

# On nonexistence of non-constant volatility in the Black-Scholes formula<sup>1</sup>

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**Abstract.** We prove that if the Black-Scholes formula holds with the spot volatility for call options with all strikes, then the volatility parameter is constant.

**Key words:** Black-Scholes formula, stochastic volatility, stochastic implied volatility, local volatility models

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## 1 Introduction

One of the most important applications of stochastic dynamics has occurred in finance as a model for the evolution of prices of stocks and their options. This model in its simple form is described by a randomly perturbed exponential growth. If  $S_t$  denotes the price of a stock at time  $t$ , then its evolution is given by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad (1)$$

with  $B_t$  denoting the Brownian motion process. The strength of the random perturbation is determined by the parameter  $\sigma$ , which is known as the volatility of the stock in finance. The above model was used by Merton, and Black and Scholes to find the price of an option on stock, such as an agreement to buy the stock at some future time  $T$  for the price  $K$  specified at time  $t < T$ . Their formula states that the price of such an option at time  $t$  is given by

$$C_t = S_t \Phi\left(\frac{\log \frac{S_t}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)} \Phi\left(\frac{\log \frac{S_t}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right), \quad (2)$$

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where  $\Phi$  denotes the standard normal distribution function. Remarkably, the parameter  $\mu$  does not enter the formula, but  $\sigma$  does, as well as  $r$ , the riskless rate available in a savings account. The Black-Scholes formula is widely used in financial markets and risk management.

The volatility of the stock is seen to be the parameter in the quadratic variation of the return on the stock process,  $\sigma^2 = d[R, R]_t/dt$ , where  $dR_t = dS_t/S_t$ , see (1). It is widely believed and experimentally verified that stocks do not have a constant volatility, rather this parameter varies with time, see e.g. [8], [4], [7], [13]. In spite of this fact, the Black-Scholes options pricing formula (2) is still used with some adjustments to the volatility. Thus the question of existence of a model with non-constant volatility in which the Black-Scholes formula remains valid is of interest in financial mathematics as well as in practical applications. A non-constant volatility model has the form

$$dS_t = \mu S_t dt + \theta_t S_t dB_t, \quad (3)$$

where  $\theta_t$  is a function of time that can be random, for example, it can be a function of stock  $S_t$ , as well as include other independent source of randomness. There is a large literature on non-constant volatility models, both deterministic and stochastic (see e.g. Fouque et.al. (2001)). In the next paragraph we need to recall the basic facts on options pricing (see e.g. Shiryaev (1999)). We can assume without loss of generality that the riskless interest rate  $r = 0$ , otherwise work with discounted prices  $S_t e^{-rt}$ .

The First Fundamental theorem of asset pricing states that a model does not admit arbitrage if and only if there exists an equivalent probability measure  $Q$  such that  $S_t$  is a  $Q$ -martingale. The price at time  $t$  of a call option that pays  $(S_T - K)^+$  at time  $T$  is given by

$$C_t = E_Q((S_T - K)^+ | \mathcal{F}_t^S), \quad (4)$$

where  $E_Q$  is the expectation under  $Q$  and  $\mathcal{F}_t^S$  is the  $\sigma$ -field generated by the process  $S_u, u \leq t$ .

Next we comment on recent work to reconcile the non-constant volatility with the use of the Black-Scholes formula by constructing the so-called stochastic implied volatility (SIV) market models. A market model is a model that returns Black-Scholes option prices. Such models are important in practical applications for calculations of non-standard options. The idea in SIV models is to use equation (3) together with the volatility surface  $\sigma(t, T, K)$ . These  $\sigma(t, T, K)$  are implied by the Black-Scholes option prices by equating the theoretical price in (4)  $C_t(T, K) = E_Q((S_T - K)^+ | \mathcal{F}_t^S)$  with the Black-Scholes price (2) with  $\sigma = \sigma(t, T, K)$ . The spot volatility  $\theta_t$  is obtained from the volatility surface  $\sigma(t, T, K)$  by  $\theta_t = \sigma(t, t, S_t)$ . This approach results in a nonlinear system of stochastic differential equations with delay, see Brace et.al. (2001), also Schonbucher (1998), Carr (2000), Brace et.al. (2002). The related (and somewhat reverse) question of mis-specification where the mis-specified volatility dominates, or is dominated by, the true volatility was discussed in El Karoui et al (1996). While existence of SIV is still an open problem, the purpose of this note

is to show that, unfortunately, non-constant volatility models are not compatible with the Black-Scholes formula.

Comment that the ‘‘Black-Scholes’’ equation with non-constant volatility  $dS_t = \theta_t S_t dB_t$  holds for a wide class of positive martingales. If  $S_t$  is a positive martingale with  $P(S_t > 0) = 1$  for any  $t \leq T$  and the predictable representation property holds then there exists a process  $\theta_t$ , such that  $dS_t = \theta_t S_t dB_t$ . We prove the statement even for wider class of processes and do not assume any dynamics on  $S_t$ , namely if for a positive semimartingale  $S_t$  and for all values of  $K$  and three different maturities the Black-Scholes formula holds with some adapted  $\theta_t$ , then  $\theta_t$  must be constant. This  $\theta_t$  can be taken as any adapted functional of the spot volatility, including the spot volatility itself.

## 2 Results

Let  $(\Omega, \mathcal{F}, \mathbb{F}, Q)$  be the filtered probability space, with the general conditions, supporting a Brownian motion  $B_t$ . Let for a constant  $\sigma > 0$ ,

$$dZ_t = \sigma Z_t dB_t, \quad Z_0 = z_0$$

and

$$C(T, t, K, \sigma, z) = \mathbb{E}[(Z_T - K)^+ | Z_t = z]. \quad (5)$$

**Theorem 1** *Let  $S_t$  and  $\theta_t$  be two adapted processes such that  $\theta_0 = \sigma$  and  $S_0 = z_0$ . Assume that  $S_t$  is strictly positive, that  $\mathcal{F}_0$  is trivial and that there exist three equidistant terminal times,  $T_1 < T_2 < T_3$  such that, for all  $K$  and all  $t \leq T_i$ ,  $i = 1, 2, 3$*

$$\mathbf{E}[(S_{T_i} - K)^+ | \mathcal{F}_t] = C(T_i, t, K, \theta_t, S_t).$$

*Then  $\theta_t^2 = \sigma^2$  for all  $t \leq T_1$ .*

The proof relies on the following result.

**Theorem 2** *Let  $M_t$  and  $X_t$  be two semimartingales,  $M_t$  is strictly positive. Assume that  $M_t$ ,  $M_t X_t$  and  $M_t X_t^2$  are local martingales. Then  $X_t \equiv X_0$ .*

Note that Theorem 1 can be formulated for three terminal non-equidistant times  $T_i$ ,  $i = 1, 2, 3$ , and in this case Theorem 2 can be generalized to the case when  $M_t$ ,  $M_t X_t$  and  $M_t X_t^\alpha$ ,  $\alpha > 1$  are local martingales. For clarity of presentation we use equidistant times.

**Proof<sup>2</sup>** of Theorem 2. We may assume without loss of generality that  $M_0 = 1$ . Let  $\{T_n, n \geq 0\}$  be a localising sequence for  $M_t$ . Then for each  $n$ ,  $M_{t \wedge T_n}$  is a true martingale. It defines on  $\mathcal{F}_{T_n}$  an equivalent measure  $\mathbf{Q}_n$ . Under  $\mathbf{Q}_n$ ,  $X_{t \wedge T_n}$  and  $X_{t \wedge T_n}^2$  are local martingales. This immediately implies that

$$\langle X^c, X^c \rangle_{t \wedge T_n} + \sum_{s \leq t \wedge T_n} (\Delta X_s)^2 = 0,$$

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<sup>2</sup>This version of the proof was suggested by Saul Jacka

where as usual  $X^c$  denotes the continuous martingale component of  $X$  and  $\Delta X_t = X_t - X_{t-}$ . That is  $X_t = X_t^c = X_0$ , on  $\{t \leq T_n\}$ . Extending  $n$  to infinity, we complete the proof.  $\square$

The proof of Theorem 1 is broken into a number of propositions. All of them assume the conditions and notations of Theorem 1.

**Proposition 3**  $S_t$  is a martingale and  $\mathbf{E}[S_t^2] < +\infty$  for all  $t$ .

**Proof** Take any  $i$ . By condition (5)  $S_{T_i}$  is integrable and

$$\begin{aligned} \mathbf{E}[S_{T_i}|\mathcal{F}_t] &= \mathbf{E}[(S_{T_i} - 0)^+|\mathcal{F}_t] = C(T_i, t, 0, \theta_t, S_t) \\ &= \mathbf{E}[(Z_{T_i} - 0)^+|Z_t = z]_{\sigma=\theta_t, z=S_t} \\ &= \mathbf{E}[Z_{T_i}|Z_t = z]_{\sigma=\theta_t, z=S_t} \\ &= Z_t|_{\sigma=\theta_t, z=S_t} = S_t, \end{aligned}$$

which proves that  $S_t$  is a martingale.

Also  $\mathbf{E}[(S_{T_i} - K)^+] = \mathbf{E}[(S_{T_i} - K)^+|\mathcal{F}_0] = C(T_i, 0, K, \theta_0, S_0) = C(T_i, 0, K, \sigma, z_0) = \mathbf{E}[(Z_{T_i} - K)^+]$ . Using Lemma 6, we deduce that  $S_{T_i}$  and  $Z_{T_i}$  have the same distribution. The integrability of  $Z_{T_i}^2$  induces that  $S_{T_i}^2$  and by Jensen's inequality that of  $S_t^2$ .  $\square$

**Proposition 4**  $\left(S_t^2 e^{\theta_t^2(T_i-t)}\right)_{t \leq T_i}$  is a martingale, for each  $i = 1, 2, 3$ .

**Proof** The proof is based on the representation of  $\mathbf{E}[X^2|\mathcal{G}]$  in terms of integrals  $\int_0^{+\infty} \mathbf{E}[(X - K)^+|\mathcal{G}]dK$  and  $\int_0^{+\infty} \mathbf{E}[(X + K)^-|\mathcal{G}]dK$  given in Lemma 7. Using this lemma we obtain

$$\begin{aligned} \mathbf{E}[S_{T_i}^2|\mathcal{F}_t] &= 2 \int_0^{+\infty} \mathbf{E}[(S_{T_i} - K)^+|\mathcal{F}_t]dK \\ &\quad + 2 \int_{-\infty}^0 (\mathbf{E}[(S_{T_i} - K)^+|\mathcal{F}_t] - \mathbf{E}[S_{T_i}|\mathcal{F}_t] + K) dK \\ &= 2 \left[ \int_0^{+\infty} C(T_i, t, K, \sigma, z)dK + \int_{-\infty}^0 (C(T_i, t, K, \sigma, z) - z + K) dK \right]_{\sigma=\theta_t, z=S_t} \\ &= 2 \left[ \int_0^{+\infty} \mathbf{E}[(Z_{T_i} - K)^+|Z_t = z]dK \right. \\ &\quad \left. + \int_{-\infty}^0 (\mathbf{E}[(Z_{T_i} - K)^+|Z_t = z] - \mathbf{E}[Z_{T_i}|Z_t = z] + K) dK \right]_{\sigma=\theta_t, z=S_t} \\ &= [\mathbf{E}[Z_{T_i}^2|Z_t = z]]_{\sigma=\theta_t, z=S_t} \\ &= [z^2 e^{\sigma^2(T_i-t)}]_{\sigma=\theta_t, z=S_t} \\ &= S_t^2 e^{\theta_t^2(T_i-t)} \end{aligned}$$

Thus for each  $i$ ,  $\left(S_t^2 e^{\theta_t^2(T_i-t)}\right)_{t \leq T_i}$  is a martingale.  $\square$

**Proposition 5**  $(\theta_t^2)_{t \leq T_1}$  is a semimartingale.

**Proof** Since  $S > 0$ ,  $e^{\theta_t^2(T_2-T_1)} = \frac{S_t^2 e^{\theta_t^2(T_2-t)}}{S_t^2 e^{\theta_t^2(T_1-t)}}$  is a semimartingale in  $t \in [0, T_1]$ .

Thus  $(\theta_t^2)_{t \leq T_1}$  is also a semimartingale.  $\square$

**Proof of Theorem 1**

Let  $M_t = S_t^2 e^{(T_1-t)\theta_t^2}$ ,  $X_t = e^{(T_2-T_1)\theta_t^2} = e^{(T_3-T_2)\theta_t^2}$ . It is easy to see using the previous propositions that  $M_t$  and  $X_t$  satisfy the conditions of Theorem 2. By its conclusion  $X_t \equiv X_0$  and  $\theta_t^2 = \sigma^2$  for all  $t \leq T_1$ , which completes the proof.  $\square$

**Remarks**

1. Note that although in Theorem 1 we do not assume any dynamics on  $S_t$ , the assumptions of the theorem imply that  $S_t$  is a strictly positive martingale, see Proposition 3. If in addition, the predictable representation property with respect to  $B_t$  holds, then  $S_t$  can be represented as a stochastic volatility model  $dS_t = h_t S_t dB_t$ , see e.g. [9], p.286.

2. Let  $dS_t = h_t S_t dB_t$ . While we may primarily think of  $\theta_t$  in Theorem 1 as the volatility  $\theta_t = h_t$ , Theorem 1 applies for any adapted functional of  $h_u$  and  $S_u$ ,  $u \leq t$ , for example, the average volatility on  $[0, t]$  is given by  $\theta_t = \sqrt{\frac{1}{t} \int_0^t h_u^2 du}$ .

3. Let  $dS_t = h_t S_t dB_t$ , where  $h_t$  is a deterministic function. Then  $S_t = S_0 \exp(\int_0^t h_s dB_s - \frac{1}{2} \int_0^t h_s^2 ds)$  has a lognormal distribution and the price of the option is given by the Black-Scholes formula  $\mathbf{E}[(S_T - K)^+ | \mathcal{F}_t] = C(T, t, K, \theta(t, T), S_t)$  with  $\theta^2(t, T) = \frac{1}{T-t} \int_t^T h_u^2 du$ , see e.g. [9]. This example does not contradict Theorem 1 since in this case  $\theta_t$  depends also on  $T$ .

### 3 Auxiliary Results

**Lemma 6** If  $\mathbf{E}[|X|] < \infty$ , then  $G(K) = \mathbf{E}[(X - K)^+ | \mathcal{G}]$  is a convex function ( $G'_-$  and  $G'_+$  are increasing, respectively left and right-continuous and  $\{K : G'_-(K) \neq G'_+(K)\}$  is countable) with

$$G'_-(K) = -\mathbf{P}[X \geq K | \mathcal{G}] \text{ and } G'_+(K) = -\mathbf{P}[X > K | \mathcal{G}].$$

**Proof** The proof easily follows from the fact that, for  $\varepsilon > 0$ ,

$$(x - (K + \varepsilon))^+ - (x - K)^+ = \begin{cases} 0 & x \leq K \\ -(x - K) & K < x < K + \varepsilon \\ -\varepsilon & x \geq K + \varepsilon \end{cases}$$

and

$$(x - (K - \varepsilon))^+ - (x - K)^+ = \begin{cases} 0 & x \leq K - \varepsilon \\ x - K + \varepsilon & K - \varepsilon < x < K \\ \varepsilon & x \geq K \end{cases}$$

Note that  $1_{K < X < K + \varepsilon} = 1_{K - \varepsilon < X < K} = 0$  for  $\varepsilon$  small enough.  $\square$

**Lemma 7** *If  $\mathbf{E}[X^2] < +\infty$ , then*

$$\begin{aligned} & \frac{1}{2}\mathbf{E}[X^2|\mathcal{G}] \\ &= \int_0^{+\infty} \mathbf{E}[(X - K)^+|\mathcal{G}]dK + \int_0^{+\infty} \mathbf{E}[(X + K)^-|\mathcal{G}]dK \\ &= \int_0^{+\infty} \mathbf{E}[(X - K)^+|\mathcal{G}]dK + \int_{-\infty}^0 (\mathbf{E}[(X - K)^+|\mathcal{G}] - \mathbf{E}[X|\mathcal{G}] + K) dK \end{aligned}$$

*In particular, if  $X$  is non-negative,*

$$\mathbf{E}[X^2|\mathcal{G}] = 2 \int_0^{\infty} \mathbf{E}[(X - K)^+|\mathcal{G}]dK.$$

**Proof** It is easily checked that for any  $x$ ,

$$x^2 = 2 \int_0^{+\infty} (x - K)^+dK + 2 \int_0^{+\infty} (x + K)^-dK.$$

The result immediately follows. □

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