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Generalized solutions in systems with active unilateral constraints[☆]

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Abstract

The concept of the generalized solution that admits well-posed representation of controlled complex behavior in systems with active unilateral phase constraints is proposed. Based on this concept, the definition of the generalized solution for this class of problems is introduced that encompasses Zeno type behavior and sliding modes along the constraint boundary. The general representation of such solutions in terms of nonlinear differential equations with a measure is derived. The latter is shown to solve a long-standing problem of providing unique extensibility of a trajectory beyond accumulation points in systems with Zeno-type behavior. An example is given, showing that the representation proposed completely captures Zeno-type behavior and provides unique extensibility of solutions without the need to truncate infinite sequences and/or switch system coefficients depending on system motion relative to the generalized coordinates of the accumulation point.

Keywords: Generalized solutions; Phase constraints; Differential equations with a measure

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1. Introduction

In recent years there has been a significant interest in modeling and control of systems characterized by interaction with actuated unilateral constraints (cf. [1,10–13,42]), such as, for example, mechanical systems with impacts. This interest is motivated by a wide range of applications that involve motions ultra-fast in comparison to the natural system dynamics, including vibro-impact mechanics [1], robotics [16], micro-electro-mechanics [8], power electronics [17,23,25], and, in general, hybrid systems with discrete transitions such as mobile sensor networks [18]. In all these applications, system dynamics can be viewed as being affected by the boundary of a constraint impinging upon a system and exerting on it an impulsive actuation capable of almost abruptly changing system velocity. Therefore, the constraints in these applications become the additional means of control that have to be incorporated both into the system design and the controller synthesis framework. The latter observation, however, significantly complicates optimal controller synthesis for this class of systems. Indeed, the control signals involved act through the constraint boundary as an impulsive feedback localized at some surface. This feature gives rise to a very particular

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system behavior — *Zeno-type* solutions [19,20,45], or finite time accumulation points, possibly followed by solution extensions beyond these points in the form of *impulsive sliding modes* [29].

Optimal control problems for systems with constraints have been considered to be approachable through *complementarity*, and there are some examples of success of the latter method [13,14,38,39,43]. The complementarity approach indeed looks natural for many impact problems, coming from Lagrangian and Hamiltonian mechanics for systems with unilateral constraints and representing collisions in terms of so-called measure type reaction forces localized at the constraint boundary [30,26]. Moreover, complementarity is computationally supported by penalty approximation of dynamic equations [32–34,40,41]. This approach, however, starts running into difficulties as the complexity of the control problems increases. Indeed, complementarity addresses these problems through additional models of contact forces arising at the constraint boundary that provide the so-called *collision mapping*, or *restitution law*, at the contact point, relating the post-impact motion to the pre-impact one. This formalization, however, gives rise to solution nonuniqueness for systems with inputs, or "data", if no additional requirement on the analyticity of the latter is imposed [2,9]. Generating an appropriate collision mapping using complementarity becomes even more problematic if the impulsive control is admitted during the contact, or singular motion, phase [3–7]; yet, the latter capability is routinely encountered in a number of systems. For example, electro-mechanical systems such as electromagnetic scanners and vibromotors with piezoceramic elements feature actuators operating with sufficiently high speed and power to produce an effect during the contact phase [37,44], and optimization of the force profile during this phase in these systems is an important step in their optimal design.

To address this problem, a new approach, termed the "physically based" or "infinitesimal dynamics" approach, has been proposed by Bentsman and Miller in [5,6]. It consists in starting with the physically well justified system description that contains in its right hand side unbounded terms representing system interactions with the constraints. This description, through a space-type transformation and a limiting procedure, is then mapped into a controlled singular phase representation given by a dynamic equation pair: the "limit" system that contains controlled solution jumps and the "controlled infinitesimal dynamics" system describing motion in the singular phase in the fictitious time. This pair permits synthesis of the optimal singular phase control laws through applying regular methods to the latter system, deriving a shift-operator along the paths of the resulting controlled system, and, finally, using this shift-operator for representing the controlled jumps in the limit system.

The theory developed in [6,27,28] is limited, however, to the class of solutions with a finite number of jumps. Based on the approach of [6,27,28], the goal of the present work is to obtain the representation of a much broader class of general solutions, including ones with an infinite number of jumps, finite time accumulation points, and unique extensibility beyond the latter, thereby solving this long-standing problem in impact and hybrid dynamics.

As the first step, the present works develops the concept of *generalized solution* as a limit of a sequence of ordinary solutions at continuity points [29] if some parameter μ characterizing the constraint elasticity tends to infinity. In this sense, all solutions that can be defined through penalty approximations are encompassed by this class. Then, the time-transformation method is developed and the representation of the generalized solution in terms of differential equations with a measure is derived. It is pointed out that even if the model of the singular phase motion gives the complete jump description, the motion along the boundary is still undetermined. The reason is that the order of constraint violation for the case of strict impact ($\sim\mu^{-1/2}$) differs from that for the case of smooth motion along the boundary ($\sim\mu^{-1}$). This gives rise to different approximation schemes, but ultimately leads to the full description of jumps and sliding modes along the boundary through the same nonlinear differential equation with a measure. The advantage of this description is that the motion along the constraint boundary could begin only if the point representing the system state smoothly touches this boundary. Therefore, the unique solution extensibility beyond the accumulation point of hitting the constraint boundary becomes natural.

The structure of the paper is as follows. Section 2 describes the concept of generalized solution in the general case that admits an infinite number of jumps. Section 3 develops the representation of the generalized solution through the use of a discontinuous time transformation. In Section 4, the representation of the jump part of generalized solution is derived through a space–time transformation. Section 5 derives the representation of sliding motion along the constraint boundary. The nonlinear differential equation with a measure describing the generalized solution that encompasses all the aforementioned modes of behavior is obtained in Section 6. Finally, Section 7 presents an example that clarifies the initiation of sliding mode along the constraint boundary and shows how the solution extends beyond the right-accumulation point. The last section presents conclusions. The notation $I\{A\}$ is used throughout the paper to denote the indicator function of a set A. The notations f'_x and f''_{xy} are used to denote partial derivatives of the first and the second order of a function f with respect to variables x and x, y, respectively.

2. Generalized solutions in dynamical systems with active unilateral constraints

Following [6,27,28], consider a dynamical system whose behavior is described on the interval $[0, T]$ by a pair of the continuous functions $(x_p(t), x_v(t)) \in R^n \times R^n$ that satisfies the differential equation

$$\begin{aligned} \dot{x}_p(t) &= F^r_p(x_p(t), x_v(t), t), \\ \dot{x}_v(t) &= F^r_v(x_p(t), x_v(t), u(t), t) + \mu F^s_v(x_p(t), x_v(t), t, \mu) I\{t : G(x_p(t), t) \le 0\}, \quad G : R^n \times [0, T] \to R \end{aligned} \tag{1}$$

with a given initial condition $(x_p(0), x_v(0)) \in R^n \times R^n$. One can consider $(x_p(t), x_v(t))$ in (1) as the generalized position and velocity, and

$$u(t) \in U \subset R^m, \tag{2}$$

where U is some compact set, as the control signal.

Assumption 1. Functions F^r_p, F^r_v are assumed to be continuous and smooth in $R^n \times R^n \times U \times [0, T]$. Function $F^s_v(x_p, x_v, t, \mu)$ is assumed to be continuous and smooth in the area $G(x_p, \tau) \le 0$. Function F^s_v is assumed to satisfy the constraints

$$F^s_v(x_p, x_v, t, \mu) = 0, \quad \text{if}$$

$$G(x_p, t) = 0 \quad \text{and} \quad \dot{G}(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{F^r_p} G(x_p, t) = G'_{x_p} F^r_p + G'_t \Big|_{(x_p, x_v, t)} = 0,$$

and the Lipschitz type condition

$$\|F^s_v(x_p, x_v, t, \mu) - F^s_v(x'_p, x'_v, t, \mu)\| \le L\{\|x_p - x'_p\| + \mu^{-1/2}\|x_v - x'_v\|\}. \tag{3}$$

The scalar function $G(x, t)$ describing the constraint is assumed to be continuous and smooth as well. Since more smoothness will be required for some of the formal statements, it is assumed that all functions F^r_p, F^r_v, F^s_v, G have continuous bounded derivatives up to the second order.

An example where Assumption 1 holds can be given by

$$F^s_v = A(x_p, t)G(x_p, t) + \mu^{-1/2} B(x_p, t)\, [G'_{x_p} F^r_p + G'_t]\Big|_{(x_p, x_v, t)} \tag{4}$$

where A and B are continuous and smooth. This expression arises in visco-elastic modeling of impact (*models of Kelvin–Voight, Maxwell, etc.*) and corresponds to the case *of Rayleigh's dissipative potentials* in the area $G(x_p, \tau) \le 0$ [22].

Assumption 2. Assume that for any given $0 \le \mu < \infty$ and a given $u(\cdot)$ the system (1) has a solution $\{x^\mu_p(\cdot), x^\mu_v(\cdot)\}$ such that

$$\sup_\mu \int_0^T \mu \|F^s_v(x^\mu_p(t), x^\mu_v(t), t, \mu)\| I\{t : G(x^\mu_p(t), t) \le 0\}\, \mathrm{d}t < \infty,$$

and

$$\lim_\mu G(x^\mu_p(t), t) \ge 0, \quad \text{a.e. on } [0, T].$$

These assumptions hold for simple mechanical systems with continuous control [32].

The propositions given next are straightforward and follow from standard estimation by Gronwall–Bellman type inequalities and Helley's theorem [36,21,31].

Proposition 1. *The sequence of solutions*

$$\{x_p^\mu(\cdot), x_v^\mu(\cdot)\} \in C^1[0,T] \times AC[0,T]$$

and satisfies the constraints

$$\sup_{\mu>0} \max_{[0,T]} \|x_p^\mu(t)\| < \infty, \qquad \sup_{\mu>0} \max_{[0,T]} \|x_v^\mu(t)\| < \infty, \qquad \sup_{\mu>0} \operatorname*{Var}_{[0,T]} \|x_p^\mu(t)\| < \infty.$$

Proposition 2. *There exists a subsequence $\mu_k \uparrow \infty$ such that the corresponding sequence $\{x_p^{\mu_k}(\cdot), x_v^{\mu_k}(\cdot)\}$ converges to some functions $\{\bar{x}_p(\cdot), \bar{x}_v(\cdot)\} \in C^1[0,T] \times BV[0,T]$ where $x_p^{\mu_k}$ converges to $\bar{x}_p$ uniformly on $[0,T]$ and $x_v^{\mu_k}$ converges to $\bar{x}_v$ at all points of continuity.*

Below, the subsequence and the sequence $\mu \uparrow \infty$ are not distinguished.

Definition 1. A pair of functions $\{\bar{x}_p(\cdot), \bar{x}_v(\cdot)\} \in C^1[0,T] \times BV[0,T]$ that exists according to Proposition 2 will be called a *generalized solution* of system (1).

Remark 1. The goal of the present work is to describe the behavior of parametrized sequences of solutions $(x_p^\mu(t), x_v^\mu(t))$ as $\mu \to \infty$. There are a number of studies where this behavior is explored for special cases and it is proved that there exists a limit, satisfying some regularity conditions. Every such treatment is, however, rather involved (see for example [32]), taking into account the special properties of the boundary and dynamic equations. The present work does not follow this line of enquiry. Instead, some conditions are assumed in advance that guarantee the existence of the limit, and the representation of the latter is derived through the use of nonlinear differential equations with a measure. This problem has been explored for the case of a single jump as well as multiple, but finite number of jumps in [27,28]. The present work, however, considers a general case of *infinite* number of jumps and sliding along the constraint boundary.

3. Representation of generalized solutions through the use of time transformation

This section develops the concept of generalized solution using the time transformation method. Details can be found in [29]. Introduce

$$\Gamma^\mu(t) = t + \int_0^t \mu \|F_v^s(x_p^\mu(\tau), x_v^\mu(\tau), \tau, \mu)\| I\{\tau : G(x_p^\mu(\tau), \tau) \le 0\}\, \mathrm{d}\tau. \tag{5}$$

Then, $\Gamma^\mu(\cdot)$ is an absolutely continuous and monotonically increasing function. Consequently, it has on the interval $[0, \Gamma^\mu(T)]$ an inverse function $\eta^\mu(\cdot)$ which has a derivative satisfying the relation

$$\dot{\eta}^\mu(s) = \alpha^\mu(s) = \left.\frac{1}{1 + \mu\|F_v^s(x_p^\mu(\tau), x_v^\mu(\tau), \tau, \mu)\| I\{\tau : G(x_p^\mu(\tau), \tau) \le 0\}}\right|_{\tau=\eta(s)} \tag{6}$$

almost everywhere in $[0, \Gamma^\mu(T)]$. Therefore,

$$0 < \alpha^\mu(s) = \dot{\eta}^\mu(s) \le 1.$$

For each $\mu > 0$ define the time-transformed functions

$$y_p^\mu(s) = x_p^\mu(\eta^\mu(s)), \qquad y_v^\mu(s) = x_v^\mu(\eta^\mu(s)),$$

which satisfy on $[0, \Gamma^\mu(T)]$ the system of equations

$$y_p^\mu(s) = x_p(0) + \int_0^s \alpha^\mu(\tau) F_p^r(y_p^\mu(\tau), y_v^\mu(\tau), \eta^\mu(\tau))\, \mathrm{d}\tau, \tag{7}$$

$$\begin{aligned} y_v^\mu(s) \;=\; & x_v(0) + \int_0^s \alpha^\mu(\tau) F_v^r(y_p^\mu(\tau), y_v^\mu(\tau), u(\eta^\mu(\tau)), \eta^\mu(\tau))\, \mathrm{d}\tau \\ & + \int_0^s \alpha^\mu(\tau) \mu F_v^s(y_p^\mu(\tau), y_v^\mu(\tau), \eta^\mu(\tau), \mu) I\{\tau : G(y_p^\mu(\tau), \eta^\mu(\tau)) \le 0\}\, \mathrm{d}\tau. \end{aligned} \tag{8}$$

Moreover, if $G^+ = \max\{0, G\}$, then

$$\int_0^{\Gamma^\mu(T)} (1 - \alpha^\mu(\tau))(G(y_p^\mu(\tau), \eta^\mu(\tau)))^+ \, \mathrm{d}\tau = 0. \tag{9}$$

Since all $\Gamma^\mu(T)$ are uniformly bounded, one can choose some $T_1 < \infty$ and define the triple $\{y_p^\mu(\cdot), y_v^\mu(\cdot), \eta^\mu(\cdot)\}$ on the interval $[0, T_1]$ by extending all functions to the latter interval using their constant values at $s = \Gamma^\mu(T)$. Then, these functions will be constants for $\tau \geq \Gamma^\mu(T)$ and continuous on the whole interval. It is then easy to check that the set of triples forms a family of equicontinuous functions, and by the Arzella–Ascoli theorem this family is precompact in the space of continuous functions with the topology of uniform convergence [21]. Therefore, there exists a subsequence of triples and times T_1^μ, such that $T_1^\mu \to \bar{T}_1$, and $\{y_p^\mu(\cdot), y_v^\mu(\cdot), \eta^\mu(\cdot)\}$ that converges to some set of absolutely continuous functions $\{\bar{y}_p(\cdot), \bar{y}_v(\cdot), \bar{\eta}(\cdot)\}$ uniformly on $[0, \bar{T}_1]$.

The importance of this construction follows from

Theorem 3. *Any generalized solution satisfying Definition 1 can be obtained by the representation*

$$\bar{x}_p(t) = \bar{y}_p(\Gamma(t)), \qquad \bar{x}_v(t) = \bar{y}_v(\Gamma(t)),$$

where

$$\Gamma(t) = \inf\{s : \bar{\eta}(s) > t\}.$$

Remark 2. The full proof is omitted, since all steps are standard and can be found in Chapters 4.1 and 5.4 of the monograph [29], though this is the first application of the aforementioned methodology to this class of problem. The principal moment in the proof is the convergence of the corresponding subsequence of $\Gamma^\mu(t) = \inf\{s : \bar{\eta}^\mu(s) > t\}$ to $\Gamma(t)$ at all points of continuity (cf. Lemma 2.5 in [29]).

Moreover, due to the uniform convergence of (y_p^μ, y_v^μ) to $(\bar{y}_p, \bar{v}_p)$ we have

$$\bar{x}_p(t) = \bar{y}_p(\Gamma(t)) = \lim_\mu y_p^\mu(\Gamma(t)), \qquad \bar{x}_v(t) = \bar{y}_v(\Gamma(t)) = \lim_\mu y_v^\mu(\Gamma(t)).$$

Thus, to derive the representation of the generalized solution, the following limits need to be calculated:

$$I_p^r(s) = \lim_\mu \int_0^s \alpha^\mu(\tau) F_p^r(y_p^\mu(\tau), y_v^\mu(\tau), \eta^\mu(\tau)) \, \mathrm{d}\tau, \tag{10}$$

$$I_v^r(s) = \lim_\mu \int_0^s \alpha^\mu(\tau) F_v^{r,\mu}(\tau) \, \mathrm{d}\tau, \tag{11}$$

$$I_v^s(s) = \lim_\mu \int_0^s \alpha^\mu(\tau) \mu F_v^{s,\mu}(\tau) I\{\tau : G^\mu(\tau) \leq 0\} \, \mathrm{d}\tau. \tag{12}$$

Here and below we use the following notations:

$$F_v^{r,\mu}(\tau) = F_v^r(y_p^\mu(\tau), y_v^\mu(\tau), u(\eta^\mu(\tau)), \eta^\mu(\tau)),$$
$$F_v^{s,\mu}(\tau) = F_v^s(y_p^\mu(\tau), y_v^\mu(\tau), \eta^\mu(\tau), \mu),$$
$$G^\mu(\tau) = G(y_p^\mu(\tau), \eta^\mu(\tau)),$$
$$\bar{G}(\tau) = G(\bar{y}_p(\tau), \bar{\eta}(\tau)).$$

3.1. *Convergence properties of α^μ*

Now consider the behavior of the corresponding subsequence $\alpha^\mu(\cdot)$. Since $\eta^\mu(\cdot)$ converges to $\bar{\eta}(\cdot)$ in a strong sense, $\alpha^\mu(\cdot)$ converges weakly to some Lebesgue measurable $\bar{\alpha}(\cdot)$, such that

$$\bar{\alpha}(s) = \dot{\bar{\eta}}(s) \in [0, 1] \quad \text{a.s.}$$

for $s \in [0, \bar{T}_1]$, where $\bar{T}_1 = \Gamma(T)$.

The next proposition shows the convergence properties of the sequence $\alpha^\mu(\cdot)$ as $\mu \uparrow \infty$.

Proposition 3. *There exists a subsequence $\alpha^{\mu}(s)$ satisfying the following relations*

$$\begin{aligned}
&\text{(a) } \lim_{\mu} \alpha^{\mu}(s) I\{s : \bar{\alpha}(s) = 1\} = \bar{\alpha}(s) I\{s : \bar{\alpha}(s) = 1\}, \quad \textit{a.e. on } [0, \bar{T}_1],\\
&\text{(b) } \lim_{\mu} \alpha^{\mu}(s) I\{s : \bar{\alpha}(s) = 0\} = \bar{\alpha}(s) I\{s : \bar{\alpha}(s) = 0\}, \quad \textit{a.e. on } [0, \bar{T}_1],\\
&\text{(c) } \lim_{\mu} I\{s : G(y_p^{\mu}(s), \eta^{\mu}(s)) > 0\} = I\{s : G(\bar{y}_p(s), \bar{\eta}(s)) > 0\};\\
&\text{(d) } \lim_{\mu} I\{s : G(y_p^{\mu}(s), \eta^{\mu}(s)) \le 0\} = I\{\tau : G(\bar{y}_p(s), \bar{\eta}(s)) = 0\}.
\end{aligned} \tag{13}$$

Proof. (a) The existence of a continuous function $I^{\varepsilon}(s) \in [0, 1]$ such that

$$\operatorname{meas}\{s : I^{\varepsilon}(s) \neq I\{s : \bar{\alpha}(s) = 0\}\} \le \varepsilon,$$

for any $\varepsilon > 0$, follows from the C-property of measurable functions [21,31]. Taking into account that $0 < \alpha^{\mu}(s) \le 1$, and

$$I\{s : \bar{\alpha}(s) = 1\}(\alpha^{\mu}(s) - \bar{\alpha}(s)) \le 0,$$

by standard calculations we have

$$-\varepsilon \le \liminf_{\mu} \int_0^{\bar{T}_1} I\{s : \bar{\alpha}(s) = 1\}(\alpha^{\mu}(s) - \bar{\alpha}(s))\, \mathrm{d}s \le 0,$$

and therefore,

$$\lim_{\mu} \int_0^{\bar{T}_1} I\{s : \bar{\alpha}(s) = 1\}(\alpha^{\mu}(s) - \bar{\alpha}(s))\, \mathrm{d}s = 0.$$

Thus, the sequence $I\{s : \bar{\alpha}(s) = 1\}(\alpha^{\mu}(s) - \bar{\alpha}(s))$ converges to 0 in Lebesgue measure, and the existence of the subsequence follows.

The proof of (b) is straightforward; (c) follows from uniform convergence of (y_p^{μ}, η^{μ}) to $(\bar{y}_p, \bar{\eta})$ and continuity of G; (d) follows from (c). □

3.2. Convergence of the integrals in (10) and (11)

Proposition 4. *For $I_p^r(s)$ we have*

$$\begin{aligned}
&\lim_{\mu} \int_0^{s} \alpha^{\mu}(\tau) F_p^r(y_p^{\mu}(\tau), y_v^{\mu}(\tau), \eta^{\mu}(\tau))\, \mathrm{d}\tau = \int_0^{s} \bar{\alpha}(\tau) F_p^r(\bar{y}_p(\tau), \bar{y}_v(\tau), \bar{\eta}(\tau))\, \mathrm{d}\tau,\\
&\lim_{\mu} \int_0^{\Gamma^{\mu}(T)} (1 - \alpha^{\mu}(\tau))(G(y_p^{\mu}(\tau), \eta^{\mu}(\tau)))^{+}\, \mathrm{d}\tau = \int_0^{\Gamma(T)} (1 - \bar{\alpha}(\tau))(G(\bar{y}_p(\tau), \bar{\eta}(\tau)))^{+}\, \mathrm{d}\tau = 0.
\end{aligned} \tag{14}$$

This result follows from the uniform convergence of $(y_p^{\mu}(\tau), y_v^{\mu}(\tau), \eta^{\mu}(\tau))$ to $(\bar{y}_p(\tau), \bar{y}_v(\tau), \bar{\eta}(\tau))$ and continuity of F_p^r and G.

Proposition 5. *Assume that set of vectors*

$$F_v^r(x, y, U, t) = \{l \in R^n : \exists u \in U, \ \textit{such that } l = F_v^r(x, y, u, t)\} \tag{15}$$

is convex and compact $\forall (x, y, t)$. Then, there exists some measurable function $\bar{u}(\cdot) : \bar{u}(t) \in U$, a.e. on $[0, \bar{T}_1]$, such that for (11) we have

$$I_v^r(s) = \lim_{\mu} \int_0^{s} \alpha^{\mu}(\tau) F_v^r(y_p^{\mu}(\tau), y_v^{\mu}(\tau), u(\eta^{\mu}(\tau)), \eta^{\mu}(\tau))\, \mathrm{d}\tau = \int_0^{s} \bar{\alpha}(\tau) F_v^r(\bar{y}_p(\tau), \bar{y}_v(\tau), \bar{u}(\tau), \eta(\tau))\, \mathrm{d}\tau, \tag{16}$$

for $\forall s \in [0, \bar{T}_1]$.

Moreover, the function $\bar{u}(\cdot)$ can be chosen to be Borel measurable.

If $u(\cdot)$ is continuous, then $\bar{u}(s) = u(\bar{\eta}(s))$.

The proof is standard and based on Phillipov's measurable selection lemma [24].

The construction of (12) is more subtle. Function $\Gamma(\cdot)$ admits [29] the representation

$$\Gamma(t) = t + V^c(t) + \sum_{\xi \le t} \Delta\Gamma(\xi),$$

where V^c is continuous and non-decreasing function, and $D_\Gamma = \{\xi : \Delta\Gamma(\xi) > 0\}$ is the countable (finite or not) set of all jump points of Γ. The interval $[0, \bar{T}_1] = [0, \Gamma(T)]$ can be represented up to negligible subsets as a union of disjoint sets

$$[0, \bar{T}_1] = \{s : \bar{\alpha}(s) = 1\} \bigcup_{\xi \le T} [\Gamma(\xi-), \Gamma(\xi)] \bigcup \{s : \eta(s) \in \operatorname{supp} V^c\}.$$

Therefore, we have the following representation of integral (12)

$$\begin{aligned} I_v^s(s) = {} & \int_0^s \alpha^\mu(\tau)\mu F_v^\mu I\{G^\mu \le 0\} I\{\bar{\alpha}(\tau) = 1\}\tau + \sum_{\xi \le \bar{\eta}(s)} \int_{\Gamma(\xi-)}^{\Gamma(\xi)} \alpha^\mu(\tau)\mu F_v^\mu I\{G^\mu \le 0\}\tau \\ & + \int_0^s \alpha^\mu(\tau)\mu F_v^\mu I\{G^\mu \le 0\} I\{\tau : \eta(\tau) \in \operatorname{supp} V^c\}\tau. \end{aligned} \tag{17}$$

On each interval $[\Gamma(\xi-), \Gamma(\xi)]$ we have $\bar{\alpha}(s) = 0$ and $\bar{\eta}(s) = \xi$. According to Theorem 3, the jumps of the generalized solution occur only at points $\xi \in D_\Gamma$ and are equal to

$$\Delta\bar{x}_p(\xi) = \bar{y}_p(\Gamma(\xi)) - \bar{y}_p(\Gamma(\xi-)), \qquad \Delta\bar{x}_v(\xi) = \bar{y}_v(\Gamma(\xi)) - \bar{y}_v(\Gamma(\xi-)).$$

However, as follows from (7) and Proposition 3, $\bar{x}_p(\cdot)$ is actually continuous at $\xi \in D_\Gamma$. At the same time

$$\Delta\bar{x}_v(\xi) = \bar{y}_v(\Gamma(\xi)) - \bar{y}_v(\Gamma(\xi-)) = \lim_\mu \int_{\Gamma(\xi-)}^{\Gamma(\xi)} \alpha^\mu(\tau)\mu F_v^\mu I\{G^\mu \le 0\}\tau.$$

4. Representation of generalized solution jumps through the use of space–time transformation

The method of representation of a single jump is described in [28] in the case of controlled singularities, and here the same assumptions are made concerning the function F^s. For the sake of simplicity, consider an uncontrolled case.

Assumption 4. Assume that for any given (x_p, t) such that $G(x_p, t) = 0$ there exists

$$\bar{F}_v(y_p, y_v, s, x_p, t) = \lim_{\mu \uparrow \infty} \mu^{1/2} F_v^s \left(\frac{y_p - x_p}{\mu^{1/2}} + x_p, y_v, t + s\mu^{-1/2}, \mu \right), \tag{18}$$

where the convergence to the limit is uniform on any compact set of variables (y_p, y_v, s, x_p, t). Then, due to the condition (3) function $\bar{F}_v$ is continuous in all variables and Lipschitz in y_p, y_v. Here (x_p, t) are considered as parameters.

Suppose also that the system of equations

$$\begin{aligned} \dot{y}_p(s) &= F_p^r(x_p(\tau), y_v(s), t), \\ \dot{y}_v(s) &= \bar{F}_v(y_p(s), y_v(s), s, x_p, t), \end{aligned} \tag{19}$$

referred to in [6] as the controlled infinitesimal dynamics equation, with initial condition $(x_p, x_v) = (x_p(t), x_v(t))$ such that

$$G(x_p, t) = 0, \qquad G'_{x_p} F_p^r + G'_t \Big|_{(x_p, x_v, t)} < 0, \tag{20}$$

has a unique solution for $s \in [t, \infty)$. Assume also that there exists the exit time

$$s^*(x_p, x_v, t) = \inf \left\{ s > t : \begin{array}{l} G'_{x_p}(x_p, t)(y_p(s) - x_p) + G'_t(x_p, t)(s - t) = 0, \\ G'_{x_p}(x_p, t) F_p^r(x_p, y_v(s), t) + G'_t(x_p, t) > 0 \end{array} \right\} \tag{21}$$

such that on the interval (t, s^*)

$$G'_{x_p}(x_p, t)(y_p(s) - x_p) + G'_t(x_p, t)(s - t) < 0. \tag{22}$$

The following theorem gives the jump representation of any generalized solution at point $\xi \in D_\Gamma$. We point out that this representation is still valid for the case of infinite set D_Γ and therefore covers the case of Zeno paths.

Theorem 5. *Let*

$$\varphi(s, x_p, x_v, t) = \begin{pmatrix} \varphi_p(s, x_p, x_v, t) \\ \varphi_v(s, x_p, x_v, t) \end{pmatrix} \tag{23}$$

be the general solution of (19) for $s \ge t$ with initial conditions

$$y_p(t) = x_p, \qquad y_v(t) = x_v, \tag{24}$$

satisfying (20). Then, the jump of the variable $\bar{x}_v(\cdot)$ at any point $\xi \in D_\Gamma$ such that $(\bar{x}_p(\xi), \bar{x}_v(\xi-))$ satisfies (20) is described by the relation

$$\Delta\bar{x}_v(\xi) = \varphi_v(s^*, x_p(\xi), x_v(\xi-), \xi) - \bar{x}_v(\xi-), \tag{25}$$

while $\bar{x}_p(t)$ is still continuous at ξ.

Proof. Let $\xi \in D_\Gamma$ and consider the interval $[\Gamma(\xi-), \Gamma(\xi)]$. Then, for $s \ge \Gamma(\xi-)$ introduce the representation

$$y_p^\mu(s) = y_p^\mu(\Gamma(\xi-)) + \mu^{-1/2}[y_p^{\mu,\xi}(s) - y_p^\mu(\Gamma(\xi-))] \tag{26}$$

and substitute it into the system (6)–(8). Thereby, we obtain the following system of equations for the triple $(y_p^{\mu,\xi}(s), y_v^\mu(s), \eta^\mu(s))$:

$$y_p^{\mu,\xi}(s) = y_p^\mu(\Gamma(\xi-)) + \int_{\Gamma(\xi-)}^{s} \frac{F_p^r\left(y_p^\mu(\Gamma(\xi-)) + \frac{y_p^{\mu,\xi}(\tau) - y_p^\mu(\Gamma(\xi-))}{\mu^{1/2}}, y_v^\mu(\tau), \eta^\mu(\tau)\right)}{\mu^{-1/2} + \mu^{1/2}|F_v^{s,\mu}(\tau)| I\{G^\mu(\tau) \le 0\}} \, \mathrm{d}\tau, \tag{27}$$

$$\begin{aligned} y_v^\mu(s) \;=\; & y_v^\mu(\Gamma(\xi-)) + \int_{\Gamma(\xi-)}^{s} \frac{\mu^{1/2} F_v^{s,\mu}(\tau) I\{G^\mu(\tau) \le 0\}}{\mu^{-1/2} + \mu^{1/2}|F_v^{s,\mu}(\tau)| I\{G^\mu(\tau) \le 0\}} \, \mathrm{d}\tau \\ & + \int_{\Gamma(\xi-)}^{s} \frac{\mu^{-1/2} F_v^r\left(y_p^\mu(\Gamma(\xi-)) + \frac{y_p^{\mu,\xi}(\tau) - y_p^\mu(\Gamma(\xi-))}{\mu^{1/2}}, y_v^\mu(\tau), u(\eta^\mu(\tau)), \eta^\mu(\tau)\right)}{\mu^{-1/2} + \mu^{1/2}|F_v^{s,\mu}(\tau)| I\{G^\mu(\tau) \le 0\}} \, \mathrm{d}\tau, \end{aligned} \tag{28}$$

$$\eta^\mu(s) = \eta^\mu(\Gamma(\xi-)) + \int_{\Gamma(\xi-)}^{s} \frac{\mu^{-1/2}}{\mu^{-1/2} + \mu^{1/2}|F_v^{s,\mu}(\tau)| I\{G^\mu(\tau) \le 0\}} \, \mathrm{d}\tau. \tag{29}$$

Now, we make a time change using the increasing function

$$\zeta^\mu(s) = \Gamma(\xi-) + \int_{\Gamma(\xi-)}^{s} \frac{1}{\mu^{-1/2} + \mu^{1/2}|F_v^{s,\mu}(\tau)| I\{G^\mu(\tau) \le 0\}} \, \mathrm{d}\tau,$$

which has an inverse, namely $s^{-1}(\zeta)$ for $\zeta \in [\Gamma(\xi-), \Gamma(\xi)]$. Now, one can rewrite the system (27)–(29) for the triple

$$(\tilde{y}_p^{\mu,\xi}(\zeta), \tilde{y}_v^\mu(\zeta), \tilde{\eta}^\mu(\zeta)) = (y_p^{\mu,\xi}(s^{-1}(\zeta)), y_v^\mu(s^{-1}(\zeta)), \eta^\mu(s^{-1}(\zeta)))$$

for $\zeta \ge \Gamma(\xi-)$. We have

$$\tilde{y}_p^{\mu,\xi}(\zeta) = y_p^\mu(\Gamma(\xi-)) + \int_{\Gamma(\xi-)}^{\zeta} F_p^r\left(y_p^\mu(\Gamma(\xi-)) + \frac{\tilde{y}_p^{\mu,\xi}(\tau) - y_p^\mu(\Gamma(\xi-))}{\mu^{1/2}}, \tilde{y}_v^\mu(\tau), \tilde{\eta}^\mu(\tau)\right) \mathrm{d}\tau, \tag{30}$$

$$\begin{aligned} \tilde{y}_v^\mu(\zeta) \;=\; & y_v^\mu(\Gamma(\xi-)) + \int_{\Gamma(\xi-)}^{\zeta} \mu^{1/2} F_v^{s,\mu}(\tau) I\{G^\mu(\tau) \le 0\} \, \mathrm{d}\tau \\ & + \int_{\Gamma(\xi-)}^{\zeta} \mu^{-1/2} F_v^r\left(\tilde{y}_p^\mu(\Gamma(\xi-)) + \frac{\tilde{y}_p^{\mu,\xi}(\tau) - y_p^\mu(\Gamma(\xi-))}{\mu^{1/2}}, \tilde{y}_v^\mu(\tau), u(\tilde{\eta}^\mu(\tau)), \tilde{\eta}^\mu(\tau)\right) \mathrm{d}\tau. \end{aligned} \tag{31}$$

$$\tilde{\eta}^{\mu}(\zeta) = \eta^{\mu}(\Gamma(\xi-)) + \mu^{-1/2}[\zeta - \Gamma(\xi-)]. \tag{32}$$

Taking into account Assumption 4, we obtain the following representation for the function $\mu^{1/2}F_v^{s,\mu}(\tau)$ under the first integral in (31)

$$\mu^{1/2}F_v^{s,\mu}(\tau) = \bar{F}_v^{s}(\tilde{y}_p^{\mu,\xi}(\tau), \tilde{y}_v^{\mu}(\tau), \tilde{\eta}^{\mu}(\tau), y_p^{\mu}(\Gamma(\xi-)), \eta^{\mu}(\Gamma(\xi-))) + O(\mu), \tag{33}$$

where $O(\mu) \to 0$ when $\mu \to \infty$.

Moreover,

$$G^{\mu}(\tau) = G(y_p^{\mu}(\Gamma(\xi-)), \eta_p^{\mu}(\Gamma(\xi-))) + \mu^{-1/2}\{G'_{x_p}[\tilde{y}_p^{\mu,\xi}(\tau) - y_p^{\mu}(\Gamma(\xi-))] + G'_t[\zeta - \Gamma(\xi-)]\} + o(1/\mu^{1/2}), \tag{34}$$

where partial derivatives have to be calculated at the point $y_p^{\mu}(\Gamma(\xi-)), \eta_p^{\mu}(\Gamma(\xi-))$ and

$$\mu^{1/2}o(\mu^{-1/2}) \to 0, \quad \text{if } \mu \to \infty.$$

Then, we observe that if $\mu \to \infty$, then

$$\begin{aligned} &y_p^{\mu}(\Gamma(\xi-)) \to \bar{y}_p(\Gamma(\xi-)), \qquad y_v^{\mu}(\Gamma(\xi-)) \to \bar{y}_v(\Gamma(\xi-)), \\ &\eta^{\mu}(\Gamma(\xi-)) \to \bar{\eta}(\Gamma(\xi-)) = \xi, \end{aligned}$$

and for given $(\tilde{y}_p^{\mu,\xi}, \tilde{y}_v^{\mu}, \tilde{\eta}^{\mu})$ functions under the integral in (30) and the function under the first integral in (31) tend to

$$F_p\left(\bar{y}_p(\Gamma(\xi-)), \tilde{y}_v^{\mu}, \xi\right)$$

and

$$\bar{F}_v(\tilde{y}_p^{\mu,\xi}, \tilde{y}_v^{\mu}, \tilde{\eta}^{\mu}, \bar{y}_p(\Gamma(\xi-)), \xi),$$

respectively.

According to Proposition 3, the chosen subsequence $\alpha^{\mu}(s) \to 0$ a.e. on the interval $[\Gamma(\xi-), \Gamma(\xi)]$, therefore,

$$\mu|F^{s,\mu}(\tau)|I\{G^{\mu}(\tau) \le 0\} \to \infty, \quad \text{a.e. on } [\Gamma(\xi-), \Gamma(\xi)],$$

which implies that

$$\lim_{\mu} I\{G^{\mu}(\tau) \le 0\} = 1, \quad \text{a.e. on } [\Gamma(\xi-), \Gamma(\xi)].$$

Moreover, from (34)

$$\mu^{1/2}G^{\mu}(\tau) = G'_{x_p}[\tilde{y}_p^{\mu,\xi}(\tau) - \bar{y}_p(\Gamma(\xi-))] + G'_t[\tau - \Gamma(\xi-)] + \mu^{1/2}o(\mu^{-1/2}). \tag{35}$$

Comparing the system (19), (30) and (31), and taking into account (33) and (34), we observe that the *rhs* of system (19) is the limit of the *rhs* of system (30) and (31). Then, assuming that $(\bar{y}_p(\Gamma(\xi-)), \bar{y}_v(\Gamma(\xi-)), \xi)$ satisfies (20), and applying the continuous dependence of the solution of differential equations on initial conditions and the *rhs* [36], we establish the convergence of the sequence $(\tilde{y}_p^{\mu,\xi}(\zeta), \tilde{y}_v^{\mu}(\zeta), \tilde{\eta}^{\mu}(\zeta))$ to the solution of system (19) uniformly on the interval $[\Gamma(\xi-), \Gamma(\xi-) + s^*]$, while at the same time the $\lim_{\mu} \mu^{1/2}G^{\mu}(\zeta) < 0$ due to Assumption 4. This implies the representation

$$\bar{y}_v(\Gamma(\xi)) = \varphi_v(s^*, \bar{y}_p(\Gamma(\xi)), \bar{y}_v(\Gamma(\xi-)), \eta(\Gamma(\xi))) - \bar{y}_v(\Gamma(\xi-)).$$

Consequently, after the inverse time-transformation, the representation (25) follows (for full details see the proof for single jump representations in the forthcoming full article [6]). The discussion of this jump representation and examples can be found in [28]. □

5. Representation of continuous motion along the boundary of constraint

This mode of motion arises if the system hits the constraint boundary, but the time derivative of deformation stays equal to zero ($\dot{G}=0$). Thus, we consider a contact force of the type (4), which in this case is formally equal to zero. However, if the constraint violation does take place, then a small boundary deformation can create a significant force, causing in the limit a perceptible contribution to the motion. According to the representation (4), consider the case when the generalized solution satisfies the condition

$$G(\bar{x}_p(t),t)=0, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{F_p^r} G(x_p,t) = G'_{x_p}F_p^r + G'_t\Big|_{(\bar{x}_p,\bar{x}_v,t)} = 0. \tag{36}$$

In the vicinity of such a point, consider system (1) with an equation for variable $G(x_p^\mu(t),t)$, which has the form

$$\begin{aligned}
&\dot{x}_p^\mu(t) = F_p^r(x_p^\mu(t), x_v^\mu(t), t),\\
&\dot{x}_v(t) = F_v^r(x_p^\mu(t), x_v^\mu(t), u(t), t) + \mu F_v^{s,\mu}(t) I\{t : G^\mu(t) \le 0\},\\
&\ddot{G}^\mu(t) \;=\; D(x_p^\mu(t), x_v^\mu(t), u(t), t) + \Big[\tilde{A}(x_p^\mu(t), x_v^\mu(t), t)\mu G^\mu(t) + \tilde{B}(x_p^\mu(t), x_v^\mu(t), t)\mu^{1/2}\dot{G}^\mu(t)\Big]\\
&\qquad\quad \times I\{t : G^\mu(t) \le 0\}
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
&\mu F_v^{s,\mu}(t) = A(x_p^\mu(t),t)\mu G^\mu(t) + B(x_p^\mu(t),t)\mu^{1/2}\dot{G}^\mu(t),\\
&D(x_p^\mu(t), x_v^\mu(t), u(t), t) \;=\; ((F_p^r)^* \vdots 1)\begin{pmatrix} G''_{x_p x_p} & G''_{x_p t}\\ (G''_{x_p t})^* & G''_{tt}\end{pmatrix}\begin{pmatrix} F_p^r \\ 1\end{pmatrix}\Bigg|_{(x_p^\mu(t), x_v^\mu(t), u(t), t)}\\
&\qquad\qquad + G'_{x_p}\Big[(F_p^r)'_{x_p}\dot{x}_p + (F_p^r)'_{x_v}F_v^r(x_p, x_v, u, t) + (F_p)'_t\Big]\Big|_{(x_p^\mu(t), x_v^\mu(t), u(t), t)}.
\end{aligned}$$

The notation $((F_p^r)^* \vdots 1)$ is used for the block row vector with the last $(n+1)$-th component equal to 1. Therefore, to derive the equation of motion for the generalized solution, the uncertainty of expressions $\mu G^\mu(t)$ and $\mu^{1/2}\dot{G}(t)$ in (37) needs to be cleared up, namely, we have to determine the limits

$$\lim_{\mu} \mu G^\mu(t), \quad \text{and} \quad \lim_{\mu} \mu^{1/2}\dot{G}(t).$$

This will be addressed on the basis of the following propositions concerning a singularly perturbed second order equation.

Proposition 6. *Consider the differential equation*

$$\ddot{G}^\mu(\tau) = U(\tau) - A\mu G^\mu(\tau) - B\mu^{1/2}\dot{G}^\mu(\tau), \quad \tau \ge t, \tag{38}$$

with initial conditions

$$G^\mu(t) = 0, \qquad \dot{G}^\mu(t) = 0,$$

where $U(\cdot)$ is uniformly bounded and has a limit from the left everywhere on $\tau > t$, and

$$A > 0, \qquad B > 0, \qquad A - (B/2)^2 > 0.$$

Then, for $\tau > t$

$$\lim_{\mu\uparrow\infty} \mu G^\mu(\tau) = \frac{U(\tau-)}{A}, \qquad \lim_{\mu\uparrow\infty} \mu^{1/2}\dot{G}^\mu(\tau) = 0, \qquad \lim_{\mu\uparrow\infty} \ddot{G}^\mu(\tau) = 0.$$

Remark 3. Here we consider $t \in [0, T]$ as a parameter.

Proof. The solution of (38) can be represented as follows

$$G^{\mu}(\tau)=\int_{t}^{\tau}U(s)\mathrm{e}^{-\lambda^{\mu}(\tau-s)}\frac{\sin\omega^{\mu}(\tau-s)}{\omega^{\mu}}\,\mathrm{d}s,$$

where

$$\begin{aligned}&\lambda^{\mu}=\mu^{1/2}\lambda, &&\omega^{\mu}=\mu^{1/2}\omega,\\ &\lambda=B/2, &&\omega=\sqrt{A-(B/2)^2}.\end{aligned}\tag{39}$$

After the time transformation $u=\mu^{1/2}(\tau-s)$ we obtain the expression

$$\mu G^{\mu}(\tau)=\int_{0}^{\mu^{1/2}(\tau-t)}U(\tau-u\mu^{-1/2})\mathrm{e}^{-\lambda u}\frac{\sin\omega u}{\omega}\,\mathrm{d}u,$$

and the estimate

$$\begin{aligned}\left|\mu G^{\mu}(\tau)-U(\tau-)\int_{0}^{\infty}\mathrm{e}^{-\lambda u}\frac{\sin\omega u}{\omega}\,\mathrm{d}u\right| &\le \int_{0}^{\infty}|U(\tau-u\mu^{-1/2})-U(\tau-)|\mathrm{e}^{-\lambda u}\left|\frac{\sin\omega u}{\omega}\right|\mathrm{d}u\\ &\quad+\int_{\mu^{1/2}(\tau-t)}^{\infty}|U(\tau-)|\mathrm{e}^{-\lambda u}\left|\frac{\sin\omega u}{\omega}\right|\mathrm{d}u.\end{aligned}\tag{40}$$

Since $U(\cdot)$ is bounded and the expression above has left limits at any $\tau>t$, we obtain the limit

$$\lim_{\mu\uparrow\infty}\mu G^{\mu}(\tau)=U(\tau-)\int_{0}^{\infty}\mathrm{e}^{-\lambda u}\frac{\sin\omega u}{\omega}\,\mathrm{d}u=\frac{U(\tau-)}{\lambda^2+\omega^2}=\frac{U(\tau-)}{A}.$$

With the help of the same arguments we obtain

$$\lim_{\mu\uparrow\infty}\mu^{1/2}\dot{G}^{\mu}(\tau)=U(\tau-)\int_{0}^{\infty}\mathrm{e}^{-\lambda u}\left[\cos\omega u-\frac{\lambda}{\omega}\sin\omega u\right]\mathrm{d}u=0.$$

The last assertion is evident. □

Remark 4. If function $U(\cdot)\in C^1(t,\infty)$ and has uniformly bounded derivative, then convergence to the limits is uniform for $\tau>t$. This conclusion follows immediately from (40) if the difference estimate under the first integral is given by $\sup_{\tau>t}|U_t'(\tau)|\,|u|\mu^{-1/2}$.

Proposition 7. *Assume that in the differential equation*

$$\ddot{G}^{\mu}(\tau)=U^{\mu}(\tau)-A^{\mu}(\tau)\mu G^{\mu}(\tau)-B^{\mu}(\tau)\mu^{1/2}\dot{G}^{\mu}(\tau),\quad \tau\ge t,\tag{41}$$

with initial conditions

$$G^{\mu}(t)=0,\qquad |\mu^{1/2}\dot{G}^{\mu}(t)|\to 0,$$

$U^{\mu}(\cdot)$, $A^{\mu}(\cdot)$, $B^{\mu}(\cdot)$ *are uniformly bounded and continuous at* $\tau=t$,

$$A(t)=\lim_{\tau\downarrow t}\lim_{\mu\uparrow\infty}A^{\mu}(\tau)>0,\qquad B(t)=\lim_{\tau\downarrow t}\lim_{\mu\uparrow\infty}B^{\mu}(\tau)>0,\qquad A(t)-(B(t)/2)^2>0,$$

and

$$U(t)=\lim_{\tau\downarrow t}\lim_{\mu\uparrow\infty}U^{\mu}(\tau)<0.$$

Then

$$\lim_{\tau\downarrow t}\lim_{\mu\uparrow\infty}\mu G^{\mu}(\tau)=\frac{U(t)}{A(t)}<0,\qquad \lim_{\tau\downarrow t}\lim_{\mu\uparrow\infty}\mu^{1/2}\dot{G}^{\mu}(\tau)=0,\qquad \lim_{\tau\downarrow t}\lim_{\mu\uparrow\infty}\ddot{G}^{\mu}(\tau)=0.$$

Proof. The proof can be obtained by using the space–time transformation $Z^{\mu}(s) = \mu G^{\mu}(t + s\mu^{-1/2})$, which gives the following equation

$$\begin{aligned} &\ddot{Z}^{\mu}(s) = U^{\mu}(t + \mu^{-1/2}s) - A^{\mu}(t + \mu^{-1/2}s)Z^{\mu}(s) - B^{\mu}(t + \mu^{-1/2}s)\dot{Z}^{\mu}(s), \\ &Z^{\mu}(0) = 0, \qquad \dot{Z}^{\mu}(0) = \mu^{1/2}\dot{G}^{\mu}(t), \end{aligned} \tag{42}$$

where t is considered as a parameter. Then, by comparing its solution with that of the equation

$$\ddot{\tilde{Z}}^{\mu}(s) = U(t) - A(t)\tilde{Z}^{\mu}(s) - B(t)\dot{\tilde{Z}}^{\mu}(s), \tag{43}$$

with initial conditions

$$\tilde{Z}^{\mu}(0) = 0, \qquad \dot{\tilde{Z}}^{\mu}(0) = \mu^{1/2}\dot{G}^{\mu}(t),$$

we find that $Z^{\mu}(s)$ converges to $\tilde{Z}^{\mu}(s)$ in each interval $[0, S]$ uniformly [15,36]. However, with the notation (39),

$$\tilde{Z}^{\mu}(s) = \dot{\tilde{Z}}^{\mu}(0)\mathrm{e}^{-\lambda s}\frac{\sin \omega s}{\omega} + \int_0^s U(t)\mathrm{e}^{-\lambda(s-\tau)}\frac{\sin \omega(s-\tau)}{\omega}\mathrm{e}^{-\lambda(s-\tau)}\,\mathrm{d}\tau,$$

and

$$\tilde{Z}^{\mu}(s) \to \frac{U(t)}{\lambda^2 + \omega^2} = \frac{U(t)}{A(t)} < 0,$$

when $\mu, s \to \infty$. If we first choose a sufficiently large S and then make $\mu \to \infty$, we obtain the result after the inverse space–time transformation. □

Remark 5. This result means that if the initial velocity of deformation is sufficiently small, the reaction force becomes stabilized, and the velocity of deformation does not play any role in the system motion. However, this is true even in more general cases. Concerning our problem, we have to consider the equation

$$\ddot{G}^{\mu}(\tau) = U^{\mu}(\tau) - [A^{\mu}(\tau)\mu G^{\mu}(\tau) + B^{\mu}(\tau)\mu^{1/2}\dot{G}^{\mu}(\tau)]I\{G^{\mu}(\tau) \le 0\}, \quad \tau \ge t. \tag{44}$$

After the same space–time transformation, we obtain for variable $Z^{\mu}(s)$ the equation

$$\ddot{Z}^{\mu}(s) = U^{\mu}(t + \mu^{-1/2}s) - [A^{\mu}(t + \mu^{-1/2}s)Z^{\mu}(s) + B^{\mu}(t + \mu^{-1/2}s)\dot{Z}^{\mu}(s)]I\{Z^{\mu}(s) \le 0\}, \tag{45}$$

and for reference variable $\tilde{Z}^{\mu}(s)$, the equation

$$\ddot{\tilde{Z}}^{\mu}(s) = U(t) - [A(t)\tilde{Z}^{\mu}(s) + B(t)\dot{\tilde{Z}}^{\mu}(s)]I\{Z^{\mu}(s) \le 0\}. \tag{46}$$

Proposition 7 shows that for sufficiently small $Z^{\mu}(0)$, the solution remains in the area $Z^{\mu} < 0$ for all $s > 0$ and tends to the limit. Generally, we can guarantee only the estimate $|\dot{Z}^{\mu}(0)| \le C < \infty$. Meanwhile, it is rather easy to calculate an explicit solution of (46) and observe that it admits only a finite number of exits from area $Z^{\mu}(s) \le 0$ and this number is restricted by some value depending on constant C only. Indeed, after each exit, the value of $\dot{Z}^{\mu}$ decreases by a factor $k < 1$, and, therefore, after some number of exits, the initial velocity becomes sufficiently small to satisfy the conditions of Proposition 7. This number of exits and returns takes some time, but this time is uniformly bounded. So, after this time elapses, we revert to the case of the previous result. The full proof of this result is lengthy and will be presented in a separate publication. However, an example will be given that demonstrates the relevant behavior in a simple case.

Returning to the representation of motion described by system (37), the following result is seen to hold.

Theorem 6. *Assume that generalized solution* $(\bar{x}_p(t), \bar{x}_v(t))$ *satisfies* (36) *and functions* $D, \tilde{A}, \tilde{B}$ *in* (37) *are continuous in all variables and satisfy the conditions*

$$\begin{aligned} &D(\bar{x}_p(t), \bar{x}_v(t), u(t), t) < 0, \\ &\tilde{A}(\bar{x}_p(t), \bar{x}_v(t), t) < 0, \end{aligned}$$

and

$$\tilde{B}(\bar{x}_p(t), \bar{x}_v(t), t) < 0.$$

Then

$$\lim_{\mu\uparrow\infty} \mu G^{\mu}(t) = -\frac{D(\bar{x}_p(t), \bar{x}_v(t), u(t), t)}{\tilde{A}(\bar{x}_p(t), \bar{x}_v(t), t)} = F^{\mathrm{sm}}(\bar{x}_p(t), \bar{x}_v(t), u(t), t), \quad \text{and} \quad \lim_{\mu\uparrow\infty} \mu G^{\mu}(t) = 0$$

at all points of continuity of the control $u(\cdot)$.

Remark 6. We use the superscript *sm* — *sliding mode* to underline that this representation of force acts during the sliding along the surface.

Remark 7. The above conditions mean:

- the constraint is repulsive in the sense given in [6],
- at the point where system touches the constraint, the external force pushes the system into the inhibited area.

Moreover, we point out that constraint violation in the case of smooth motion along the boundary has the order ($\sim\mu^{-1}$), while in the case of strict impact this order is ($\sim\mu^{-1/2}$) (see Eq. (26)).

6. Representation of generalized solution by nonlinear differential equation with a measure

Now, we are ready to derive the equation for the pair $(\bar{x}_p(t), \bar{x}_v(t))$.

Theorem 7. (a) *Assume that conditions of Proposition 5 and Theorems 5 and 6 hold, and* $(\bar{x}_p(t), \bar{x}_v(t))$ *is the generalized solution of controlled system (1) satisfying unilateral constraints*

$$G(\bar{x}_p(t), t) \geq 0, \quad \forall t \in [0, T].$$

Then there exist:

(1) *Borel measurable control* $\bar{u}(\cdot)$, *such that* $\bar{u}(t) \in U$, *a.e. on* $[0, T]$;
(2) *non-negative measure* $V(\mathrm{d}t)$ *with representation*

$$V([0,t]) = V^c([0,t]) + \sum_{\xi\leq t} V(\{\xi\}),$$

satisfying the complementary-slackness constraint

$$\int_0^T G(\bar{x}_p(t), t)\mathrm{d}V(t) = 0$$

such that generalized solution $(\bar{x}_p(t), \bar{x}_v(t))$ *satisfies the following system of equations with a measure*

$$\bar{x}_p(t) = x_p(0) + \int_0^t F_p^r(\bar{x}_p(\tau), \bar{x}_v(\tau), \tau)\,\mathrm{d}\tau, \tag{47}$$

$$\begin{aligned} \bar{x}_v(t) \;=\; & x_v(0) + \int_0^t F_v^r(\bar{x}_p(\tau), \bar{x}_v(\tau), \bar{u}(\tau), \tau)\,\mathrm{d}\tau + \int_0^t A(\bar{x}_p(\tau), \tau) F^{\mathrm{sm}}(\bar{x}_p(\tau), \bar{x}_v(\tau), \bar{u}(\tau), \tau)\mathrm{d}V^c(\tau) \\ & + \sum_{\xi\leq t} [\varphi_v(s^*, \bar{x}_p(\xi), \bar{x}_v(\xi-), \xi) - \bar{x}_v(\xi-)] I\{G(\bar{x}_p(\xi), \xi) = 0, \dot{G}(\bar{x}_p(\xi), \bar{x}_v(\xi-), \xi) < 0\}. \end{aligned} \tag{48}$$

(b) *Starting from any point* τ *where*

$$G(\bar{x}_p(\tau), \tau) = 0, \qquad \dot{G}(\bar{x}_p(\tau), \bar{x}_v(\tau), \tau) = 0,$$

provided the Lipschitz property of function F^{sm} *in* $(\bar{x}_p, \bar{x}_v)$ *holds, the generalized solution is uniquely extensible until the time*

$$t^* = \inf\left\{ t > \tau : \frac{D(t)}{\tilde{A}(t)} \geq 0 \right\},$$

if $t^* > \tau$.

Proof. (a) At all points of continuity we have

$$\bar{x}_p(t) = \lim_{\mu\uparrow\infty} x_p^{\mu}(t) = \lim_{\mu\uparrow\infty} y_p^{\mu}(\Gamma^{\mu}(t)) = \lim_{\mu\uparrow\infty} y_p^{\mu}(\Gamma(t)),$$

since $\Gamma^{\mu}(t) \to \Gamma(t)$ at all points of continuity and the family $y^{\mu}(s)$ is equicontinuous on $[0, T_1]$. The same relation is valid for $\bar{x}_p(t)$. By applying the inverse time transformation $t = \eta(\tau)$ to the first integral in (14), we obtain the first equation of (47), and with the aid of the same time-transformation in the integral (16) we obtain the first integral in (48). To calculate the remaining terms we have to calculate the limit of the second integral in (8). Thus, we have

$$\int_0^{\Gamma(t)} \alpha^{\mu}(\tau)F_v^{s,\mu}(\tau)I\{G^{\mu}(\tau) \le 0\}\,\mathrm{d}\tau = \int_0^{\Gamma(t)} \alpha^{\mu}(\tau)F_v^{s,\mu}(\tau)I\{G^{\mu}(\tau) \le 0\}I\{\bar{G}(\tau) > 0\}\,\mathrm{d}\tau$$
$$+ \int_0^{\Gamma(t)} \alpha^{\mu}(\tau)F_v^{s,\mu}(\tau)I\{G^{\mu}(\tau) \le 0\}I\{\bar{G}(\tau) = 0\}\,\mathrm{d}\tau = I_1^{\mu}(t) + I_2^{\mu}(t).$$

Due to Proposition 3

$$I_1^{\mu}(t) \to 0.$$

Further,

$$\{\bar{G}(\tau) = 0\} = \{\tau : \eta(\tau) \in \operatorname{supp} V^c\} \bigcup_{\xi \le \Gamma(t)} [\Gamma(\xi-), \Gamma(\xi)].$$

By Theorem 5

$$\lim_{\mu} \int_{\Gamma(\xi-)}^{\Gamma(\xi)} \alpha^{\mu}(\tau)\mu F_v^{s,\mu}(\tau)I\{G^{\mu}(\tau) \le 0\}\,\mathrm{d}\tau = [\varphi_v(s^*, \bar{x}_p(\xi), \bar{x}_v(\xi-), \xi) - \bar{x}_v(\xi-)]$$
$$\times I\{G(\bar{x}_p(\xi), \xi) = 0, \dot{G}(\bar{x}_p(\xi), \bar{x}_v(\xi-), \xi) < 0\}.$$

Next, we observe that

$$I\{\eta(\tau) \in \operatorname{supp} V^c\} = I\{\bar{G}(\tau) = 0, \dot{\bar{G}}(\tau) = 0\} = I\left\{\{\tau : 0 \le \bar{\alpha}(\tau) < 1\} \setminus \bigcup_{\xi \le T}[\Gamma(\xi-), \Gamma(\xi)]\right\}.$$

Then, by transforming I_2^{μ} through the use of the inverse time transformation $\tau = \eta^{\mu}(s)$, applying Theorem 6 to calculate the integral over the set $\{\bar{G}(\tau) = 0, \dot{\bar{G}}(\tau) = 0\}$, and defining the absolutely continuous measure $V^c(\mathrm{d}t)$ by relation

$$V^c([0, T]) = \int_0^{\Gamma(t)} \bar{\alpha}(\tau)I\{\bar{G}(\tau) = 0, \dot{\bar{G}}(\tau) = 0\}\,\mathrm{d}\tau = \int_0^t I\{\tau : G(\bar{x}_p(\tau), \tau) = 0, \dot{G}(\bar{x}_p(\tau), \bar{x}_v(\tau), \tau) = 0\}\,\mathrm{d}\tau \tag{49}$$

and the control $\bar{u}(\cdot)$ using the procedure of measurable selection, we complete the proof of assertion (a) of the theorem.

(b) As follows from Theorem 6, on the interval $[\tau, t^*]$ the generalized solution satisfies the systems of ordinary differential equations

$$\begin{aligned}
&\dot{\bar{x}}_p(t) = F_p^r(\bar{x}_p(t), \bar{x}_v(t), t),\\
&\dot{\bar{x}}_v(t) = F_v^r(\bar{x}_p(t), \bar{x}_v(t), u(t), t) + A(\bar{x}_p(t), t)F^{\mathrm{sm}}(\bar{x}_p(t), \bar{x}_v(t), u(t), t)I\{t : \bar{G}(t) = 0, \dot{\bar{G}}(t) = 0\},\\
&\ddot{\bar{G}}(t) = 1 - I\left\{t : \frac{D(\bar{x}_p(t), \bar{x}_v(t), u(t), t)}{\tilde{A}(\bar{x}_p(t), \bar{x}_v(t), t)} < 0\right\} = 0
\end{aligned} \tag{50}$$

which is a limit of the *rhs* of (37) on the set $\{t \in \operatorname{supp} V^c\} = \{\bar{G}(\tau) = 0, \dot{\bar{G}}(\tau) = 0\}$. System of equations (50) has Lipschitzian *rhs*, therefore its solution is unique, and the theorem is proven. □

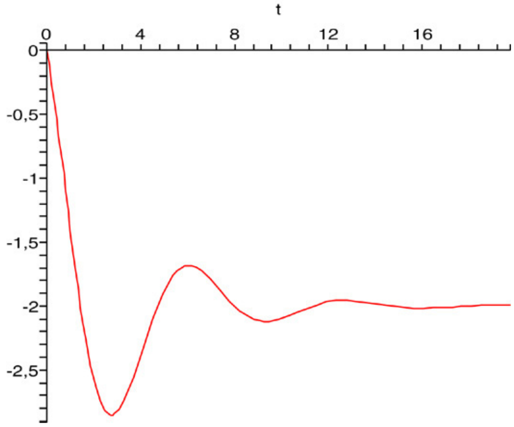

Fig. 1. Stabilization of a bouncing ball for finite μ.

7. Example

We consider the classical problem of a bouncing ball subject to constant external force with a visco-elastic model of contact. The system equation in this case has the form

$$\begin{aligned}&\dot{x}_p^{\mu}(t) = x_v^{\mu}(t),\\ &\dot{x}_v^{\mu}(t) = -F - \mu[x_p^{\mu}(t) + 2\xi x_v^{\mu}(t)\mu^{-1/2}]I\{t : x_p^{\mu}(t) \le 0\},\end{aligned} \tag{51}$$

where $F = \text{const} > 0$, and $0 < \xi < 1$. We consider the behavior of this system as $\mu \to \infty$. This system provides the penalty approximation of a well known example of a bouncing ball which stabilizes after an infinite number of impacts, but within a finite time interval. However, for any finite $\mu < \infty$ the ball stabilizes within the inhibited area $x_p^{\mu} \le 0$ after a *finite* number of impacts. Indeed, if at some impact point τ the value of the velocity satisfies inequality $|x_v^{\mu}(\tau)| \le C(F, \xi)/\mu^{1/2}$, then the ball stays in the area $x_p^{\mu} \le 0$, never leaving it afterwards. Here, the constant $C(F, \xi)$ does not depend on μ. This behavior is illustrated in Fig. 1, where the ball stabilizes around the point $x_p^{\mu} = -F/\mu$, exactly in accordance with Proposition 7.

Moreover, one can estimate the time of the last impact. Indeed, if the velocity $|x_v^{\mu}(\tau)|$ before some impact is sufficiently large and the ball comes out of the inhibited domain after the phase of contact with the obstacle then:

(1) the time Δ which is necessary to exit the inhibited domain satisfies the inequality

$$\Delta \le \frac{2\pi}{\mu^{1/2}\sqrt{1-\xi^2}};$$

(2) velocity $x_v^{\mu}(\tau + \Delta)$ after leaving the inhibited domain satisfies the inequality

$$|x_v^{\mu}(\tau + \Delta)| < |x_v^{\mu}(\tau)| \exp\left\{-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right\} < \alpha|x_v^{\mu}(\tau)|,$$

where $\alpha < 1$;

(3) the next impact after the exit occurs after the time

$$\frac{2|x_v^{\mu}(\tau + \Delta)|}{F} = \alpha\frac{2|x_v^{\mu}(\tau)|}{F}.$$

Therefore, the time of the last impact T_{μ}^{*} is estimated as follows

$$T_{\mu}^{*} \le \frac{2\pi k^{*}}{\mu^{1/2}\sqrt{1-\xi^2}} + \frac{2|x_v^{\mu}(0)|}{F}\sum_{k=0}^{k^{*}}\alpha^{k}, \tag{52}$$

where integer k^* is such that

$$\log\left(\frac{C(F,\xi)}{\mu^{1/2}|x_v^\mu(0)|}\right)\bigg/\log\alpha \le k^* \le \log\left(\frac{C(F,\xi)}{\mu^{1/2}|x_v^\mu(0)|}\right)\bigg/\log\alpha + 1.$$

If $\mu \to \infty$, then $k^* \to \infty$, but the first term in the *rhs* of (52) goes to zero, T_μ^* remains uniformly bounded, and

$$\lim_{\mu\to\infty} T_\mu^* = \frac{2|x_v^\mu(0)|}{F}\sum_{k=0}^{\infty}\alpha^k = \frac{2|x_v^\mu(0)|}{F(1-\alpha)},$$

which coincides with the accumulation point of the generalized solution.

Therefore, the behavior of the prelimit system is the same as described by Theorem 7, and in this example we show explicitly how the solution can be extended beyond the accumulation point. Of course, in the general case the external force is not a constant, but if it is still continuous, its behavior within the small vicinity of accumulation carries no significance, and the qualitative behavior of the system is the same.

Thus, the representation proposed completely captures Zeno-type behavior and provides unique extensibility of solutions without the need to truncate infinite sequences and/or switch system coefficients depending on the system motion relative to the generalized coordinates of the accumulation point.

8. Conclusions and discussion

Thus, a representation of generalized motion that encompasses Zeno and sliding mode type paths has been obtained. This representation is different from the standard one given by complementarity conditions, since an explicit representation of the measure in terms of external control and phase coordinates (49) has been derived. In the standard case of complementarity, this measure has to be defined in terms of the solution of some additional variational problem as in the case of Lagrangian systems [35].

Moreover, the continuous part of this measure is absolutely continuous in the case of smooth boundaries. We emphasize that our result covers a practically important case of the unique generalized path continuation beyond accumulation points, when the system starts sliding along the boundary after multiple impacts, possibly infinitely many (*Zeno-type path*), if the motion along the boundary is described by differential equations with Lipschitzian *rhs* (Theorem 7). Another conclusion is that in the case of viscoelastic contact forces (of type (4)), the sliding motion is smooth, and for big values of μ it continues along the boundary due to a permanently existing deformation, but not as a result of a bang–bang type regime with infinitely increasing frequency of micro exits and returns (Propositions 6 and 7 and Theorem 6).

Even though the conditions of Theorem 7 look cumbersome and hardly tested, one can give one very important practical example of a penalized mechanical system with loss of energy at impacts if the constraint boundary is a hyperplane and the contact force is a linear combination of deformation and its velocity. In this case, as follows from the results of a very detailed analysis given in [32], all conditions which are necessary for the application of the proposed approach hold.

The representation developed is expected to provide the basis for optimal control problems within the class of generalized solutions of general type, including investigation of the existence of the optimal solution and necessary optimality conditions for impulsive control problems [29] and problems with active singularities [27,7].

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