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Nonlinear Analysis 65 (2006) 999–1017
Nonlinear Analysis
www.elsevier.com/locate/na

Optimal control problems in hybrid systems with active singularities[☆]

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Received 22 August 2005; accepted 23 August 2005

Abstract

The optimal control problem for systems with controlled unilateral phase constraints is considered. The definition of the generalized solutions is introduced, the transformation method for the original optimal control problem within the class of generalized solution to a standard optimal control problem is proposed, and the necessary optimality conditions are found.

Keywords: Impulse controls; Generalized solutions; Phase constraints; Optimization; Maximum principle

1. Introduction

In recent years there has been a significant interest in modeling [18–22,24,25] and control [9–11] of impact mechanical systems. This interest is motivated by a wide range of applications, including vibro-impact mechanics [1], robotics [13], and microelectromechanics [8]. The optimal control problems for this class of systems, however, have remained largely unexplored. The reason for this appears to be the difficulty in appropriately blending together impact mechanics and impulsive control. To address this problem, in a series of recent works the present authors have introduced a new class of systems — *systems with active singularities*, i.e., singularities

☆ This work is supported by NSF grants CMS-0000458 and CMS-0324630 and the UIUC Grainger Center for Electric Machinery and Electromechanics, and by Russian Basic Research Foundation Grant N 05-01-00508.
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doi:10.1016/j.na.2005.08.033

that arise due to system interaction with controlled, or active, state constraints characterized by the almost infinite "elasticity" [3–7], and proposed a novel physically based approach to their representation. As a result, one can obtain the description of system dynamics in terms of a nonlinear differential equation with measure. Singular motions are represented in the latter with the aid of a shift-operator along the paths of a so-called "limit", or "controlled infinitesimal dynamics", system [5–7]. In this paper, a class of optimal control problems for systems with active singularities is formulated and the properties of the corresponding optimal solutions are established.

The first step in developing an optimal control synthesis technique for the class of systems considered is meaningfully defining a solution of a system with optimally controlled singularity. A good starting point for this effort is a rigorous mathematical framework for the impulsive control law synthesis developed in [17]. The approach considered in [17] is based on the concept of a generalized solution for systems with impulsive controls as a limit of a sequence of ordinary solutions at continuity points, leading to the infinitesimal solution generator in the form of a nonlinear differential equation with a measure. This helps create the physically based concept of generalized solution which can be approximated in the common sense by ordinary ones.

In this paper, however, we consider the class of unilaterally constrained systems arising in impact mechanics. In this case, characterized by phase (unilateral) constraints, the problem of defining a solution appears to be more complicated and alternative concepts of generalized solution need to be introduced. Furthermore, the control problem looks more challenging as well, since the conditions for the existence of an optimal control law and the necessary optimality conditions need to reflect the additional complexity of the controlled impact. To address these problems, the present work employs a technique well tested for problems with state constraints — admitting small (in some sense) violations (*relaxation*) of constraints. Then, a solution can be realized through an approach similar to the so-called penalization method, where the absolutely rigid constraint is replaced by its elastic approximation, and the generalized solution is obtained in the limit as the elasticity coefficient goes to infinity [19]. The difference from the penalization method, however, is that for the problems considered, penalty functions do not arise.

In [17] two forms of generalized solutions are suggested, namely: strong and weak ones, distinguished by their behaviors at points where the solution hits the constraints. For strong solutions we admit the uniformly small constraint violation only, whereas for weak solutions we admit violations small in the L_1 sense. Both solution types have physical meaning depending on the application area. With this approach, one can establish that the limit, i.e., the (*generalized*) solution with discontinuous velocities, satisfies the differential equation with a measure, localized at the constraint boundary. It has become commonplace to employ this differential equation with a measure as a model for description of hybrid systems and systems with phase constraints, using, for example, such techniques as complementarity, characterized by augmenting the differential equation with algebraic relations between the pre-impact and the post-impact velocities. However, in the case of controllable dynamics these solutions are not unique! The sufficient condition for uniqueness which cannot be relaxed even in very simple cases is as follows: *All applied forces have to be analytical functions of time and coordinates* [2]. Though the necessity for this condition was demonstrated by a number of examples, *it is absolutely unreal for applied control problems.* In view of a well-posed limit equation setting that could guarantee uniqueness of solutions in the control problems, heuristically matching the solutions generated by the controlled limit models to those generated by the systems obtained from limit models through penalization has been proposed. Solutions generated by the latter are well-posed, and one can select the generalized solution of the limit system corresponding to that of its penalized approximation. This approach, however, is still not entirely satisfactory, since it does not introduce the dynamics necessary for fully capturing the subtleties of impact phenomena. Thus, developing well-posed controlled limit impact systems is at present an open problem.

To clearly bring out the problem, the present work considers some well known examples, and demonstrates a typical use of the penalization method for selecting a solution out of those produced from a typical limit description. Then, a technique is proposed that looks promising for addressing this problem: the time–space transformation method [4], that permits describing the solution jumps in a well-posed manner, and on this basis producing necessary optimality conditions for the optimal control problem in the controlled singular phase.

The structure of the paper is as follows. Section 2 describes a general model of a discrete–continuous controlled system with controlled singularities and states the general optimal control problem for this system. The setting proposed is demonstrated on an example of a juggling mechanical system. Then, various impact models, namely a classical one and models with controlled singularities, are considered, and a strong dependence of the resulting post-impact velocities on the profiles of the applied impulsive forces is demonstrated. Section 3 presents the method of discontinuous time transformation [17] and extends it to the new class of optimal control problems, formulated in Section 2. It is shown that by using this method one can reduce the originally singular control problem to a relatively standard one in terms of so-called multiprocesses [12]. The last section presents conclusions, where the applicability of the results obtained to classical problems of optimal control with unilateral constraints of the impact type is assessed.

2. Generalized solutions in dynamical systems with controlled singularities

2.1. Model description

Consider a dynamical system whose behavior is described on the interval $[0, T]$ by a pair of continuous functions $(x_p(t), x_v(t)) \in R^n \times R^n$, which satisfies the differential equation

$$\begin{aligned} \dot{x}_p(t) &= F_p^r(x_p(t), x_v(t), t), \\ \dot{x}_v(t) &= F_v^r(x_p(t), x_v(t), u(t), t) \\ &\quad + F^s(x_p(t), x_v(t), w(t), t, \mu) I\{t : G(x_p(t), t) \le 0\}, \end{aligned} \tag{1}$$

with a given initial condition $(x_p(0), x_v(0)) \in R^n \times R^n$. One can consider in (1) $(x_p(t), x_v(t))$ as the generalized state and velocity, and

$$u(t) \in U \subset R^m, \qquad w(t) \in W \subset R^k, \tag{2}$$

where U, W are some compact sets, are the control signals.

Function $F_v^s(x_p, x_v, w, t, \mu)$ is supposed to be continuous and smooth in the area $G(x_p, \tau) \le 0$, and to satisfy the constraints

$$F_v^s(x_p, x_v, w, t, \mu) = 0, \qquad \text{if}$$

$$G(x_p, t) = 0, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{F_p^r} G(x_p, t) = G'_x F_p^r + G'_t \Big|_{(x_p, x_v, t)} = 0$$

and the Lipshitz-type condition

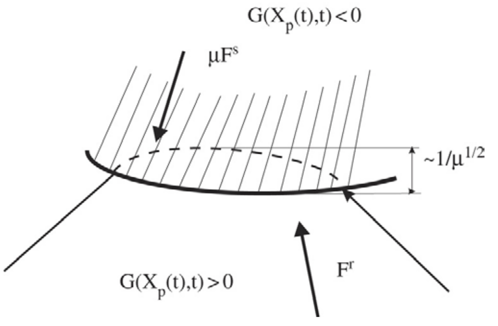

Fig. 1. Motion of a dynamic system with unilateral constraints if $\mu < \infty$.

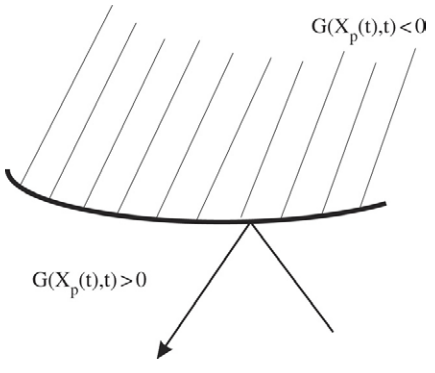

Fig. 2. Limit motion of a dynamic system with unilateral constraints when $\mu = \infty$.

$$\|F_v^s(x_p, x_v, w, t, \mu) - F_v^s(x_p', x_v', w, t, \mu)\| \\ \le L\{\|x_p - x_p'\| + \mu^{-1/2}\|x_v - x_v'\|\}. \tag{3}$$

Assume that for any given $0 \le \mu < \infty$ the joint system (1) has a unique solution for any given measurable controls $u(\cdot)$, $w(\cdot)$. A condition of the type (3) is satisfied for well-known models of mechanical systems with viscoelastic obstacles (Kelvin–Foight, Maxwell, etc., models) and can usually be described by Rayleigh's dissipative potentials in the area $G(x_p, \tau) \le 0$ [14].

In the limit $\mu \uparrow \infty$, system (1) could still have a solution. If such a limit solution exists, one can treat it as *the generalized solution of a dynamical system with unilateral constraints*, which would then be described by the limiting form of the joint system as *the differential equation with delta functions in the rhs, or, more generally, with a measure*. The motion of system (1) for finite μ and the corresponding motion of its limit system for $\mu \uparrow \infty$ are depicted in Figs. 1 and 2, respectively.

2.2. Description of the single-jump motion

Let the system start from the initial condition at $t = 0$ and $x_p(0), x_v(0)$ such that $G(x_p(0), 0) > 0$ and let τ be the first point where the system engages with the constraint, so that

$$G(x_p(\tau), \tau) = 0, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{F_p^r} G(x_p(\tau), \tau) < 0, \tag{4}$$

and the control $w(t,\mu)$ is coordinated with the duration of the impact and, therefore, has the form

$$w(t,\mu)=\begin{cases} w_\tau((t-\tau)\mu^{1/2}), & \text{if } t\geq\tau,\\ 0, & \text{otherwise.}\end{cases} \tag{5}$$

The typical case in mechanics is the case of viscoelastic constraints where the orders of the constraint violation and the period of the constraint violation are $\sim\mu^{-1/2}$ (see Figs. 1 and 2). Therefore, the following space–time transformation in the vicinity of the jump point is natural.

For $s>0$ we put

$$\begin{aligned} y_p^\mu(s) &= x_p(\tau)+\mu^{1/2}[x_p(\tau+\mu^{-1/2}s)-x_p(\tau)],\\ y_v^\mu(s) &= x_v(\tau+\mu^{-1/2}s),\\ t &= \tau+\mu^{-1/2}s. \end{aligned} \tag{6}$$

Then the new variables $\{y_p^\mu(s), y_v^\mu(s)\}$ satisfy the system of differential equations

$$\begin{aligned} \dot y_p^\mu(s) &= F_p^r\left(\frac{y_p^\mu(s)-x_p(\tau)}{\mu^{1/2}}+x_p(\tau), y_v^\mu(s), \tau+\mu^{-1/2}s\right),\\ \dot y_v^\mu(s) &= \mu^{1/2}F_v^s\left(\frac{y_p^\mu(s)-x_p(\tau)}{\mu^{1/2}}+x_p(\tau), y_v^\mu(s), w_\tau(s), \tau+\mu^{-1/2}s, \mu\right)\\ &\quad +\mu^{-1/2}F_v^r\left(\frac{y_p^\mu(s)-x_p(\tau)}{\mu^{1/2}}+x_p(\tau), y_v^\mu(s), u(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s\right) \end{aligned} \tag{7}$$

with initial conditions $y_p^\mu(0)=x_p(\tau)$, $y_v^\mu(0)=x_v(\tau-)$.

Due to the continuity of F_p^r there exists

$$\begin{aligned} &\bar F_p(y_p, y_v, x_p, \tau)\\ &\quad=\lim_{\mu\uparrow\infty} F_p^r\left(\frac{y_p-x_p}{\mu^{1/2}}+x_p, y_v, \tau+\mu^{-1/2}s\right)=F_p^r(x_p, y_v, \tau). \end{aligned} \tag{8}$$

Assume also that for given admissible control $w(\cdot)$ there exists

$$\begin{aligned} &\bar F_v(y_p, \bar y_v, w_\tau(s), s, x_p, \tau)\\ &\quad=\lim_{\mu\uparrow\infty} \mu^{1/2}F^s\left(\frac{y_p-x_p}{\mu^{1/2}}+x_p, y_v, w(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s, \mu\right), \end{aligned} \tag{9}$$

defined for all (x_p,τ) such that $G(x_p,\tau)=0$. Then due to the condition (3), functions $\bar F_p$, $\bar F_v$ are continuous with respect to all variables and Lipschitz in y_p, y_v.

Suppose that the system of limit equations describing the transformed singular phase motion

$$\begin{aligned} \dot y_p(s) &= F_p^r(x_p(\tau), y_v(s), \tau),\\ \dot y_v(s) &= \bar F_v(y_p(s), y_v(s), w_\tau(s), s, x_p(\tau), \tau), \end{aligned} \tag{10}$$

referred to in [6] as the *controlled infinitesimal dynamics equation*, for $s\in[0,\infty)$ with initial condition $y_p(0)=x_p(\tau)$, $y_v(0)=x_v(\tau-)$, has a solution such that there exists the exit time

$$s^*=\inf\left\{s>0: \begin{array}{l} G_t'\big|_{(x_p(\tau),\tau)}s+G_x'\big|_{(x_p(\tau),\tau)}(y_p(s)-x_p(\tau))=0,\\ G_t'\big|_{(x_p(\tau),\tau)}+G_x'\big|_{(x_p(\tau),\tau)}F_p^r(x_p(\tau), y_v(s), \tau)>0 \end{array}\right\} \tag{11}$$

such that on the interval $(0, s^*)$

$$\bar{G}(x_p(\tau), \tau, y_p(s), s) = \bar{G}(y, \eta, z, s) = G'_t\big|_{(y,\eta)} s + G'_x\big|_{(y,\eta)}(z - y) < 0. \tag{12}$$

Then, if $\mu \to \infty$,

$$(y_p^\mu(s), y_v^\mu(s)) \to (y_p(s), y_v(s)) \qquad \text{uniformly on } [0, s^* + \varepsilon],$$

and for all sufficiently large μ there exists the exit time

$$s_\mu^* = \inf\left\{ s > 0 : \begin{array}{l} G(x_p(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s) = 0, \\ (G'_t + G'_x F_p^r)\big|_{(x_p(\tau+\mu^{-1/2}s), x_v(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s)} > 0 \end{array} \right\} \tag{13}$$

such that

$$s_\mu^* \to s^*.$$

The following theorem [6] describes the single jumps of the generalized solution.

Theorem 1. *If the above assumptions hold, then for sufficiently small $\varepsilon > 0$ on the interval $[0, \tau + \varepsilon)$, the solution of the system (1) converges to some discontinuous functions $(\bar{x}_p(t), \bar{x}_v(t))$, such that*

$$\bar{x}_p(t) = x_p(t), \qquad \bar{x}_v(t) = x_v(t), \qquad t < \tau,$$

and

$$\bar{x}_p(\tau+) = \lim_{\mu\uparrow\infty} x_p(\tau + \mu^{-1/2}s_\mu^*) = x_p(\tau)$$
$$\bar{x}_v(\tau+) = \lim_{\mu\uparrow\infty} y_v(\tau + \mu^{-1/2}s_\mu^*) = y_v(s^*).$$

Let

$$\varphi(\cdot, s, w_\tau(\cdot), \tau) = \begin{pmatrix} \varphi_p(\cdot, s, w_\tau(\cdot), \tau) \\ \varphi_v(\cdot, s, w_\tau(\cdot), \tau) \end{pmatrix} \tag{14}$$

be the general solution of (10) for $s \geq 0$ with initial conditions

$$y_p(0) = x_p(\tau), \qquad y_v(0) = x_v(\tau). \tag{15}$$

Then, the jump of the variable $x_v(\cdot)$ at a point τ is described by the relation

$$\Delta x_v(\tau_i) = \varphi_v(y_p(0), y_v(0), s^*, w_\tau(\cdot), \tau) - y_v(0). \tag{16}$$

Thereby, $\Delta x_v(\tau)$ is defined as the y_v-component of the shift-operator along the paths of system (14) up to the exit time $s^* = s^*(y_p(0), y_v(0))$. Therefore, the generalized solution is described by a nonlinear generalized differential equation with Dirac measure

$$\begin{aligned} \dot{x}_p(t) &= F_p^r(x_p(t), x_v(t), t), \\ \dot{x}_v(t) &= F_v^r(x_p(t), x_v(t), u(t), t) + \Psi_v(x_p(\tau), x_v(\tau-), w_\tau(\cdot), \tau)\delta(t - \tau), \end{aligned} \tag{17}$$

where

$$\Psi_v(y_p, y_v, w_\tau(\cdot), \tau) = \varphi_v(y_p, y_v, s^*, w_\tau(\cdot), \tau) - y_v. \tag{18}$$

2.3. Example: Representation of the juggling system motion

In order to demonstrate this approach we derive a jump representation for a ball–racket juggling system. Consider the system with the state vector $z = (x_{p1}, x_{p2}, X_p, x_{v1}, x_{v2}, X_v)$, where (x_{p1}, x_{p2}) are the horizontal and the vertical coordinates of the moving ball, (x_{v1}, x_{v2}) are the corresponding velocities, and (X_p, X_v) are the coordinate and velocity of the obstacle surface in the vertical direction. Then, the free motion area is given by the relation

$$G(z) = x_{p2} - X_p \geq 0. \tag{19}$$

2.3.1. General motion equations

First, consider the motion along the vertical axis only. Suppose that in the area free of constraint the motion is described by the equations

$$\begin{aligned} \dot{x}_{p1}(t) &= x_{v1}(t), & \dot{x}_{v1}(t) &= 0, \\ \dot{x}_{p2}(t) &= x_{v2}(t), & \dot{x}_{v2}(t) &= -g, \\ \dot{X}_p(t) &= X_v(t), & \dot{X}_v(t) &= \frac{F(t)}{M}. \end{aligned} \tag{20}$$

In the inhibited area $G(z) = x_{p2} - X_p \leq 0$ the motion is described by the equation

$$\begin{aligned} \dot{x}_{p1}(t) &= x_{v1}(t), & \dot{x}_{v1}(t) &= -\nu\mu \frac{F^s(z(t), w(t,\mu))}{m} \operatorname{sign} x_{v1}(s), \\ \dot{x}_{p2}(t) &= x_{v2}(t), & \dot{x}_{v2}(t) &= -g + \mu \frac{F^s(z(t), w(t,\mu), \mu)}{m}, \\ \dot{X}_p(t) &= X_v(t), & \dot{X}_v(t) &= \frac{F(t)}{M} - \mu \frac{F^s(z(t), w(t,\mu), \mu)}{M} \end{aligned} \tag{21}$$

where:

- $\mu F^s(z(t), \mu)$ is a viscoelastic force during the contact of ball and racket described by the relation

$$\begin{aligned} &F^s(z(t), w(t,\mu), \mu) \\ &\quad = -(x_{p2}(t) - X_p(t)) - 2\mu^{-1/2}\xi(x_{v2}(t) - X_v(t)) + \mu^{-1/2} w(t,\mu), \end{aligned} \tag{22}$$

- m, M are the masses of the ball and racket, respectively,
- g is the gravitational acceleration,
- $\nu \geq 0$ is the dry friction coefficient,
- $F(t)$, such that $|F(t)| \leq F_0 < \infty$, is an external control force acting on the racket,
- $\mu > 0$ is the elasticity coefficient and $0 \leq \xi \leq 1$ is the damping,
- $w(t, \mu)$ is the control of the singular motion acting during the impact phase and having the representation

$$w(t,\mu) = w_\tau((t-\tau)\mu^{1/2}) \qquad \text{for } t \geq \tau,$$

where τ is the instant of the impact and $w_\tau(\cdot)$ is some integrable function.

Eqs. (20) and (21) describe the continuous motion in the case of $\mu < \infty$. The objective, however, is to obtain the velocity jump representation corresponding to the limit motion as $\mu \to \infty$.

Since the motion in the vertical direction does not depend on coordinates (x_{p1}, x_{v1}), one can obtain the jump representation independently. First, consider the motion in the vertical direction with the corresponding reduced state vector $\bar{z}(t) = (x_{p2}, X_p, x_{v2}, X_v)$. Applying Theorem 1 and calculating $\bar{F}$ using formula (10) yields the following system for new variables $(y_{p2}(s), Y_p(s), y_{v2}(s), Y_v(s))$ describing the motion in the enlarged space–time scale:

$$\begin{pmatrix} \dot{y}_{p2}(s) \\ \dot{Y}_p(s) \\ \dot{y}_{v2}(s) \\ \dot{Y}_v(s) \end{pmatrix} = \begin{pmatrix} y_{v2}(s) \\ Y_v(s) \\ -\frac{1}{m}[(y_{p2}(s) - Y_p(s)) + 2\xi(y_{v2}(s) - Y_v(s))] \\ \frac{1}{M}[(y_{p2}(s) - Y_p(s)) + 2\xi(y_{v2}(s) - Y_v(s)) + w_1(s)] \end{pmatrix}. \tag{23}$$

The system (23) has to be solved for $s > 0$ with initial conditions

$$y_{p2}(0) = Y_p(0) = x_{p2}(\tau), \qquad y_{v2}(0) = x_{v2}(\tau) < Y_v(0) = X_v(\tau), \tag{24}$$

up to the exit time defined by the relation (11)

$$s^* = \inf\left\{ s > 0 : \begin{matrix} y_{p2}(s) - Y_p(s) = 0, \\ y_{v2}(s) - Y_v(s) > 0 \end{matrix} \right\}. \tag{25}$$

The variable

$$Z(s) = y_{p2}(s) - Y_p(s),$$

which characterizes the constraint violation satisfies the equation

$$\frac{d^2}{ds^2}Z(s) = \frac{M+m}{Mm}\left[-Z(s) - 2\xi\frac{d}{ds}Z(s)\right] + \frac{w_\tau(s)}{M+m}, \tag{26}$$

with initial conditions

$$Z(0) = 0, \qquad \frac{d}{ds}Z(0) < 0.$$

We consider the case when

$$|\xi| < \left(\frac{mM}{M+m}\right)^{1/2} = \frac{1}{k} \tag{27}$$

which guaranties the existence of s^* in (25) in most reasonable cases.

2.3.2. *Jump representation in the vertical direction: continuous racket velocity*

First we consider the case of continuous racket motion, so $w_1(s) = 0$. In this case the solution of (26) has the form

$$Z(s) = \exp\{-\lambda s\}\dot{Z}(0)\frac{\sin\omega s}{\omega},$$
$$\dot{Z}(s) = \exp\{-\lambda s\}\dot{Z}(0)\left(\cos\omega s - \frac{\lambda\sin\omega s}{\omega}\right),$$

where

$$\omega = k\sqrt{1 - k^2\xi^2}, \qquad \lambda = \xi k.$$

Then, there exists

$$s^* = \frac{\pi}{\omega}$$

satisfying (25), and by Theorem 1 one can calculate the jumps of variables $x_{v2}(\tau)$ and $X_v(\tau)$. For y_{v2}, integration of the system (23) and (26) yields the representation

$$y_{v2}(s) = x_{v2}(\tau-) + \frac{M}{M+m}[\dot{Z}(s) - x_{v2}(\tau-) + X_v(\tau-)] + \frac{1}{M+m}\int_0^s w_\tau(u)\,\mathrm{d}u. \tag{28}$$

The substitution of s^* into relations $y_{v2}(s^*) = x_{v2}(\tau)$ and $Y(s^*) = X_v(\tau)$ gives the following formulas for the velocity increments:

$$\begin{aligned} \Delta x_{v2}(\tau) &= -\frac{M(1+k_r)}{M+m}[x_{v2}(\tau-) - X_v(\tau-)], \\ \Delta X_v(\tau) &= \frac{m(1+k_r)}{M+m}[x_{v2}(\tau-) - X_v(\tau-)], \end{aligned} \tag{29}$$

with the restitution coefficient

$$k_r = \exp\left\{-\frac{\pi\xi}{\sqrt{1-k^2\xi^2}}\right\}. \tag{30}$$

Remark 1. These conventional relations for the change of velocity during the impact and corresponding restitution coefficient can be found in [10,19,24]. The principal assumption is the continuity of the racket velocity. Of course one can observe that these relations can be derived with the aid of the usual approach based on conservation laws. Therefore, in this case the approach based on consideration of a singular phase of motion gives the same result as a classical one. However, in the case of active singularity one can admit the discontinuity of this velocity during the impact. Therefore, the profile of the velocity change becomes critical. We consider two different cases: in the first one the change of the racket velocity is uniformly distributed during the phase of contact. In the second case the abrupt change of the racket velocity is concentrated at some instant within the phase of contact. Thereby, this abrupt change plays a role of some impulsive control within the phase on contact.

2.3.3. Jump representation in the vertical direction: discontinuous racket velocity

In the first case of the uniform distribution of the impulsive effort during the contact phase we have in Eqs. (26) and (28) $w_\tau(s) = W = \text{const}$. An explicit representation of the exit time s^* can be derived, but is rather cumbersome; however, we give the results of numerical solutions obtained with the following data:

$$\begin{aligned} &M = 100, \qquad m = 1, \qquad \xi = 0.2, \\ &x_{v2}(\tau-) = -10, \qquad X_v(\tau-) = 5, \qquad W = 300. \end{aligned} \tag{31}$$

The numerical results are presented in Fig. 3.

One can observe that the instant s^* exists and $s^* \approx 3.8$. The increments of the velocities can be directly calculated as follows:

$$\Delta x_{v2}(\tau) = y_{v2}(s^*) - x_{v2}(\tau-) \approx 32,$$

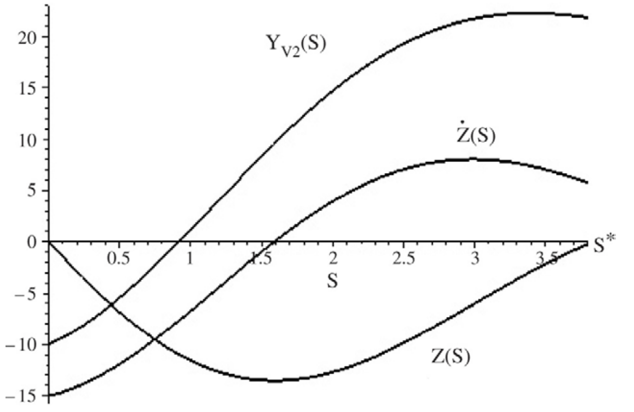

Fig. 3. Motion of the "limit" system with a "uniform" break of the racket velocity during the impact.

$$\Delta X_v(\tau) = y_{v2}(s^*) - Z(s^*) - X_v(\tau-) \approx 12.$$

In the second case of abrupt change of the racket's velocity we have in Eqs. (26) and (28) $w_\tau(s) = P\delta(s - s_1)$, where s_1 is the instant of the impulsive action application within the singularity phase and P is the value of the impulse applied. On can easily derive the representations

$$Z(s) = \exp\{-\lambda s\}\dot{Z}(0)\frac{\sin\omega s}{\omega} - \frac{P}{M}\exp\{-\lambda(s - s_1)\}\frac{\sin\omega(s - s_1)}{\omega},$$

$$\dot{Z}(s) = \exp\{-\lambda s\}\dot{Z}(0)\left(\cos\omega s - \frac{\lambda\sin\omega s}{\omega}\right) - \frac{P}{M}\exp\{-\lambda(s - s_1)\}\left(\cos\omega(s - s_1) - \frac{\lambda\sin\omega(s - s_1)}{\omega}\right),$$

an explicit representation of the exit time

$$s^* = \frac{\pi}{\omega} + \arctan\frac{\frac{P}{M}\exp\{\lambda s_1\}\sin\omega s_1}{\dot{Z}(0) - \frac{P}{M}\exp\{\lambda s_1\}\cos\omega s_1},$$

and the velocity after the impact calculated with the aid of a modified relation (28) as

$$x_{v2}(\tau) = y_{v2}(s^*) = x_{v2}(\tau-) + \frac{M}{M+m}\left[\dot{Z}(s^*) - x_{v2}(\tau-) + X_v(\tau-)\right] + \frac{P}{M+m}. \tag{32}$$

For the modelling we use the same data as in the case of uniform action distribution (see (31)) and assign

$$P = Ws^* = 300 * 3.8 = 1140,$$

which is equal to the external force impulse during the phase of contact. The dependence of an exit time and the terminal ball velocity on the time s_1 of the impulse application is shown below in Figs. 4 and 5.

The main observation is that the maximum terminal velocity can be achieved by application of the impulse as closely as possible to the beginning of the singular phase and, vice versa,

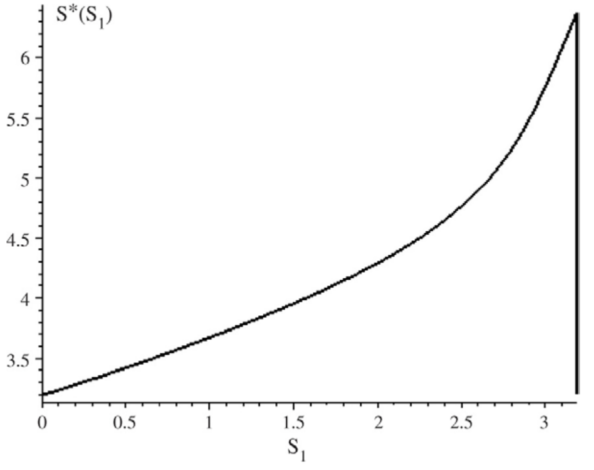

Fig. 4. Dependence of the exit time s^* on the instant s_1 of abrupt velocity change during the phase of contact.

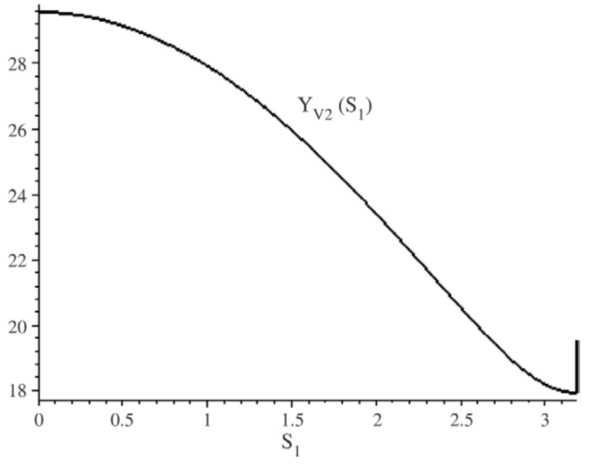

Fig. 5. Dependence of the exit velocity y_{v2} on the instant s_1 of abrupt velocity change during the phase of contact.

the minimum can be achieved by application of the impulse as closely as possible to the end of the singular phase. Moreover, comparing the values of the terminal velocity in Fig. 3 and the maximal value of the terminal velocity at $s_1 = 1$ in Fig. 5, the impulsive force looks more effective than the uniform distribution. The modelling results corresponding to the intermediate case, i.e. $s_1 = 1$, are shown in Fig. 6.

2.3.4. Motion in the horizontal direction with dry friction

By using the space–time transformation of (x_{p1}, x_{v1}) we obtain the following system of equations in the new space–time scale:

$$\begin{pmatrix} \dot{y}_{p1}(s) \\ \dot{y}_{v1}(s) \end{pmatrix} = \begin{pmatrix} y_{v1}(s) \\ -\nu y_{v2}(s)\text{sign}\,\{y_{v1}(s)\} \end{pmatrix}, \tag{33}$$

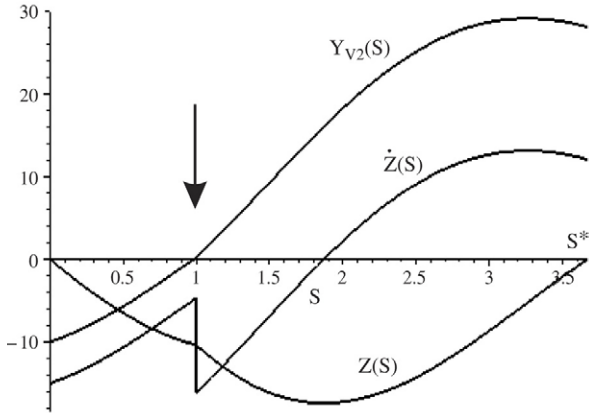

Fig. 6. Motion of the “limit” system with an “abrupt” break of the racket velocity during the impact. The arrow shows the instant $s_1 = 1$ of the abrupt velocity change during the phase of contact. At this moment the abrupt change of the racket velocity takes place.

where by definition sign $\{0\} = 0$. The solution of (33) has the form

$$y_{v1}(s) = y_{v1}(0) - \nu \int_0^{\min\{s,\tau^*\}} \dot{y}_{v2}(\tau)\, d\tau = y_{v1}(0) - \nu[y_{v2}(\min\{s, \tau^*\}) - y_{v2}(0)],$$

where

$$\tau^* = \inf\{\tau > 0 : y_{v1}(\tau) = 0\}.$$

Therefore,

$$y_{v1}(s^*) = y_{v1}(0) - \nu[y_{v2}(\min\{s^*, \tau^*\}) - y_{v2}(0)],$$

and taking into account the initial condition $y_{v1}(0) = x_{v1}(\tau-)$ by Theorem 1 one can calculate the velocity increment in the horizontal direction as

$$\Delta x_{v1}(\tau) = \begin{cases} -\nu \Delta x_{v2}(\tau), & \text{if } [x_{v1}(\tau-) - \nu \Delta x_{v2}(\tau)] x_{v1}(\tau-) \geq 0, \\ -x_{v1}(\tau-), & \text{otherwise}, \end{cases} \tag{34}$$

where $\Delta x_{v2}(\tau)$ is defined by the relation (29). The interesting observation is that the increment of the horizontal velocity admits an explicit representation which depends only on the increment of the vertical velocity and does not depend on the profile of the applied external force. This is not surprising, since the increment of the vertical velocity is proportional to the impulse of the normal component of the contacting force which, in turn, defines the impulse of the dry friction force, i.e. the horizontal component of the contacting force.

2.4. *Description of the discrete–continuous motion with the sequence of jumps*

Consider a discrete–continuous dynamical system whose behavior is described on the interval $[0, T]$ by a pair of piecewise continuous functions

$$(x_p(t), x_v(t)) \in R^n \times R^n,$$

which satisfies the differential equation (1) with control $u(\cdot)$ satisfying (2) with a given initial condition

$$(x_p(0), x_v(0)) \in R^n \times R^n.$$

Assume that this equation describes the continuous system evolution up to the attainment of the switching surface defined by the relation

$$G(x_p(t), t) = 0. \tag{35}$$

Suppose that originally $G(x_p(0), 0) \geq 0$, and one can define a sequence of intersection times

$$0 < \tau_1 < \cdots < \tau_l < \cdots < T$$

as follows:

$$\tau_1 = \begin{cases} \inf\limits_{0<t\leq T} \left\{ t : G(x_p(t), t) = 0, \quad \left.\dfrac{\mathrm{d}}{\mathrm{d}t}\right|_{F_p^r} G(x_p(\tau), \tau) < 0 \right\}, \\ \infty, \quad \text{if the set is empty,} \end{cases} \tag{36}$$

and

$$\tau_i = \begin{cases} \inf\limits_{\tau_{i-1}<t\leq T} \left\{ t : G(x_p(t), t) = 0, \quad \left.\dfrac{\mathrm{d}}{\mathrm{d}t}\right|_{F_p^r} G(x_p(\tau), \tau) < 0 \right\}, \\ \infty, \quad \text{if the set is empty.} \end{cases} \tag{37}$$

As can be seen, $\tau_i, i = 1, 2, \ldots$, are the times of exit from the domain $\{(x_p, t) : G(x_p, t) \geq 0\}$. We consider τ_i to be the times of path singularity, i.e. the path $x_v(\cdot)$ to be discontinuous at instant $t = \tau_i$.

In order to control this discontinuity, we introduce the sequence of admissible controls $w_{\tau_i}(\cdot)$, such that the corresponding $\Delta x_v(\tau_i)$ are defined as the y_v-components of the shift-operator along the paths of system (14) until the exit time $s^* = s^*(x_p(\tau_i), x_v(\tau_i-))$. Therefore, the solution of the discrete–continuous system is described by a nonlinear generalized differential equation with Dirac measures

$$\begin{aligned} \dot{x}_p(t) &= F_p^r(x_p(t), x_v(t), t), \\ \dot{x}_v(t) &= F_v^r(x_p(t), x_v(t), u(t), t) \\ &\quad + \sum_{\tau_i \leq t} \Psi_v(x_p(\tau_i), x_v(\tau_i-), w_{\tau_i}(\cdot), \tau_i)\delta(t - \tau_i). \end{aligned} \tag{38}$$

Representation of a jump in terms of the shift-operator along the paths of some auxiliary system given by (18) is very important for analysis of the optimal control problems, since it permits addressing the following tasks [16]:

- derivation of the representation of the generalized solution with a finite number of jumps in terms of the differential equations with a measure, such as (38);
- rigorous statement of the optimal control problem for systems with active singularities;
- reduction of the optimal control problem for the transformed system to the multiprocess form [12], and then derivation of the optimality conditions in the maximum principle form.

The latter two tasks are addressed in the next section.

3. Optimal control problem for discrete–continuous systems with controllable singularities

Consider the optimal problem for system (38) on a finite time interval as a problem of minimizing the performance criterion

$$J[x_p(\cdot), x_v(\cdot), u(\cdot), w_{\tau_i}(\cdot), i = 1, 2, \dots] = \phi_0(x_p(T), x_v(T)) \tag{39}$$

where ϕ_0 is some continuously differentiable function.

In spite of the regularity assumptions in the system description, this problem belongs to a class of extremely irregular ones due to the possible lack of solution extensibility. The latter could arise for a variety of reasons:

(a) the set of points s in the above s^* definition (11) being empty or the infimum being equal to infinity; this case leads to capture of the system by the constraint;
(b) the number of jump points going to infinity, giving rise to so-called accumulation points [15];
(c) a "sliding mode" existing along the set $G(x, t) = 0$ as in systems with discontinuous right-hand sides [23].

Each of these irregular cases requires a separate investigation that is mostly beyond of the scope of this paper. The focus of the present work is on developing an approach to the optimal control problem (39) that, through the use of a special time transformation, permits reduction of the problem to a particular form of multiprocess optimization [12].

3.1. Time transformation

Assume that for some control function $u(\cdot)$ all solutions of (38) are defined on the interval $[0, T]$, and that any solution $\{x_p(t), x_v(t)\}$ has a finite number of jumps at points $\{\tau_i, i = 1, \dots, N\}$. This means that for every $i = 1, \dots, N$ the set of controls $\{w_{\tau_i}(\cdot)\}$ and the set of $s_i^* = s^*(x_p(\tau_i), x_v(\tau_i-)) < \infty$, such that

$$x_v(\tau_i) = \varphi_v(x_p(\tau_i), x_v(\tau_i-), s_i^*, w_{\tau_i}(\cdot), \tau_i),$$

are defined. Consider the time interval $[0, T_1]$, where

$$T_1 = T + \sum_{i=1}^{N} s_i^*, \tag{40}$$

and define on $[0, T_1]$ the function

$$\alpha(s) = \begin{cases} 0, & \text{if } s \in \left[\tau_i + \sum_{k<i} s_k^*, \tau_i + \sum_{k\le i} s_k^*\right), \\ 1, & \text{otherwise.} \end{cases} \tag{41}$$

Also, define on $[0, T_1]$ the auxiliary system

$$\begin{aligned}
\dot{y}_p(s) &= \alpha(s) F_p^r(y_p(s), y_v(s), \eta(s)), \\
\dot{y}_v(s) &= \alpha(s) F_v^r(y_p(s), y_v(s), u_1(s), \eta(s)) \\
&\quad + (1 - \alpha(s)) \bar{F}_v(z_p(s), y_v(s), w_i(s), \theta(s), y_p(s), \eta(s)), \\
\dot{z}_p(s) &= (1 - \alpha(s)) F_p^r(z_p(s), y_v(s), \eta(s)), \\
\dot{\eta}(s) &= \alpha(s), \\
\dot{\theta}(s) &= 1 - \alpha(s)
\end{aligned} \tag{42}$$

with initial conditions

$$y_p(0) = z_p(0), \qquad y_v(0) = x_v(0), \qquad \eta(0) = 0,$$

and with intermediate conditions

$$z_p(\bar{s}_i) = y_p(\bar{s}_i), \qquad \theta(\bar{s}_i) = 0, \tag{43}$$

at points

$$\bar{s}_i = \tau_i + \sum_{k<i} s_k^* = \inf\{s : \eta(s) = \tau_i\},$$

where α switches from 1 to 0, and with controls

$$u_1(s) = u(\eta(s)),$$
$$w_i(s) = \begin{cases} w_{\tau_i}(s - \bar{s}_i), \\ \text{if } s \in \left[\bar{s}_i, \bar{s}_i + s_i^*\right); \\ \text{arbitrary, but in } W \text{ otherwise.} \end{cases} \tag{44}$$

Then, the correspondence between the auxiliary and the original systems, (42) and (38), respectively, is specified by the following theorem.

Theorem 2. *For any solution* $\{x_p(t), x_v(t)\}$ *of* (38) *define the function* $\eta(s)$ *by the relation* (41) *and its inverse by the relation*

$$\Gamma(t) = \inf\{s : \eta(s) > t\},$$

with $\Gamma(T) = T_1$ *by definition.*

Let $\{y_p(\cdot), y_v(\cdot), z_p(\cdot), \eta(\cdot), \theta(\cdot)\}$ *be the corresponding solution of* (42)*. Then*

$$x_p(t) = y_p(\Gamma(t)), \qquad x_v(t) = y_v(\Gamma(t)).$$

This result permits deriving the optimality conditions by using the following approach. According to the Theorem 2 the auxiliary optimal control problem for system (42) with intermediate conditions (43), terminal condition $\eta(T_1) = T$, and performance criterion

$$J[y_p(\cdot), y_p(\cdot), z_p(\cdot), \eta(\cdot), \theta(\cdot), u_1(\cdot), w_i(\cdot), i = 1, 2, \ldots] = \phi_0(y_p(T_1), y_v(T_1))$$

is equivalent to the original problem in the following sense: if the number of jumps is fixed and given as $N < \infty$, then the optimal solution of the original problem gives, after the time transformation, the optimal solution of the auxiliary optimal control problem and vice versa. Therefore, the necessary optimality condition in the auxiliary problem can be transformed into an optimality condition in the original problem with the aid of the inverse discontinuous time transformation [17]. Since the auxiliary optimal control problem belongs to a class of multiprocess problems [12], the optimality conditions in the maximum principle form are straightforward.

3.2. Necessary optimality conditions

Let $N < \infty$. As follows from Theorem 2, the trajectory $\{x_p(\cdot), x_v(\cdot)\}$ can be described as follows:

- there exists a control $u(t) \in U$ a.e. on $[0, T]$;

- the set of controls $w_{\tau_i}(s) \in W$ is defined on the intervals $[0, s_i^*]$, where $i = 1, \ldots, N$;
- the set of variables $\{z_p^i(\cdot), z_v^i(\cdot)\}$, $i = 1, \ldots, N$, is defined on the intervals $[0, s_i^*]$, and satisfies the equations

$$\dot{z}_p^i(s) = F_p^r(x_p(\tau_i), z_v^i(s), \tau_i),$$
$$\dot{z}_v^i(s) = \bar{F}_v(z_p^i(s), z_v^i(s), w_{\tau_i}(s), s, x_p(\tau_i), \tau_i)$$

with initial and terminal conditions

$$z_p^i(0) = x_p(\tau_i),$$
$$z_v^i(0) = x_v(\tau_i -), \qquad z_v^i(s_i^*) = x_v(\tau_i);$$
$$G(z_p^i(0), \tau_i) = G(x_p(\tau_i), \tau_i) = 0,$$
$$\bar{G}(x(\tau_i), \tau_i, z_p^i(s_i^*), s_i^*) = 0;$$

such that the path $\{x_p(\cdot), x_v(\cdot)\}$ satisfies the equations

$$x_p(t) = x_p(0) + \int_0^t F_p^r(x_p(s), x_v(s), s)\,\mathrm{d}s,$$
$$x_v(t) = x_v(0) + \int_0^t F_v^r(x_p(s), x_v(s), u(s), s)\,\mathrm{d}s + \sum_{\tau_i \le t} \Delta x_v(\tau_i),$$

with jumps defined by relations

$$\Delta x_v(\tau_i) = z_v^i(s_i^*) - z_v^i(0).$$

Then, the optimality condition of the path $\{x_p(\cdot), x_v(\cdot)\}$ has the following form of the maximum principle:

Theorem 3. *Assume that the set*

$$\{x_p(\cdot), x_v(\cdot), u(\cdot), z_p^i(\cdot), z_v^i(\cdot), w_{\tau_i}(\cdot), \tau_i, s_i^*\}$$

defines the optimal path in the problem with the cost function (39). Then, there exists the set of elements

$$\{\psi_p(\cdot), \psi_v(\cdot), \psi_{zp}^i(\cdot), \psi_{zv}^i(\cdot), \lambda_0, \lambda_i, \nu_i\}$$

that are not equal to zero simultaneously, where:

(1) $\lambda_0, \lambda_i, \nu_i, i = 1, \ldots, N$, *are the constants,*
(2) *vector functions* $\{\psi_p(\cdot), \psi_v(\cdot)\}$ *defined on the interval* $[0, T]$ *satisfy the equations*

$$\begin{aligned}
\psi_p(t) \;&=\; -\lambda_0\{\phi_0\}'_{x_p}(x_p(T), x_v(T)) + \int_t^T [\langle \bar{\psi}_p(s), \{F_p^r\}'_{x_p}(x_p(s), x_v(s), s)\rangle \\
&\quad + \langle \psi_v(s), \{F_v^r\}'_{x_p}(x_p(s), x_v(s), u(s), s)\rangle]\,\mathrm{d}s + \sum_{\tau_i \ge t} \Delta\psi_p(\tau_i), \\
\psi_v(t) \;&=\; -\lambda_0\{\phi_0\}'_{x_v}(x_p(T), x_v(T)) + \int_t^T [\langle \bar{\psi}_p(s), \{F_p^r\}'_{x_v}(x_p(s), x_v(s), s)\rangle \\
&\quad + \langle \psi_v(s), \{F_v^r\}'_{x_v}(x_p(s), x_v(s), u(s), s)\rangle]\,\mathrm{d}s + \sum_{\tau_i \ge t} \Delta\psi_v(\tau_i),
\end{aligned} \tag{45}$$

where

$$\bar{\psi}_p(t) = \psi_p(t) + \sum_{\tau_i \geq t} (\psi^i_{xp} + \bar{\psi}^i_{zp}(t)), \tag{46}$$

and

$$\psi^i_{xp} = -\lambda_i G'_{x_p}(x_p(\tau_i), \tau_i), \tag{47}$$

$$\bar{\psi}^i_{zp}(t) = \begin{cases} -\nu_i G'_{x_p}(x_p(\tau_i), \tau_i), & \textit{if } t \geq \tau_i, \\ \psi^i_{zp}(0), & \textit{if } t < \tau_i, \end{cases} \tag{48}$$

(3) *the jumps of variables* $\{\psi_p, \psi_v\}$ *at points* τ_i *are equal to*

$$\begin{aligned} \Delta\psi_p(\tau_i) &= \psi^i_{zp}(s^*_i) - \psi_{zp}(0), \\ \Delta\psi_v(\tau_i) &= \psi^i_{zv}(s^*_i) - \psi_{zv}(0), \end{aligned} \tag{49}$$

and

(4) *variables* $\{\psi^i_{zp}(\cdot), \psi^i_{zv}(\cdot)\}$ *are defined on the intervals* $[0, s^*_i]$ *and satisfy the equations*

$$\begin{aligned} \psi^i_{zp}(s) &= -\nu_i G'_{x_p}(x_p(\tau_i), \tau_i) \\ &\quad + \int_s^{s^*_i} \langle \psi^i_{zv}(u), \{\bar{F}_v\}'_{z_p}(z^i_p(u), z^i_v(u), w_{\tau_i}(u), u, x_p(\tau_i), \tau_i)\rangle \, \mathrm{d}u, \\ \psi^i_{zv}(s) &= \psi_v(\tau_i) + \int_s^{s^*_i} \langle \psi^i_{zv}(u), \{\bar{F}_v\}'_{z_v}(z^i_p(u), z^i_v(u), w_{\tau_i}(u), u, x_p(\tau_i), \tau_i)\rangle \\ &\quad + \langle \psi^i_{zp}(u), \{F^r_p\}'_{z_v}(x_p(\tau_i), z^i_v(u), \tau_i)\rangle \, \mathrm{d}u. \end{aligned} \tag{50}$$

The optimal trajectory and the optimal controls satisfy the relations:

- *on the set* $[0, T] \setminus \{\tau_i, i = 1, \ldots, N\}$,

$$\begin{aligned} &\langle \psi_v(t), F^r_v(x_p(t), x_v(t), u(t), t)\rangle \\ &\quad \geq \langle \psi_v(t), F^r_v(x_p(t), x_v(t), u, t)\rangle \geq 0, \end{aligned} \tag{51}$$

for all $u \in U$,

- *on the intervals* $[0, s^*_i]$,

$$\begin{aligned} &\langle \psi^i_{zv}(s), \bar{F}_v(z^i_p(s), z^i_v(s), w_{\tau_i}(s), s, x_p(\tau_i), \tau_i)\rangle \\ &\quad \geq \langle \psi^i_{zv}(s), \bar{F}_v(z^i_p(s), z^i_v(s), w, s, x_p(\tau_i), \tau_i)\rangle \end{aligned} \tag{52}$$

for all $w \in W$.

Thereby, optimality conditions of two types are obtained: the first one for the regular motion phase, i.e. (51), and the second one for active singularity control, which also satisfies the maximum principle, but only on the intervals describing fast motion (52). The evolution of adjoint variables satisfies the system of nonlinear differential equations with measures, described in terms of integral relations (45) with the intermediate conditions (46)–(48).

4. Conclusions

The presence of an active singular phase in the system motion characterized by its own dynamics drastically modifies the ordinary optimality conditions. The multi-scale description of this phase gives the opportunity to express the "fast" phase of motion in explicit terms in terms of the shift-operator along the solutions of the controlled infinitesimal dynamics equation. Moreover, if the motion in the singular phase does not admit the control activity (i.e., function $\bar{F}_v$ does not depend on any control w), the form of necessary condition is still invariant with respect to the "fast" motion description and depends only on the jump representation of the final velocity. In other words, the jumps (cf. item (49) in Theorem 3) can be expressed explicitly without adjoint differential equations (50) in Theorem 3, but with the aid of the shift-operator jump representation (14). Indeed, if control $w(\cdot)$ is absent in the r.h.s. of $\bar{F}_v$, then the system of Eq. (50) admits an explicit solution in terms of the derivatives of the shift-operator jump representation. This means that this type of optimality condition is also applicable to classical optimal control problems with passive unilateral constraints producing jumping-type motions. The authors intend to consider this class of problems in further publications.

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