty} \mathbf{P}\big\{Q_j^N(t) = l\big\} &= \alpha(t) \lim_{N\to\infty} \mathbf{P}\Big\{\sup_{s \leqslant t}\big[\pi_j^N(s) - S_{j,N}(s)\big] = l\Big\} \\ &\quad + \big[1 - \alpha(t)\big] \lim_{N\to\infty} \mathbf{P}\Big\{\sup_{s \leqslant t}\big[\Pi_j^N(s) - S_{j,N}(s)\big] = l\Big\},\end{aligned}$$

where $\alpha(t) \leqslant 1$ for all $t > 0$. This means that the limiting nonstationary queue-length distribution for satellite nodes exists, and theorem 2 is, therefore, proved. □

7. Main results for nonstationary probabilities at the first $k-1$ satellite nodes. I. Up- and down-crossing approach

The main purpose of this section is to prove the following:

Theorem 3. Let

$$s^*_{j,N} = \inf\{s > 0\colon\ S_{j,N}(s) = S_{j,N}(t)\}, \quad j = 1, 2, \ldots, k-1.$$

Then,

$$\lim_{N\to\infty} \mathbf{P}\{Q_j^N(s^*_{j,N}-) = 0\} = 1 - \rho_j(t), \tag{7.1}$$

$$\lim_{N\to\infty} \mathbf{P}\{Q_j^N(t) = l\} = \frac{1}{\rho_j(t)} \lim_{N\to\infty} \mathbf{P}\{Q_j^N(s^*_{j,N}-) = l+1\}, \quad l = 0, 1, \ldots, \tag{7.2}$$

where $\rho_j(t) = \rho_j[1 - q(t)]$, and the function $q(t)$ is defined in (4.17).

Proof. Denote

$$S^*_{j,N}(u) = \inf\{s > 0\colon\ S_{j,N}(s) = S_{j,N}(u)\}, \quad j = 1, 2, \ldots, k-1.$$

Then

$$s^*_{j,N} = S^*_{j,N}(t).$$

Prove, first, the following representation:

$$\lim_{N\to\infty} \frac{1}{\mu_j N} \mathbf{E} \int_0^t \mathbf{I}\{Q_j^N(s-) = l\} \mathrm{d}S_{j,N}(s) = \lim_{N\to\infty} \int_0^t \mathbf{P}\{Q_j^N(S^*_{j,N}(s)-) = l\} \mathrm{d}s,$$
$$l = 0, 1, \ldots, \tag{7.3}$$

where $S^*_{j,N}(s)-$ implies $\lim_{\varepsilon\to 0}[S^*_{j,N}(s) - \varepsilon]$.

Take the small semi-interval $\mathcal{U} = (u, u + \mathrm{d}u]$. As earlier in section 2 let $x^N_{j,n} = \sum_{i=1}^n \xi^N_{j,i}$. Denote

$$n_1 = \min\{n\colon\ x^N_{j,n} \in \mathcal{U}\},$$
$$n_2 = \max\{n\colon\ x^N_{j,n} \in \mathcal{U}\}.$$

Then, keeping in mind that $N^{-1}[S_{j,N}(u + \mathrm{d}u) - S_{j,N}(u)] \xrightarrow{\mathbf{P}} \mu_j \,\mathrm{d}u$ (see lemma 1), accordind to theorem by Cesaro (see, e.g., Polya and Szego [27]) we obtain

$$\lim_{N\to\infty} \mathbf{P}\{Q_j^N(S^*_{j,N}(u + \mathrm{d}u)-) = l\}$$
$$= \mathbf{P}\text{-}\lim_{N\to\infty} \frac{1}{[S_{j,N}(u + \mathrm{d}u) - S_{j,N}(u)]} \sum_{i=n_1}^{n_2} \mathbf{P}\{Q_j^N(x^N_{j,i}-) = l\}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \frac{1}{\mu_j N\,\mathrm{d}u}\,\mathbf{E}\sum_{i=n_1}^{n_2} \mathbf{P}\bigl\{Q_j^N\bigl(x_{j,i}^N-\bigr)=l\bigr\}\\
&= \lim_{N\to\infty} \frac{1}{\mu_j N\,\mathrm{d}u}\,\mathbf{E}\sum_{i=n_1}^{n_2} \mathbf{I}\bigl\{Q_j^N\bigl(x_{j,i}^N-\bigr)=l\bigr\}\\
&= \frac{1}{\mathrm{d}u}\lim_{N\to\infty} \frac{1}{\mu_j N}\,\mathbf{E}\bigl(\mathbf{I}\bigl\{Q_j^N\bigl(u^*-\bigr)=l\bigr\}\bigl[S_{j,N}(u+\mathrm{d}u)-S_{j,N}(u)\bigr]\bigr),\\
&\quad l=0,1,\ldots, \qquad (7.4)
\end{aligned}
$$

where $u^* \in \mathcal{U}$. Thus, (7.3) follows.

Analogously, denoting

$$
A_{j,N}^*(u) = \inf\bigl\{s>0\colon\ A_{j,N}(s)=A_{j,N}(u)\bigr\},\quad j=1,2,\ldots,k-1,
$$

one can prove the following representation:

$$
\lim_{N\to\infty}\frac{1}{\lambda_j N}\,\mathbf{E}\int_0^t \mathbf{I}\bigl\{Q_j^N(s-)=l\bigr\}\mathrm{d}A_{j,N}(s) = \lim_{N\to\infty}\int_0^t \mathbf{P}\bigl\{Q_j^N\bigl(A_{j,N}^*(s)-\bigr)=l\bigr\}\mathrm{d}s,
$$
$$
l=0,1,\ldots, \qquad (7.5)
$$

where $A_{j,N}^*(s)-$ implies $\lim_{\varepsilon\to 0}[A_{j,N}^*(s)-\varepsilon]$.

Next, for all $l=1,2,\ldots$, $j=1,2,\ldots,k-1$ and $t>0$ we have the following relationship connecting the number of up- and down-crossings:

$$
\sum_{i=1}^{A_{j,N}(t)} \mathbf{I}\bigl\{Q_j^N\bigl(t_{j,i}^N-\bigr)=l-1\bigr\} = \sum_{i=1}^{S_{j,N}(t)} \mathbf{I}\bigl\{Q_j^N\bigl(x_{j,i}^N-\bigr)=l\bigr\} + \mathbf{I}\bigl\{Q_j^N(t)\geqslant l\bigr\}, \qquad (7.6)
$$

where $\{\tau_{j,i}^N\}_{i\geqslant 1}$ is the sequence of interarrival times of satellite node j, and

$$
t_{j,n}^N = \sum_{i=1}^{n} \tau_{j,i}^N.
$$

It follows from (7.3), (7.5) and (7.6) that

$$
\lim_{N\to\infty}\frac{1}{N}\,\mathbf{E}\int_0^t \mathbf{I}\bigl\{Q_j^N(s-)=l-1\bigr\}\mathrm{d}A_{j,N}(s) = \lim_{N\to\infty}\frac{1}{N}\,\mathbf{E}\int_0^t \mathbf{I}\bigl\{Q_j^N(s-)=l\bigr\}\mathrm{d}S_{j,N}(s),
$$
$$
l=1,2,\ldots. \qquad (7.7)
$$

Rewrite the left-hand side of (7.7) as

$$
\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\,\mathbf{E}\int_0^t \mathbf{I}\bigl\{Q_j^N(s-)=l-1\bigr\}\mathrm{d}A_{j,N}(s)\\
&\quad= \lim_{N\to\infty}\frac{1}{N}\,\mathbf{E}\int_0^t \mathbf{I}\bigl\{Q_j^N(s-)=l-1\bigr\}\mathrm{d}A_{j,N}^p(s)\\
&\qquad+ \lim_{N\to\infty}\frac{1}{N}\,\mathbf{E}\int_0^t \mathbf{I}\bigl\{Q_j^N(s-)=l-1\bigr\}\mathrm{d}\bigl[A_{j,N}(s)-A_{j,N}^p(s)\bigr]. \qquad (7.8)
\end{aligned}
$$

Note, that as $N \to \infty$,

$$\frac{1}{N}\left[A_{j,N}(s) - A^p_{j,N}(s)\right]$$

vanishes in probability (see section 4) and, hence, the term

$$\frac{1}{N}\mathbf{E}\int_0^t \mathbf{I}\left\{Q_j^N(s-) = l-1\right\}\mathrm{d}\left[A_{j,N}(s) - A^p_{j,N}(s)\right]$$

vanishes as $N \to \infty$. Therefore, from (7.8) we obtain

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\mathbf{E}\int_0^t \mathbf{I}\left\{Q_j^N(s-) = l-1\right\}\mathrm{d}A_{j,N}(s)\\ &\quad= \lim_{N\to\infty}\frac{1}{N}\mathbf{E}\int_0^t \mathbf{I}\left\{Q_j^N(s-) = l-1\right\}\mathrm{d}A^p_{j,N}(s).\end{aligned} \tag{7.9}$$

Taking into consideration that

$$\lim_{N\to\infty}\frac{1}{N}A^p_{j,N}(t) = \lambda_j\int_0^t \left[1 - q(s)\right]\mathrm{d}s \tag{7.10}$$

(see (3.8), (4.17) and (5.3)) and substituting it for (7.9) we obtain

$$\lim_{N\to\infty}\frac{1}{N}\mathbf{E}\int_0^t \mathbf{I}\left\{Q_j^N(s-) = l-1\right\}\mathrm{d}A_{j,N}(s) = \lambda_j\int_0^t \mathbf{P}\left\{Q_j^N(s) = l-1\right\}\left[1 - q(s)\right]\mathrm{d}s. \tag{7.11}$$

Therefore, substituting (7.11) for (7.7) we obtain

$$\lim_{N\to\infty}\lambda_j\int_0^t \mathbf{P}\left\{Q_j^N(s) = l-1\right\}\left[1-q(s)\right]\mathrm{d}s = \lim_{N\to\infty}\frac{1}{N}\mathbf{E}\int_0^t \mathbf{I}\left\{Q_j^N(s-) = l\right\}\mathrm{d}S_{j,N}(s),$$
$$l = 1, 2, \ldots,$$

and taking into consideration (7.3) yields

$$\lim_{N\to\infty}\rho_j\int_0^t \mathbf{P}\left\{Q_j^N(s) = l-1\right\}\left[1-q(s)\right]\mathrm{d}s = \lim_{N\to\infty}\int_0^t \mathbf{P}\left\{Q_j^N\left(S^*_{j,N}(s)-\right) = l\right\}\mathrm{d}s,$$
$$l = 1, 2, \ldots.$$

Thus, relation (7.2) of theorem follows. Relation (7.1) follows from the condition

$$\sum_{l=0}^{\infty}\lim_{N\to\infty}\mathbf{P}\left\{Q_j^N(t) = l\right\} = 1.$$

Theorem is proved. □

Special case (Kogan and Liptser [16]). If the sequence $\{\xi^N_{j,i}\}_{i\geqslant 1}$ consists of independent identically and exponentially distributed random variables, then we have

$$\lim_{N\to\infty}\mathbf{P}\left\{Q_j^N(t) = l\right\} = \left[1 - \rho_j(t)\right]\rho_j^l(t), \quad l = 0, 1, \ldots.$$

Proof. Indeed, keeping in mind that

$$\lim_{N\to\infty}\frac{1}{N}\mathbf{E}\int_0^t \mathbf{I}\big\{Q_j^N(s-)=l\big\}\mathrm{d}S_{j,N}(s)=\lim_{N\to\infty}\frac{1}{N}\mathbf{E}\int_0^t \mathbf{I}\big\{Q_j^N(s-)=l\big\}\mathrm{d}S^p_{j,N}(s),\quad l=0,1,\ldots,$$

limit relation (7.3) can be rewritten as follows:

$$\lim_{N\to\infty}\int_0^t \mathbf{P}\big\{Q_j^N\big(S^*_{j,N}(s)-\big)=l\big\}\mathrm{d}s=\lim_{N\to\infty}\frac{1}{\mu_j N}\mathbf{E}\int_0^t \mathbf{I}\big\{Q_j^N(s-)=l\big\}\mathrm{d}S^p_{j,N}(s),\quad l=0,1,\ldots.$$

Taking into consideration that in the given case $S^p_{j,N}(t)=\mu_j Nt$, we have

$$\lim_{N\to\infty}\mathbf{P}\big\{Q_j^N\big(s^*_{j,N}-\big)=l\big\}=\lim_{N\to\infty}\mathbf{P}\big\{Q_j^N(t)=l\big\},\quad l=0,1,\ldots. \tag{7.12}$$

The property of (7.12) means that *departures see time averages*. Substituting (7.12) for (7.2) we obtain the desired result. The statement is proved. □

8. Main results for nonstationary probabilities at the first $k-1$ satellite nodes. II. Martingale approach

8.1. The Doob–Meyer decomposition for the indicators of queue-length processes

In this section we derive the Doob–Meyer semimartingale decomposition for the process

$$I^N_{j,l}(t)=\mathbf{I}\big\{Q_{j,N}(t)=l\big\},\quad j=1,2,\ldots,k-1;\ l=0,1,\ldots,N.$$

Since a technique of this section is standard and well known, see, e.g., Kogan and Liptser [16], Kogan et al. [17] we shall try to be possibly brief, omitting the technical details.

Taking into consideration that $I^N_{j,-1}(t)=\mathbf{I}\{Q_j^N(t)=-1\}=0$ and denoting the jump of process $I^N_{j,l}(t)$ by $\Delta I^N_{j,l}(t)$ as well as the jumps of the processes $A_{j,N}(t)$, $S_{j,N}(t)$ and $Q_j^N(t)$ by $\Delta A_{j,N}(t)$, $\Delta S_{j,N}(t)$ and $\Delta Q_j^N(t)$, respectively, we have

$$\begin{aligned}\Delta I^N_{j,l}(t)&=\mathbf{I}\big\{Q_j^N(t-)+\Delta Q_j^N(t)=l\big\}-\mathbf{I}\big\{Q_j^N(t-)=l\big\}\\&=I^N_{j,l-1}(t-)\Delta A_{j,N}(t)+I^N_{j,l+1}(t-)\Delta S_{j,N}(t)\mathbf{I}\big\{Q_j^N(t-)>0\big\}\\&\quad+I^N_{j,l}(t-)\big[1-\Delta A_{j,N}(t)\big]\big[1-\Delta S_{j,N}(t)\mathbf{I}\big\{Q_j^N(t-)>0\big\}\big]-I^N_{j,l}(t-).\end{aligned}\tag{8.1}$$

Since

$$\sum_{s\leqslant t}\Delta I^N_{j,l}(s)=I^N_{j,l}(t)-I^N_{j,l}(0),$$

and the jumps of $A_{j,N}(t)$ and $S_{j,N}(t)$ are disjoint, the process $I^N_{j,l}(t)$ can be represented as follows:

$$I^N_{j,l}(t) = I^N_{j,l}(0) + \int_0^t \big[I^N_{j,l-1}(s-) - I^N_{j,l}(s-)\big]\,\mathrm{d}A_{j,N}(s) \\ + \int_0^t \big[I^N_{j,l+1}(s-) - I^N_{j,l}(s-)\big]\mathbf{I}\big\{Q^N_j(s-) > 0\big\}\mathrm{d}S_{j,N}(s). \qquad (8.2)$$

Next, using Doob–Meyer decomposition, from (8.2) we obtain

$$I^N_{j,l}(t) = I^N_{j,l}(0) + \int_0^t \big[I^N_{j,l-1}(s-) - I^N_{j,l}(s-)\big]\,\mathrm{d}A^p_{j,N}(s) \\ + \int_0^t \big[I^N_{j,l+1}(s-) - I^N_{j,l}(s-)\big]\mathbf{I}\big\{Q^N_j(s-) > 0\big\}\mathrm{d}S^p_{j,N}(s) + M^N_{j,l}(t), \qquad (8.3)$$

with the local square integrable martingale

$$M^N_{j,l}(t) = \int_0^t \big[I^N_{j,l-1}(s-) - I^N_{j,l}(s-)\big]\,\mathrm{d}\big[A_{j,N}(s) - A^p_{j,N}(s)\big] \\ + \int_0^t \big[I^N_{j,l+1}(s-) - I^N_{j,l}(s-)\big]\mathbf{I}\big\{Q^N_j(s-) > 0\big\}\mathrm{d}\big[S_{j,N}(s) - S^p_{j,N}(s)\big]. \qquad (8.4)$$

For the predictable quadratic characteristic of that martingale we have

$$\big\langle M^N_{j,l}\big\rangle_t = \int_0^t \big[I^N_{j,l-1}(s-) - I^N_{j,l}(s-)\big]\,\mathrm{d}A^p_{j,N}(s) \\ + \int_0^t \big[I^N_{j,l+1}(s-) - I^N_{j,l}(s-)\big]\mathbf{I}\big\{Q^N_j(s-) > 0\big\}\big[1 - \Delta S^p_{j,N}(s)\big]\mathrm{d}S^p_{j,N}(s) \\ \leqslant \int_0^t \big[I^N_{j,l-1}(s-) - I^N_{j,l}(s-)\big]\,\mathrm{d}A^p_{j,N}(s) + \int_0^t \big[I^N_{j,l+1}(s-) - I^N_{j,l}(s-)\big]\,\mathrm{d}S^p_{j,N}(s). \qquad (8.5)$$

8.2. *Proof of theorem for nonstationary probabilities*

This section suggests another proof of theorem 3. Namely, we prove the following statement.

Lemma 5. For any fixed $t > 0$ and any smooth function $f(s)$, $0 \leqslant s \leqslant t$,

$$\lim_{N\to\infty} \frac{1}{N}\,\mathbf{E}\int_0^t f(s)\mathbf{I}\big\{Q^N_j(s-) = 0\big\}\mathrm{d}S_{j,N}(s) = \mu_j \int_0^t f(s)\big[1 - \rho_j(s)\big]\mathrm{d}s, \\ j = 1, 2, \ldots, k-1, \qquad (8.6)$$

$$\lim_{N\to\infty} \frac{1}{N}\,\mathbf{E}\int_0^t f(s)\mathbf{I}\big\{Q^N_j(s-) = l+1\big\}\mathrm{d}S_{j,N}(s) \\ = \lim_{N\to\infty} \lambda_j \int_0^t f(s)\big[1 - q(s)\big]\mathbf{P}\big\{Q^N_j(s) = l\big\}\mathrm{d}s, \\ j = 1, 2, \ldots, k-1;\ l = 0, 1, \ldots. \qquad (8.7)$$

Note. By virtue of (7.3), lemma 5 implies theorem 3.

Proof of lemma 5. It is sufficient to prove (8.7), because according to note, (8.6) follows from (8.7). Let us prove (8.7). For this purpose use the Doob–Meyer semimartingale decomposition (8.3) for $I_{j,l}^N(t) = \mathbf{I}\{Q_j^N(t) = l\}$. Consider first the special case of $l = 0$ because representation for that case differs from the case $l \geqslant 1$. We have

$$\begin{aligned}\frac{1}{N}\int_0^t f(s)\,\mathrm{d}I_{j,0}^N(s) = &-\int_0^t \lambda_j f(s)\left[1-\sum_{i=1}^k q_i^N(s)\right] I_{j,0}^N(s-)\,\mathrm{d}s \\ &+\frac{1}{N}\int_0^t f(s)\big[I_{j,1}^N(s-) - I_{j,0}^N(s-)\big]\mathbf{I}\big\{Q_j^N(s-) > 0\big\}\mathrm{d}S_{j,N}^p(s) \\ &+\frac{1}{N}\int_0^t f(s)\,\mathrm{d}M_{j,0}^N(s). \end{aligned}\tag{8.8}$$

From (8.8) we obtain the following relation:

$$\begin{aligned}\frac{1}{N}\int_0^t f(s) I_{j,1}^N(s-)\,\mathrm{d}S_{j,N}^p(s) = &\int_0^t \lambda_j f(s)\left[1-\sum_{i=1}^k q_i^N(s)\right] I_{j,0}^N(s-)\,\mathrm{d}s \\ &-\frac{1}{N}\int_0^t f(s)\,\mathrm{d}M_{j,0}^N(s) - \frac{1}{N}\int_0^t f(s)\,\mathrm{d}I_{j,0}(s) \\ = &K_1^N + K_2^N + K_3^N. \end{aligned}\tag{8.9}$$

As $N \to \infty$, let us consider asymptotic behavior for those three terms. By (5.2), (5.3) and Lebesgue theorem on bounded convergence we have

$$\begin{aligned}\mathbf{P}\text{-}\lim_{N\to\infty} K_1^N &= \mathbf{E}\lim_{N\to\infty}\lambda_j\int_0^t f(s)\big[1-q(s)\big] I_{j,0}^N(s-)\,\mathrm{d}s \\ &= \lim_{N\to\infty}\lambda_j\int_0^t f(s)\big[1-q(s)\big]\mathbf{P}\big\{Q_j^N(s) = 0\big\}\mathrm{d}s. \end{aligned}\tag{8.10}$$

Next, by (8.5),

$$\begin{aligned}\mathbf{E}\big(K_2^N\big)^2 &= \frac{1}{N^2}\mathbf{E}\int_0^t f^2(s)\,\mathrm{d}\big\langle M_{j,0}^N\big\rangle_s \\ &\leqslant \frac{1}{N}\mathbf{E}\int_0^t \lambda_j f^2(s)\left[1-\sum_{i=1}^k q_i^N(s)\right] I_{j,1}^N(s-)\,\mathrm{d}s + \frac{1}{N^2}\mathbf{E}\int_0^t f^2(s)\,\mathrm{d}S_{j,N}^p(s). \end{aligned}$$

Taking into account (5.2), (5.3), it is readily seen that, as $N \to \infty$, the term

$$\frac{1}{N}\mathbf{E}\int_0^t \lambda_j f^2(s)\left[1-\sum_{i=1}^k q_i^N(s)\right] I_{j,1}^N(s-)\,\mathrm{d}s$$

vanishes. Using a partial integration for the term

$$\frac{1}{N^2}\mathbf{E}\int_0^t f^2(s)\,\mathrm{d}S^p_{j,N}(s)$$

we obtain

$$\frac{1}{N^2}\mathbf{E}\int_0^t f^2(s)\,\mathrm{d}S^p_{j,N}(s) = \frac{1}{N^2}\mathbf{E}\left[f^2(t)S^p_{j,N}(t) - 2\int_0^t f(s)f'(s)S^p_{j,N}(s)\,\mathrm{d}s\right],$$

and this term vanishes as well, and thus,

$$\mathbf{P}\text{-}\lim_{N\to\infty} K_2^N = 0. \tag{8.11}$$

Next, by partial integration

$$\frac{1}{N}\int_0^t f(s)\,\mathrm{d}I^N_{j,0}(s) = \frac{1}{N}\left[f(t)I^N_{j,0}(t) - \int_0^t f'(s)I^N_{j,0}(s)\,\mathrm{d}s\right],$$

and, therefore,

$$\mathbf{P}\text{-}\lim_{N\to\infty} K_3^N = 0. \tag{8.12}$$

Combining (8.10)–(8.12), we can see that the limit in probability for the right-hand side of (8.9), as $N\to\infty$, is equal to

$$\lim_{N\to\infty}\lambda_j\int_0^t f(s)\mathbf{P}\big\{Q_j^N(s)=0\big\}\mathrm{d}s.$$

Keeping in mind, that both $N^{-1}S_{j,N}(t)$ and $N^{-1}S^p_{j,N}(t)$ have the same limit in probability for the left-hand side of (8.9) we obtain

$$\lim_{N\to\infty}\frac{1}{N}\mathbf{E}\int_0^t f(s)I^N_{j,1}(s-)\,\mathrm{d}S^p_{j,N}(s) = \lim_{N\to\infty}\frac{1}{N}\mathbf{E}\int_0^t f(s)I^N_{j,1}(s-)\,\mathrm{d}S_{j,N}(s),$$

and the statement (8.7) under $l=0$ follows.

Let us now prove the statement (8.7) under $l\geqslant 1$. For this purpose use the semimartingale decomposition (8.3) for $I^N_{j,l}(t)=\mathbf{I}\{Q_j^N(t)=l\}$, $l\geqslant 1$. We obtain

$$\begin{aligned}\frac{1}{N}\int_0^t f(s)\,\mathrm{d}I^N_{j,l}(s) &= \int_0^t \lambda_j f(s)\left[1-\sum_{i=1}^k q_i^N(s)\right]\big[I^N_{j,l-1}(s-)-I^N_{j,l}(s-)\big]\mathrm{d}s\\ &\quad+\frac{1}{N}\int_0^t f(s)\big[I^N_{j,l+1}(s-)-I^N_{j,l}(s-)\big]\mathbf{I}\big\{Q_j^N(s-)>0\big\}\mathrm{d}S^p_{j,N}(s)\\ &\quad+\frac{1}{N}\int_0^t f(s)\,\mathrm{d}M^N_{j,l}(s).\end{aligned} \tag{8.13}$$

From (8.13) we can readily obtain the relation

$$\begin{aligned}
&\frac{1}{N}\int_0^t f(s)\big[I_{j,l+1}^N(s-)-I_{j,l}^N(s-)\big]\,\mathrm{d}S_{j,N}^p(s)\\
&\quad=-\int_0^t \lambda_j f(s)\left[1-\sum_{i=1}^k q_i^N(s)\right]\big[I_{j,l-1}^N(s-)-I_{j,l}^N(s-)\big]\,\mathrm{d}s\\
&\qquad-\frac{1}{N}\int_0^t f(s)\,\mathrm{d}M_{j,l}^N(s)-\frac{1}{N}\int_0^t f(s)\,\mathrm{d}I_{j,l}^N(s)\\
&\quad=K_1^N+K_2^N+K_3^N. \qquad (8.14)
\end{aligned}$$

As $N\to\infty$, let us consider asymptotic behavior for those three terms. By (5.2), (5.3) and Lebesgue theorem on bounded convergence we have

$$\begin{aligned}
&\mathbf{P}\text{-}\lim_{N\to\infty} K_1^N=-\mathbf{E}\lim_{N\to\infty}\lambda_j\int_0^t f(s)\big[1-q(s)\big]\big[I_{j,l-1}^N(s)-I_{j,l}^N(s)\big]\,\mathrm{d}s\\
&\quad=-\lim_{N\to\infty}\lambda_j\int_0^t f(s)\big[1-q(s)\big]\big[\mathbf{P}\big\{Q_j^N(s)=l-1\big\}-\mathbf{P}\big\{Q_j^N(s)=l\big\}\big]\,\mathrm{d}s. \qquad (8.15)
\end{aligned}$$

Next, by (8.5), as $N\to\infty$,

$$\begin{aligned}
\mathbf{E}\big(K_2^N\big)^2&=\frac{1}{N^2}\,\mathbf{E}\int_0^t f^2(s)\,\mathrm{d}\big\langle M_{j,l}^N\big\rangle_s\\
&\leqslant\frac{1}{N}\,\mathbf{E}\int_0^t \lambda_j f^2(s)\left[1-\sum_{i=1}^k q_i^N(s)\right]I_{j,l}^N(s)\,\mathrm{d}s+\frac{1}{N^2}\,\mathbf{E}\int_0^t f^2(s)\,\mathrm{d}S_{j,N}^p(s)\to 0
\end{aligned}$$

(the detailed explanation is the same as for (8.11)), i.e.,

$$\mathbf{P}\text{-}\lim_{N\to\infty} K_2^N=0. \qquad (8.16)$$

Next, by partial integration

$$\frac{1}{N}\int_0^t f(s)\,\mathrm{d}I_{j,l}^N(s)=\frac{1}{N}\left[f(t)I_{j,l}^N(t)-\int_0^t f'(s)I_{j,l}^N(s)\,\mathrm{d}s\right],$$

and therefore,

$$\mathbf{P}\text{-}\lim_{N\to\infty} K_3^N=0. \qquad (8.17)$$

Combining (8.15)–(8.17) for the **P**-limit of the right-hand side of (8.14) as $N\to\infty$ we have

$$\mathbf{P}\text{-}\lim_{N\to\infty}\sum_{i=1}^3 K_i^N=\mathbf{P}\text{-}\lim_{N\to\infty} K_1^N. \qquad (8.18)$$

Keeping in mind that both $N^{-1}S_{j,N}(t)$ and $N^{-1}S^{p}_{j,N}(t)$ have the same limit in probability for the left-hand side of (8.14) we obtain

$$\begin{aligned}&\mathbf{P}\text{-}\lim_{N\to\infty}\frac{1}{N}\int_0^t f(s)\big[I^N_{j,l+1}(s-)-I^N_{j,l}(s-)\big]\,\mathrm{d}S^p_{j,N}(s)\\&\quad=\mathbf{P}\text{-}\lim_{N\to\infty}\frac{1}{N}\int_0^t f(s)\big[I^N_{j,l+1}(s-)-I^N_{j,l}(s-)\big]\,\mathrm{d}S_{j,N}(s),\end{aligned}$$

and the statement (8.7) of this lemma follows by applying the Lebesgue theorem on bounded convergence. Lemma 5 is comletely proved. □

9. Fluid and diffusion approximations for a bottleneck node

Let us study now the moderate and heavy usage regimes for the bottleneck node. Denote

$$Y^N(t)=\sqrt{N}\,\big[q^N_k(t)-q(t)\big].$$

Theorem 4 (Diffusion approximation with reflection). Let us slightly modify one of our previous notation. Namely, assume that

$$\frac{1}{N}\,\mathbf{E}\xi^N_{k,1}=\frac{1}{\mu_{k,N}},$$

and

$$\lim_{N\to\infty}\sqrt{N}\,(\mu_{k,N}-\lambda_k)=0. \tag{9.1}$$

Then the sequence $Y^N=[Y^N(t)]_{t\geqslant 0}$, $N\geqslant 1$, converges weakly (in the Skorohod–Lindvall topology of space $\mathbf{D}$) to a non-negative diffusion process $Y=[Y(t)]_{t\geqslant 0}$ with normal reflection at zero defined by the Itô equation:

$$Y(t)=-\int_0^t\lambda_k Y(s)\,\mathrm{d}s+\sqrt{2\lambda_k}\,W(t)+\boldsymbol{\Psi}(t),$$

where $W(t)$ is a Wiener process, and $\boldsymbol{\Psi}(t)$ is the functional of the normal reflection.

Proof. Under condition (9.1) the function $q(t)=0$ for all t and, consequently,

$$\begin{aligned}Y^N(t)&=\sqrt{N}\,x_k(t)-\inf_{s\leqslant t}\sqrt{N}\,x^N_k(s)\\&=\boldsymbol{\Phi}_t\big(\sqrt{N}\,x^N_k\big)=\boldsymbol{\Phi}_t\big(U^N\big),\quad U^N=\sqrt{N}\,x^N_k(t).\end{aligned}$$

We must show that the sequence $U^N = (U_t^N)_{t\geqslant 0}$, $N \geqslant 1$, converges in Skorohod–Lindvall topology to a process $U = (U_t)_{t\geqslant 0}$ of diffusion type defined by the Itô equation

$$U_t = -\int_0^t \lambda_k \Phi_s(U)\,\mathrm{d}s + \sqrt{2\lambda_k}\,W^U(t) \tag{9.2}$$

with respect to Wiener process $W^U(t)$. In view of (4.3) and (3.8) under $j = k$ we have the following representation for U_t^N:

$$U_t^N = \lambda_k t\sqrt{N} - \int_0^t \lambda_k \left[\sqrt{N}\sum_{l=1}^{k-1} q_l^N(s) + \Phi_s\big(U^N\big)\right]\mathrm{d}s \\ - \frac{1}{\sqrt{N}}\,S_{k,N}^p(t) + \sqrt{N}\,m_{k,N}(t). \tag{9.3}$$

The jumps of U^N are no more than $1/\sqrt{N}$. By the Lipshitz condition (4.16), equation (9.2) has a unique solution. Note that by (9.1)

$$\mathbf{P}\text{-}\lim_{N\to\infty}\left[\frac{S_{k,N}^p(t)}{\sqrt{N}} - \lambda_k t\sqrt{N}\right] = 0. \tag{9.4}$$

Indeed, by (4.8) and (9.1)

$$\mathbf{E}\left|\frac{S_{k,N}^p(t)}{\sqrt{N}} - \lambda_k t\sqrt{N}\right| = \sqrt{N}\,\mathbf{E}\left|\frac{S_{k,N}^p(t)}{N} - \lambda_k t\right| = \sqrt{N}\,|\mu_{k,N} - \lambda_k t| \to 0,$$

and the convergence (9.4) follows.

Therefore, one can apply theorem 8.3.1(c) by Liptser and Shiryayev [24] checking two of its conditions:

$$\mathbf{P}\text{-}\lim_{N\to\infty}\sup_{s\leqslant t}\left|\int_0^s \lambda_k \sum_{l=1}^{k-1}\sqrt{N}\,q_l^N(u)\,\mathrm{d}u\right| = 0 \tag{9.5}$$

and

$$\mathbf{P}\text{-}\lim_{N\to\infty}\sup_{s\leqslant t}\left|\left\langle\sqrt{N}\,m_{k,N}\right\rangle_s - 2\lambda_k s\right| = 0. \tag{9.6}$$

The convergence (9.5) holds by (5.2). The convergence (9.6) holds by (5.2), (5.3), (4.4), since

$$\mathbf{P}\text{-}\lim_{N\to\infty}\left\langle\sqrt{N}\,m_{k,N}\right\rangle_t = 2\lambda_k t,$$

and moreover,

$$\mathbf{P}\text{-}\lim_{N\to\infty}\sup_{s\leqslant t}\left|\left\langle\sqrt{N}\,m_{k,N}\right\rangle_s - 2\lambda_k s\right| = 0.$$

Thus, the desired convergence holds and with the aid of continuous mapping method (see, e.g., Jacod and Shiryayev [14] or Liptser and Shiryayev [24])

$\{\Phi_t(U^N)\}_{t\geqslant 0}$ converges weakly to $\{\Phi_t(U)\}_{t\geqslant 0}$ as $N \to \infty$, i.e., $Y^N(t) = \Phi_t(U^N)$ converges weakly to

$$Y(t) = \Phi_t(U) = U(t) - \inf_{s\leqslant t} U_s.$$

Therefore,

$$Y(t) = -\int_0^t \lambda_k Y(s)\,\mathrm{d}s + \sqrt{2\lambda_k}\,W^U(t) - \inf_{s\leqslant t} U_s.$$

According to Liptser and Shiryayev [24, theorem 10.2.1] or to Anulova and Liptser [5, theorem 5] the process $Y(t)$ can be represented in the form of the Itô equation with the normal reflection:

$$Y(t) = -\int_0^t \lambda_k Y(s)\,\mathrm{d}s + \sqrt{2\lambda_k}\,W(t) + \Psi(t)$$

with the Wiener process $W(t)$ and the functional of the normal reflection $\Psi(t)$ adapted to the filtration denerated by $Y = \{Y(t)\}_{t\geqslant 0}$. Theorem 4 is proved. □

Theorem 5 (Fluid and Gaussian diffusion approximation). Let us assume that $\lambda_k > \mu_k$. Then for any fixed $t > 0$ and $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbf{P}\Big\{\sup_{s\leqslant t}\big|q_k^N(s) - q(s)\big| \geqslant \varepsilon\Big\} = 0. \tag{9.7}$$

(Recall that $q_k^N(t) = N^{-1}Q_k^N(t)$, and $q(t)$ is defined in (4.17).) The sequence $Y^N = [Y^N(t)]_{t\geqslant 0}$, $N \geqslant 1$, converges weakly (in the Skorohod–Lindvall topology of space **D**) to Gaussian diffusion process $Y = [Y(t)]_{t\geqslant 0}$ defined by the Itô equation

$$Y(t) = -\int_0^t \lambda_k Y(s)\,\mathrm{d}s + \int_0^t \sqrt{\lambda_k\big[1 - q(s)\big] + \mu_k}\,\mathrm{d}W(s)$$

with respect to a Wiener process $W(t)$.

Proof. Indeed, the statement (9.7) follows immediately from (4.17) and (5.3). Let us prove the residual part of the theorem. Along with the process Y^N let us introduce the process $V^N = [V_t^N]_{t\geqslant 0}$, where

$$V_t^N = \sqrt{N}\,\big[x_{k,N}(t) - x_k(t)\big].$$

By (4.17) it is clear that

$$\begin{aligned} Y^N(t) - V_t^N &= \sqrt{N}\,\big[q_k^N(t) - q(t)\big] - \sqrt{N}\,\big[x_{k,N}(t) - x_k(t)\big] \\ &= \sqrt{N}\,\big[q_k^N(t) - x_{k,N}(t)\big]. \end{aligned} \tag{9.8}$$

It follows from (9.8) and (5.6) that

$$\mathbf{P}\text{-}\lim_{N\to\infty} \sup_{s\leqslant t} \big|Y^N(s) - V_s^N\big| = 0. \tag{9.9}$$

Hence, by Liptser and Shiryayev [24, problem 6.2.1], both sequences Y^N and V^N converge weakly to the same limit (if that limit does exist). Let us prove the weak convergence of V^N to Y. From (4.7) and (4.15) we have the following representation for V_t^N:

$$V_t^N = -\int_0^t \lambda_k \sum_{l=1}^k \left[\Phi_s\left(\sqrt{N}\, x_{l,N}\right) - \Phi_s\left(\sqrt{N}\, x_l\right)\right] \mathrm{d}s \\ - \frac{1}{N} S_{k,N}^p(t) + \mu_k t + \sqrt{N}\, m_{k,N}(t). \tag{9.10}$$

Keeping in mind that

$$\sum_{l=1}^k \left[\Phi_s\left(\sqrt{N}\, x_{l,N}\right) - \Phi_s\left(\sqrt{N}\, x_l\right)\right] = \sqrt{N}\left[q_k^N(s) - x_k(s)\right] + \sum_{l=1}^{k-1} \sqrt{N}\, q_l^N(s) \\ = \sum_{l=1}^{k-1} \sqrt{N}\, q_l^N(s) + \sqrt{N}\left[q_k^N(s) - x_{k,N}(s)\right] + V_s^N,$$

rewrite (9.10) as

$$V_t^N = -\int_0^t \left\{\lambda_k V_s^N + \lambda_k \left[\sum_{l=1}^{k-1} \sqrt{N}\, q_l^N(s) + \sqrt{N}\left[q_k^N(s) - x_{k,N}(s)\right]\right]\right\} \mathrm{d}s \\ - \frac{1}{N} S_{k,N}^p(t) + \mu_k t + \sqrt{N}\, m_{k,N}(t).$$

Taking into consideration that the jumps of V^N are no more than $1/\sqrt{N}$, and $N^{-1} S_{k,N}^p(t)$ converges in probability to $\mu_k t$, in order to prove the weak convergence of V^N to Y, we can apply theorem 8.3.1(c) by Liptser and Shiryayev [24] checking for any fixed $t > 0$ only two of its conditions:

$$\mathbf{P}\text{-}\lim_{N\to\infty} \sup_{s\leqslant t} \left|\int_0^s \left\{\lambda_k \left[\sum_{l=1}^{k-1} \sqrt{N}\, q_l^N(u) + \sqrt{N}\left[q_k^N(u) - x_{k,N}(u)\right]\right]\right\} \mathrm{d}u\right| = 0 \tag{9.11}$$

and

$$\mathbf{P}\text{-}\lim_{N\to\infty} \sup_{s\leqslant t} \left|\left\langle \sqrt{N}\, m_{k,N}\right\rangle_s - \int_0^s \lambda_k\left[1 - q(u)\right] \mathrm{d}u - \frac{1}{N} S_{k,N}^p(s)\right| = 0. \tag{9.12}$$

Indeed, (9.11) follows immediately from lemma 3, and (9.12) follows, in turn, from lemma 2, since

$$\left\langle \sqrt{N}\, m_{k,N}\right\rangle_t = \lambda_k \int_0^t \left[1 - \sum_{j=1}^k q_j^N(s)\right] \mathrm{d}s + \frac{1}{N} S_{k,N}^p(t)$$

(see relation (4.4)) and, therefore,

$$\sup_{s\leqslant t}\left|\left\langle\sqrt{N}\,m_{k,N}\right\rangle_s-\int_0^s\lambda_k\big[1-q(u)\big]\,\mathrm{d}u-\frac{1}{N}\,S^p_{k,N}(s)\right|$$
$$=\sup_{s\leqslant t}\left|\int_0^s\lambda_k\left[\sum_{j=1}^k q_j^N(u)-q(u)\right]\mathrm{d}s\right|.$$

Theorem 5 is proved. □

10. Discussion for the future work and extension of the model

In this section we discuss an extension of the model in the following two cases as the service time in the hub and, therefore, inputs to satellite node are time dependent and the inputs to satellite nodes are generated by a process with strictly stationary and ergodic increments. In those both cases we omit the technical details and pay attention to general behavior of the model.

10.1. A model with time dependent inputs to satellite nodes

Although this model seems to be the straightforward generalization of our model it has some interesting specific features.

Consider the queue-length process at satellite node j defined by (2.1). Let the process $D_{j,N}(t)$ is the same as in (2.2) while the process $A_{j,N}(t)$ is defined as follows. Assume that $\{P_{j,i}(t)\}$, $i=1,2,\dots,N$, is a collection of independent nonhomogeneous Poisson processes with rate $\lambda_j(t)$. Then,

$$A_{j,N}(t)=\int_0^t\sum_{i=1}^N\mathbf{I}\left\{N-\sum_{l=1}^k Q_l^N(s-)\geqslant i\right\}\mathrm{d}P_{j,i}(s). \tag{10.1}$$

Next,

$$A^p_{j,N}(t)=\int_0^t\left\{N-\sum_{l=1}^k Q_l^N(s-)\right\}\mathrm{d}P^p_{j,1}(s), \tag{10.2}$$

where $P^p_{j,1}(t)$ is a compensator of the point process $P_{j,1}(t)$, and

$$P^p_{j,1}(t)=\lambda_j(t)t \tag{10.3}$$

(for details see Dellacherie [13] or Liptser and Shiryayev [23, theorem 1.6.1]). Therefore, substituting (10.3) for (10.2) we have

$$A^p_{j,N}(t)=\int_0^t\left\{N-\sum_{l=1}^k Q_l^N(s-)\right\}\mathrm{d}\big[s\lambda_j(s)\big] \tag{10.4}$$

and, further,

$$\frac{1}{N}A^p_{j,N}(t)=\int_0^t\left\{1-\sum_{l=1}^k q_l^N(s-)\right\}\mathrm{d}\big[s\lambda_j(s)\big],\quad q_l^N(s)=\frac{1}{N}Q_l^N(s). \tag{10.5}$$

Thus, analogously to (4.15) we have

$$x_j(t)=\int_0^t\left\{1-\sum_{l=1}^k\Phi_s(x_l)\right\}\mathrm{d}\big[s\lambda_j(s)\big]-\mu_j t,\quad j=1,2,\dots,k, \tag{10.6}$$

(we use here the same notation as in the usual case). And if $\lambda_j(t)$ is a continuous function in point t, the solution of (10.6) depends on whether $\lambda_j(t)<\mu_j$ or $\lambda_j(t)\geqslant\mu_j$. The first inequality leads to nonbottleneck case, the second one to bottleneck case. It is interesting to study a behavior around bifurcation points where a node is changed from nonbottleneck to bottleneck case.

10.2. A general model with state dependent service times at the hub

In this section we consider the situation where the hub is a single-server queueing system with service depending on queue-length generated in turn by point process with strictly stationary and ergodic increments. That is,

$$\Pi(t)=\sum_{i=1}^{\infty}\mathbf{I}\{t_i\leqslant t\},\qquad t_n=\sum_{i=1}^n\tau_i,$$

where $\{\tau_i\}_{i\geqslant1}$ is a strictly stationary and ergodic sequence of random variables, having the expectation λ^{-1}. A service time at the hub depends on the number of customers residing here as follows. If immediately before a service time of ith customer there are K customers at the hub, we set that service time to be equal to $K^{-1}\tau_i$.

Note, that the case of independent identically and exponentially distributed random variables $\{\tau_i\}$ reduces this model with service at the hub depending on queue-length to the model where the hub is an infinite-server queueing system, as it was considered in the previous sections of the paper.

Let us consider now satellite node j where the queue-length process is defined by (2.1) and the departure process by (2.2).

To define an arrival process we should first construct an auxiliary process with strictly stationary and ergodic increments by thinning the initial process $\Pi(t)$. To do that let us introduce the sequence $\{\tau_{j,i}\}_{i\geqslant1}$ as follows:

$$\tau_{j,1}=\tau_1+\tau_2+\cdots+\tau_{\nu_{j,1}},$$
$$\tau_{j,n}=\tau_{\nu_{j,1}+\nu_{j,2}+\cdots+\nu_{j,n-1}+1}+\cdots+\tau_{\nu_{j,1}+\nu_{j,2}+\cdots+\nu_{j,n}},$$

where $\{\nu_{j,i}\}_{i\geqslant1}$ is the sequence of independent identically distributed positive integers, $\mathbf{P}\{\nu_{j,1}=n\}=p_j(1-p_j)^{n-1}$.

Let now

$$\pi_j(t) = \sum_{i=1}^{\infty} \mathbf{I}\{t_{j,i} \leqslant t\}, \quad t_{j,n} = \sum_{i=1}^{n} \tau_{j,i}.$$

It is clear that $\{\tau_{j,i}\}_{i\geqslant 1}$ is a strictly stationary and ergodic sequence of random variables, and the process $\pi_j(t)$, being a thinning of the process $\Pi(t)$, satisfies the necessary properties.

Next, define an arrival process as follows:

$$A_{j,N}(t) = \int_0^t \mathrm{d}\pi_j\big(\big[Q_0^N(s_*-)\big]s\big) = \int_0^t \mathrm{d}\pi_j\left(\left[N - \sum_{l=1}^{k} Q_l^N(s_*-)\right]s\right), \tag{10.7}$$

where $s_* \leqslant s$ is the instant of the last departure before time s of a customer from the hub. It is obvious that

$$A_{j,N}(t) \leqslant \widetilde{A}_{j,N}(t) = \int_0^t \mathrm{d}\pi_j\left(\left[N - \sum_{l=1}^{k} Q_l^N(s-)\right]s\right) \tag{10.8}$$

with probability 1. Inequality (10.8) holds because during time interval (s_*, s) customers can return from satellite nodes to the hub, and the queue-length at the hub during that time interval can only increase.

It follows also from (10.8) that

$$A_{j,N}^p(t) \leqslant \widetilde{A}_{j,N}^p(t), \tag{10.9}$$

where $A_{j,N}^p(t)$ and $\widetilde{A}_{j,N}^p(t)$ are the compensators for the processes $A_{j,N}(t)$ and $\widetilde{A}_{j,N}(t)$, respectively, and

$$\langle M_{j,N}\rangle_t \leqslant A_{j,N}^p(t) + S_{j,N}^p(t) \leqslant \widetilde{A}_{j,N}^p(t) + S_{j,N}^p(t). \tag{10.10}$$

Next, denoting $\pi_j^p(t)$ the compensator for $\pi_j(t)$ and taking into consideration that

$$\pi_j^p(t) = \sum_{i=1}^{\infty} \int_{t\wedge\tau_{j,i-1}}^{t\wedge\tau_{j,i}} \frac{\mathrm{d}F_{j,i}(s)}{1 - F_{j,i}(s-)}, \tag{10.11}$$

where $F_{j,i}(x) = \mathbf{P}\{\tau_{j,i} \leqslant x \mid \tau_{j,i-1}, \ldots, \tau_{j,0}\}$ and $\tau_{j,0}$ is assumed to be zero (see, e.g., Liptser and Shiryayev [23,24], Davis [12]), from (10.8) we obtain

$$\widetilde{A}_{j,N}^p(t) = \int_0^t \mathrm{d}\pi_j^p\left(\left[N - \sum_{l=1}^{k} Q_l^N(s-)\right]s\right). \tag{10.12}$$

Equations (10.7), (10.8) and (10.12) are very general for analysis similar to that for the model considered earlier in the previous sections. Nevertheless, those equations seem useful to prove the statement on stability for this network with the aid of development of the method in the sixth section of the paper.

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References

[1] V.M. Abramov, *Investigation of a Queueing System with Service Depending on Queue-Length* (Donish, Dushanbe, 1991) (in Russian).

[2] V.M. Abramov, On the asymptotic distribution of the maximum number of infectives in epidemic models with immigration, J. Appl. Probab. 31 (1994) 606–613.

[3] V.M. Abramov, Queueing systems with autonomous service: up- and down-crossings approach, submitted for publication (1999).

[4] V.M. Abramov, Some results for large closed queueing networks with and without bottleneck: up- and down-crossings approach, in preparation.

[5] S.V. Anulova and R.Sh. Liptser, Diffusion approximation for the processes with normal reflection, Theory Probab. Appl. 35 (1990) 413–423.

[6] A.T. Bharucha-Reid, *Elements of the Theory of Markov Processes and their Applications* (McGraw-Hill, New York, 1960).

[7] P. Bilingsley, *Convergence of Probability Measures* (Wiley, New York, 1968).

[8] A.A. Borovkov, *Stochastic Processes in Queueing Theory* (Springer, Berlin, 1976).

[9] H. Chen and A. Mandelbaum, Discrete flow networks: Bottleneck analysis and fluid approximations, Math. Oper. Res. 16 (1991) 408–446.

[10] H. Chen and A. Mandelbaum, Discrete flow networks: Diffusion approximations and bottlenecks, Ann. Probab. 19 (1991) 1463–1519.

[11] J.W. Cohen, On up- and downcrossings, J. Appl. Probab. 14 (1977) 405–410.

[12] M.H.A. Davis, *Linear Estimation and Stochastic Control* (Chapman and Hall, London, 1977).

[13] C. Dellacherie, *Capacités et Processus Stochastiques* (Springer, Berlin, 1972).

[14] J. Jacod and A.N. Shiryayev, *Limit Theorems for Stochastic Processes* (Springer, Berlin, 1987).

[15] Y. Kogan, Another approach to asymptotic expansions for large closed queueing networks, Oper. Res. Lett. 11 (1992) 317–321.

[16] Y. Kogan and R.Sh. Liptser, Limit non-stationary behavior of large closed queueing networks with bottlenecks, Queueing Systems 14 (1993) 33–55.

[17] Y. Kogan, R.Sh. Liptser and M. Shenfild, State-dependent Beneŝ buffer model with fast loading and output rates, Ann. Appl. Probab. 5 (1995) 97–120.

[18] Y. Kogan, R.Sh. Liptser and A.V. Smorodinskii, Gaussian diffusion approximation of closed Markov model of computer networks, Problems Inform. Transmission 22 (1986) 38–51.

[19] T. Konstantopoulos, S.N. Papadakis and J. Walrand, Functional approximation theorems for controlled queueing networks, in: *32nd IEEE CDC*, San Antonio, TX (1993) pp. 3544–3549.

[20] T. Konstantopoulos, S.N. Papadakis and J. Walrand, Functional approximation theorems for controlled renewal processes, J. Appl. Probab. 31 (1994) 765–776.

[21] E.V. Krichagina, R.Sh. Liptser and A.A. Puhalskii, Diffusion approximation for system with arrivals depending on queue, and with arbitrary service, Theory Probab. Appl. 33 (1988) 114–124.

[22] T. Lindvall, Weak convergence of probability measures and random functions in the functional space $\mathbf{D}[0,\infty]$, J. Appl. Probab. 10 (1973) 109–121.

[23] R.Sh. Liptser and A.N. Shiryayev, *Statistics of Random Processes*, Vols. I, II (Springer, Berlin, 1978).
[24] R.Sh. Liptser and A.N. Shiryayev, *Theory of Martingales* (Kluwer, Dordrecht, 1989).
[25] A. Mandelbaum and G. Pats, State dependent stochastic networks. Part I. Approximations and applications with continuous diffusion limits, Ann. Appl. Probab. 8 (1998) 569–646.
[26] G. Polya and G. Szego, *Aufgaben und Lehrsatze aus der Analysis. Erster Band: Reihen, Integralrechnung, Functionentheorie* (Springer, Berlin, 1964).
[27] A.V. Skorohod, Stochastic equations for diffusion processes in a bounded region, Theory Probab. Appl. 6 (1961) 264–274.
[28] C. Stone, Weak convergence in stochastic processes defined on semi-infinite time intervals, Proc. Amer. Math. Soc. 14 (1963) 694–696.
[29] H. Tanaka, Stochastic differential equations with reflected boundary condition in convex regions, Hiroshima Math. J. 9 (1979) 163–177.
[30] W. Whitt, Open and closed models for networks of queues, AT&T Bell. Lab. Tech. J. 63 (1984) 1911–1979.