Maximum principle for singular stochastic control problems

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Abstract

In this paper, an optimal singular stochastic control problem is considered. For this model, it is obtained a general stochastic maximum principle by using a time transformation. This is the first version of the stochastic maximum principle that covers the singular control problem in the nonlinear case.

1 Introduction

Notation

$\mathbb{N}_N$ is the set of the first $N$ integers, that is $\mathbb{N}_N = \{1, \cdots, i, \cdots, N\}$.
$\mathbb{N}^* \doteq \{k \in \mathbb{N} : k > 0\}$ and $\mathbb{R}_+ \doteq \{x \in \mathbb{R} : x \geq 0\}$.
The $i$th component of a vector $M$ is denoted by $M_i$.
The symbol $|.|$ is used to denote the norm of vectors and matrices.
($^\top$) denotes the transpose operation.
$0_n \in \mathbb{R}^n$ is the zero vector.
For a vector $x$ in $\mathbb{R}^r$, $\text{diag}[x]$ denotes a square matrix of order $r \times r$ having the elements of $x$ along its diagonal and zeros elsewhere.
The indicator function of a set $A$ is defined as $I_A(x)$.
The function $\delta$ defined on $\mathbb{N} \times \mathbb{N}$ is such that $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.
For $x \in \mathbb{R}$, $x^+$ is defined by $x^+ = \frac{1}{x}$ if $x \neq 0$; and $x^+ = 0$ if $x = 0$.
If $X$ is a metric space then $\mathcal{B}(X)$ denotes its associated borel $\sigma$-field.

On a probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$, the mathematical expectation will be denoted by $E_P[.]$ and if $\{A(t)\}$ is a right continuous increasing process on $[0, T]$ then $\mathcal{D}_A$ denotes the Doleans’ measure associated to $\{A(t)\}$ and is defined on $([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F})$ by $\mathcal{D}_A(A) = E_P[\int_{[0, T]} I_A(t, \omega) dA(t)]$ for all $A \in \mathcal{B}([0, T]) \otimes \mathcal{F}$.

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For an $\mathbb{R}^r$-valued process $\{u(t)\}$ of bounded variation, $\text{Var}_{[0,t]}[u]$ denotes a vector in $\mathbb{R}^r$ such that $ith$ component is equal to $\text{Var}_{[0,t]}[u_i]
$ For $A \in \mathcal{B}(\mathbb{R})$ the Lebesgue-Stieltjes measure of the set $A$ induced by a real-valued process $\{\Gamma(t)\}$ of bounded variation will be denoted by $\Gamma(A)$.

In order to define the state processes, let us introduce the following data:

- $T$ and $M$ are fixed real numbers.
- $K$ is a subset of $\mathbb{R}^r$.
- $\mathfrak{B} = \{(x, y) \in \mathbb{R}^r \rightarrow \mathbb{R}^r : (\forall j \in \mathbb{N}_r), 0 \leq x_j, \sum_{j=1}^{r} x_j \leq 1, |y_j| \leq 1$ and $\text{diag}[x]y \in K\}$.
- $A : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.
- $B : [0, T] \rightarrow \mathbb{R}^{n \times r}$.
- $D : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$.
- $F : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}$.
- $\zeta$ is a fixed vector in $\mathbb{R}^n$.
- $h : \mathbb{R}^r \rightarrow \mathbb{R}$ such that $h(x) = 1 - \sum_{j=1}^{r} x^j$.
- $l : \mathbb{R}^r \times \mathbb{R}^r \rightarrow \mathbb{R}^r$ such that $l_j(x, y) = x_j |y_j|$.
- $N : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $N(x, y) = (x - T)^2 + I_{\{y \leq M\}}(y - M)^3$.

The following assumptions will be used in the paper:

2 Problem statement

In this section, we formulate the stochastic control problem presented in the introduction using the formulation described in [2] and in [4].

Definition 2.1 A singular control is defined by the following term:

$$C \doteq (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, \{u(t)\}, \{W(t)\}, \{x(t)\})$$

where

(i) $(\Omega, \mathcal{F}, P)$ is a complete probability space with a right continuous complete filtration $\{\mathcal{F}_t\}$.

(ii) $\{W(t)\}$ is a standard $m$-dimensional $\{\mathcal{F}_t\}$-Brownian motion.

(iii) $\{u(t)\}$ is a $\mathbb{R}^r$-valued, corol, $\{\mathcal{F}_t^W\}$-progressively measurable process such that

$$\forall A \in \mathcal{B}([0, T]) \otimes \mathcal{F}, \quad \int_0^T I_A du(t) \in K, \quad \left|\text{Var}_{[0,T]}[u]\right| \leq M.$$
(iv) \( \{x(t)\} \) is an \( \mathbb{R}^n \)-valued, \( \mathcal{F}_t \)-progressively measurable process such that (\( \forall t \in [0,T] \))

\[
x(t) = \zeta + \int_0^t A(s, x(s)) \text{d}s + \int_{[0,t]} B(s) \text{d}u(s) + \int_0^t D(s, x(s)) \text{d}W(s),
\]

and \( x(0-) = \zeta \).

We write \( \mathcal{C} \) for the set of controls satisfying the previous conditions.

The cost is given by

\[
J[C] = \mathbb{E}_p[F(x(T), \text{Var}[u])].
\]

The set \( \mathcal{E}^a \) of admissible controls is defined by

\[
\mathcal{E}^a = \{ C \in \mathcal{C} : J[C] < \infty \}.
\]

The singular control problem is defined by the minimization of \( J[C] \) on \( \mathcal{E}^a \).

3 Preliminary results

Proposition 3.1 Assume the existence of an optimal singular control denoted by

\[
C^* = (\Omega^*, \mathcal{F}^*, P^*, \{\mathcal{F}_t^*\}, \{u^*(t)\}, \{W^*(t)\}, \{x^*(t)\}),
\]

such that \( \{u^*(t)\} \) is \( \{\mathcal{F}_t^*\} \)-progressively measurable.

Then, there exists a continuous time change \( \{\eta^*(t)\} \) such that

i) \( (\Omega^*, \mathcal{F}^*, P^*, \{\mathcal{F}_{\eta^*(t)}^*\}) \) is a probability space satisfying the usual hypotheses.

ii) On \( (\Omega^*, \mathcal{F}^*, P^*, \{\mathcal{F}_{\eta^*(t)}^*\}) \), there exists a \( \mathcal{B} \)-valued, \( \{\mathcal{F}_{\eta^*(t)}^*\} \)-progressively measurable process \( \{((\alpha^*(t), \theta^*(t))\} \) such that

\[
\eta^*(t) = \int_0^t h(\alpha^*(s)) \text{d}s,
\]

\[
J[C^*] = \mathbb{E}_{p^*}[F(\xi^*(T + M), \mu^*(T + M))],
\]

\[
\mathbb{E}_{p^*}[N(\eta^*(T + M), \mu^*(T + M))] = 0.
\]

where the processes \( \{\xi^*(t)\} \) and \( \{\mu^*(t)\} \) are solution of the following equations

\[
\xi^*(t) = \zeta + \int_0^t A(\eta^*(s), \xi^*(s)) h(\alpha^*(s)) \text{d}s + \int_{[0,t]} B(\eta^*(s)) \text{diag}[\alpha^*(s)] \theta^*(s) \text{d}s + \int_0^t D(\eta^*(s), \xi^*(s)) \sqrt{h(\alpha^*(s)) h^+(\alpha^*(s))} \text{d}W^*(\eta^*(s)),
\]

\[
\mu^*(t) = \int_0^t l(\alpha^*(s), \theta^*(s)) \text{d}s.
\]

Proof: On \( (\Omega^*, \mathcal{F}^*, P^*, \{\mathcal{F}_t^W\}) \), let us define the process \( t + \sum_{j=1}^r \text{Var}_j \{u_j^*\} \) by \( \{\Gamma(t)\} \). The process \( \{\eta(t)\} \) being the right inverse of \( \{\Gamma(t)\} \), we have that \( \Gamma(T) \) is an \( \{\mathcal{F}_{\eta(t)}^\Gamma\} \)-stopping time. Moreover,
using the fact that $\left| \text{Var}_{[0,T]}[u^*] \right| \leq M$, we have that $\Gamma(T) \leq T + M$. Clearly, the probability space $(\Omega^*, \mathcal{F}^*, P^*, \{\mathcal{F}_{\eta(t)}^*\})$ satisfies the usual hypotheses. Using Propositions 3.1 and 3.2 in [1], it follows that on $(\Omega^*, \mathcal{F}^*, P^*, \{\mathcal{F}_{\eta(t)}^*\})$, there exist a $\mathcal{B}$-valued, $\{\mathcal{F}_{\eta(t)}^*\}$-progressively measurable process $\{\alpha(t), \theta(t)\}$ such that

$$ \eta(t) = \int_0^t h(\alpha(s))ds, \quad (11) $$

$$ \text{Var}_{[0,t]}[u^*] = \int_0^\Gamma l(\alpha(s), \theta(s))ds, \quad u^*(t) = \int_0^\Gamma \text{diag}[\alpha(s)]\theta(s)ds. \quad (12) $$

Let us introduce the process

$$ \alpha^*(t) \triangleq \alpha(t)I_{[0,\Gamma(T)]} + e_1I_{[\Gamma(T), T + M]}, \quad (13) $$

$$ \theta^*(t) \triangleq \theta(t)I_{[0,\Gamma(T)]}, \quad (14) $$

$$ \eta^*(t) \triangleq \int_0^t h(\alpha^*(s))ds, \quad (15) $$

where $e_1$ is the first vector in the canonical basis of $\mathbb{R}^r$.

Since

$$ \eta^*(t) = \eta(t) \wedge T + [t - (T + M)] \vee 0, \quad (16) $$

it follows that $\{\eta^*(t)\}$ is a time change on $(\Omega^*, \mathcal{F}^*, P^*, \{\mathcal{F}_{\eta(t)}^*\})$. It is easy to show that $\Gamma(T)$ is an $\{\mathcal{F}_{\eta(t)}^*\}$-stopping time. Moreover, using the fact that $\mathcal{F}_{\eta(t)}^* \subset \mathcal{F}_{\eta(t)}^*$ and Theorem T57 in [7, p. 105], we obtain that $\{\alpha^*(t), \theta^*(t)\}$ is a $\{\mathcal{F}_{\eta(t)}^*\}$-progressively measurable process. Clearly, the probability space $(\Omega^*, \mathcal{F}^*, P^*, \{\mathcal{F}_{\eta(t)}^*\})$ satisfies the usual hypotheses and the stochastic differential equations (9), (10) admit a unique solution.

By definition of the process $\{\alpha^*(t), \theta^*(t)\}$, we have that

$$ \eta^*(T + M) = \eta^*(\Gamma(T)), \quad \xi^*(T + M) = \xi^*(\Gamma(T)), \quad \mu^*(T + M) = \mu^*(\Gamma(T)). \quad (17) $$

Following the proof of Theorem 4.2 in [1], it is easy to show that

$$ \eta^*(\Gamma(T)) = T, \quad \xi^*(\Gamma(T)) = x^*(T), \quad \mu^*(\Gamma(T)) = \text{Var}_{[0,T]}[u^*]. \quad (18) $$

Combining equations (17) and (18), the result follows.

Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ be a probability space supporting a standard $m$-dimensional Brownian motion $\{\bar{W}_t\}$ and denote by $\{\bar{\mathcal{F}}_t\}$, its natural filtration. Define by $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$, the usual augmentation of the probability space $(\Omega^* \times \bar{\Omega}, \mathcal{F}^* \otimes \bar{\mathcal{F}}, P^* \otimes \bar{P}, \mathcal{F}_{\eta(t)}^* \otimes \bar{\mathcal{F}}_t)$. A random variable $X^*$ defined on $(\Omega^*, \mathcal{F}^*, P^*)$ may be viewed as defined on $(\Omega, \mathcal{G}, Q)$ by setting $X(\omega^*, \bar{\omega}) = X^*(\omega^*)$ for $(\omega^*, \bar{\omega}) \in \Omega^* \times \bar{\Omega}$. Consequently, let us introduce on $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$ the following processes

$$ \bar{\alpha}(\omega^*, \bar{\omega}, t) \triangleq \alpha^*(\omega^*, t), \quad \bar{\theta}(\omega^*, \bar{\omega}, t) \triangleq \theta^*(\omega^*, t), \quad \bar{\eta}(\omega^*, \bar{\omega}, t) \triangleq \eta^*(\omega^*, t), $$

$$ \bar{\xi}(\omega^*, \bar{\omega}, t) \triangleq \xi^*(\omega^*, t), \quad \bar{\mu}(\omega^*, \bar{\omega}, t) \triangleq \mu^*(\omega^*, t), \quad \bar{W}(\omega^*, \bar{\omega}, t) \triangleq W^*(\omega^*, t). $$
Proposition 3.2  On $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$, the process $\{\overline{V}(t)\}$ defined by

$$\overline{V}(t) \doteq \int_0^t \sqrt{h^+(\alpha(s))} d\overline{W}(\overline{\eta}(s)) + \int_0^t \sqrt{1 - h(\alpha(s))h^+(\alpha(s))} d\overline{W}(s)$$  \hspace{1cm} (19)

is a standard $m$-dimensional $\{\mathcal{G}_t\}$-Brownian motion. The process $\{\overline{\xi}(t)\}$ is the unique solution of the following equation

$$\overline{\xi}(t) \doteq \zeta + \int_0^t A(\overline{\eta}(s), \xi(s))h(\alpha(s)) ds + \int_0^t B(\overline{\eta}(s)) \text{diag}[\alpha(s)] \overline{\theta}(s) ds$$

$$+ \int_0^t D(\overline{\eta}(s), \xi(s)) \sqrt{h(\alpha(s))} d\overline{W}(s).$$  \hspace{1cm} (20)

Moreover,

$$J[\overline{V}] = E_Q[F(\overline{\xi}(T + M), \overline{\eta}(T + M))],$$  \hspace{1cm} (21)

$$E_Q[N(\overline{\eta}(T + M), \overline{\eta}(T + M))] = 0.$$  \hspace{1cm} (22)

**Proof:** The process $\{\overline{V}(t)\}$ defined by equation (19) is a continuous $\{\mathcal{G}_t\}$-local martingale. Moreover,

$$\langle \overline{V}^i, \overline{V}^j \rangle(t) = \delta_{ij} \left[ \int_0^t h^+(\alpha(s)) d\overline{\eta}(s) + \int_0^t [1 - h^+(\alpha(s))h^+(\alpha(s))] ds \right].$$

However, using the definition of $\{\overline{\eta}(t)\}$ and equation (15), it follows that $\langle V^i, V^j \rangle_t = \delta_{ij} t$ which, by P. Levy’s characterization Theorem, gives that $\{\overline{V}(t)\}$ is a standard $m$-dimensional $\{\mathcal{G}_t\}$-Brownian motion.

Using Theorem 6 in [8, page 194], it follows that the solution $\{\xi(t)\}$ of equation (20) exists and is unique. From the definition of $\{\overline{V}(t)\}$, it is easy to show that $\{\xi(t)\}$ is the unique solution of the following equation:

$$\xi(t) \doteq \zeta + \int_0^t A(\eta(s), \xi(s))h(\alpha(s)) ds + \int_0^t B(\eta(s)) \text{diag}[\alpha(s)] \theta(s) ds$$

$$+ \int_0^t D(\eta(s), \xi(s)) \sqrt{h(\alpha(s))} dW(\eta(s)).$$  \hspace{1cm} (23)

However, the process $\{\xi^*(t)\}$ is solution of equation (9). From the definition of the processes $\{\overline{\xi}(t)\}$, $\{\overline{\eta}(t)\}$, $\{\overline{\theta}(t)\}$, $\{\overline{\eta}(t)\}$, it is easy to obtain that $\{\overline{\xi}(t)\}$ satisfy the following equation

$$\overline{\xi}(t) \doteq \zeta + \int_0^t A(\eta(s), \xi(s))h(\alpha(s)) ds + \int_0^t B(\eta(s)) \text{diag}[\alpha(s)] \theta(s) ds + \overline{M}(t)$$  \hspace{1cm} (24)

where $\{\overline{M}(t)\}$ is the process defined on $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$ by

$$\overline{M}(t)(\omega^*, \tilde{\omega}) \doteq M^*(t)(\omega^*)$$  \hspace{1cm} (25)

and $\{M^*(t)\}$ is the local martingale defined on $(\Omega^*, \mathcal{F}^*, P^*, \{\mathcal{F}^W_{\eta^*(t)}^*\})$ by

$$M^*(t) \doteq \int_0^t D(\eta^*(s), \xi^*(s)) \sqrt{h(\alpha^*(s))} h^+(\alpha^*(s)) dW^*(\eta^*(s)).$$  \hspace{1cm} (26)
Using Theorem 5.10 in [6, p. 180], it can be shown that

\[ \mathcal{M}(t) = \int_0^t D(\eta(s), \xi(s)) \sqrt{h(\alpha(s))h^+(\alpha(s))} dW(\eta(s)). \]  

(27)

Combining equations (24) and (27), we obtain that \( \{\xi(t)\} \) is solution of equation (23). Therefore, \( \{\xi(t)\} \) is the unique solution of equation (20).

Finally, equations (21) and (22) follow easily from equations (7) and (8) and the definition of the probability \( Q \).

\[ \square \]

On the probability space \((\Omega, \mathcal{G}, Q)\), define the filtration \( \mathcal{H}_t = \mathcal{F}^{\mathcal{W}^*}_{\eta(t)} \otimes \{\emptyset, \Omega\} \).

The set of auxiliary control \( \mathcal{U} \) is the set of \( \{\mathcal{H}_t\}\)-progressively measurable processes defined on \((\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})\) and taking their value in \( \mathcal{B} \).

For any \( \{(\alpha(t), \theta(t))\} \) in \( \mathcal{U} \), the auxiliary state process \( (\eta(t), \xi(t), \mu(t)) \) is defined \( \{(\eta(t), \xi(t), \mu(t))\} \) by

\[ \eta(t) = \int_0^t h(\alpha(s)) ds, \]  

(28)

\[ \xi(t) = \zeta + \int_0^t A(\eta(s), \xi(s)) h(\alpha(s)) ds + \int_0^t B(\eta(s)) \text{diag}[\alpha(s)] \theta(s) ds \]  

\[ + \int_0^t D(\eta(s), \xi(s)) \sqrt{h(\alpha(s))} d\mathcal{W}(s), \]  

(29)

\[ \mu(t) = \int_0^t \alpha(s) ds. \]  

(30)

Note that for any \( \{(\alpha(t), \theta(t))\} \) in \( \mathcal{U} \), the previous system admits a unique solution. Moreover, we have \( E_Q[g(\xi(T + M), \mu(T + M))] \leq \infty \).

The associated cost functional is defined by

\[ \mathcal{M}[\alpha, \theta] = E_Q[F(\xi(T + M), \mu(T + M))]. \]  

(31)

**Definition 3.3** The set of admissible auxiliary control \( \mathcal{U}_{ad} \) is defined by the set of processes \( \{(\alpha(t), \theta(t))\} \in \mathcal{U} \) such that the corresponding auxiliary state process \( \{(\eta(t), \xi(t), \mu(t))\} \) satisfies the following constraint

\[ E_Q[N(\eta(T + M), \mu(T + M))] = 0. \]  

(32)

The auxiliary control problem is to minimize the cost (31) over \( \mathcal{U}_{ad} \).

**Proposition 3.4** The process \( \{(\overline{\alpha(t)}, \overline{\theta(t)})\} \) is an optimal auxiliary control.

Moreover, \( \{(\overline{\alpha(t)}, \overline{\theta(t)})\} \) and the corresponding optimal auxiliary state \( \{(\overline{\eta}(t), \overline{\xi}(t), \overline{\mu}(t))\} \) are \( \{\mathcal{H}_t\}\)-progressively measurable processes.

**Proof:** Using Proposition 3.2, it is easy to check that the auxiliary control \( \{(\overline{\alpha(t)}, \overline{\theta(t)})\} \) belongs to \( \mathcal{U}_{ad} \). Moreover, we have

\[ \mathcal{M}[\overline{\alpha}, \overline{\theta}] = J[C^*]. \]  

(33)
The processes \( \{(\bar{\pi}(t), \bar{\theta}(t))\} \) and \( \{(-\bar{\pi}(t), -\bar{\theta}(t))\} \) are clearly \( \mathcal{H}_t \)-progressively measurable. Following Theorem 4.6 in [1], we have that for any control \( \{ (\alpha(t), \theta(t)) \} \in \mathcal{U}_{ad} \), there exists a control \( C \in \mathcal{C}^a \) such that

\[
J[C] \leq M[\alpha, \theta].
\]  

(34)

Combining equations (33) and (34), the result follows.

**Lemma 3.5** Let \( \phi \) be a real-valued random variable defined on \( (\Omega, \mathcal{F}, P^*) \) such that \( EP^*[\phi^2] < \infty \). Then there exists a \( \mathbb{R}^m \)-valued, \( \{\mathcal{F}_{\eta(t)}^{W^*}\} \)-progressively measurable process \( \{J(s)\} \) such that

\[
EP^*\left[ \int_0^{T+M} |J(s)|^2 h(\alpha^*(s)) ds \right] < \infty,
\]

(35)

and

\[
EP^*[\phi | \mathcal{F}_{\eta(t)}^{W^*}] = EP^*[\phi] + \sum_{j=1}^{m} \int_0^t J_j(s) dW^*(\eta^*(s)),
\]

(36)

for all \( t \leq T + M \).

**Proof:** Combining the optional stopping Theorem and the martingale representation Theorem (see Theorem 4.15 in [5, p. 182]), we obtain that there exists a \( \mathbb{R}^m \)-valued, \( \{\mathcal{F}_{W^*}^{\eta(t)}\} \)-progressively measurable process \( \{H(s)\} \) such that

\[
EP^*\left[ \int_0^{T} |H(s)|^2 ds \right] < \infty,
\]

(37)

and

\[
EP^*[\phi | \mathcal{F}_{\eta(t)}^{W^*}] = EP^*[\phi] + \sum_{j=1}^{m} \int_0^{\eta^*(t)} H_j(s) dW^*(s),
\]

(38)

for all \( t \leq T \). Now using Proposition 4.8 in [5, p. 176], we obtain that for all \( j \)

\[
\int_0^{\eta^*(t)} H_j(s) dW^*(s) = \int_0^{t} H_j(\eta^*(s)) dW^*(\eta^*(s)),
\]

(39)

so

\[
EP^*[\phi | \mathcal{F}_{\eta(t)}^{W^*}] = EP^*[\phi] + \sum_{j=1}^{m} \int_0^{t} H_j(\eta^*(s)) dW^*(\eta^*(s)).
\]

(40)

Moreover, using equation (37), we have

\[
EP^*\left[ \int_{\eta^*(0)}^{\eta^*(T+M)} |H(\eta^*(\Gamma^*(s)))|^2 ds \right] < \infty,
\]

(41)

giving

\[
EP^*\left[ \int_0^{T+M} |H(\eta^*(s))|^2 h(\alpha^*(s)) ds \right] < \infty,
\]

(42)

and the result follows.

\[\square\]
4 Main results

**Proposition 4.1** On the probability space \((\Omega^*, \mathcal{F}^*, P^*, \{\mathcal{F}^W_{\eta^*(t)}\})\), let us assume that the function \(f : \Omega^* \times [0, T + M] \times \mathbb{R}^k \times \mathbb{R}^{k \times m} \rightarrow \mathbb{R}^k\) is \(\mathcal{M} \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^{k \times m})\)-measurable where \(\mathcal{M}\) denotes the progressive \(\sigma\)-field when the probability space \((\Omega^*, \mathcal{F}^*, P^*)\) is equipped with the filtration \(\{\mathcal{F}^W_{\eta^*(t)}\}\) and satisfies

\[f(t, y, z) \leq L \|y\| + |z| \quad P^* \otimes \lambda - \text{a.s.}\]

Then for any given \(Y^* \in L^2_{\mathcal{F}^W_{\eta^*(T+M)}}(\Omega^*; \mathbb{R}^k)\), the following backward stochastic differential equation

\[Y^*(t) = Y^* - \int_t^{T+M} f(s, Y^*(s), \sqrt{h(\alpha^*(s))}Z^*(s))ds - \int_t^{T+M} Z^*(s)dW^*(\eta^*(s)) \quad (43)\]

admits a unique solution

\[(Y^*(t), Z^*(t)) \in L^2_{\mathcal{F}^W_{\eta^*(T+M)}}(\Omega^*, C([0, T + M]; \mathbb{R}^k)) \times L^2_{\mathcal{F}^W_{\eta^*(T+M)}}(\Omega^*; [0, T + M]; \mathbb{R}^{k \times m}).\]

**Proof:** For any fixed \(\{(y(t), z(t))\} \in L^2_{\mathcal{F}^W_{\eta^*(T+M)}}(\Omega^*, C([0, T + M]; \mathbb{R}^k)) \times L^2_{\mathcal{F}^W_{\eta^*(T+M)}}(\Omega^*; [0, T + M]; \mathbb{R}^{k \times m})\), we have that \(f(t, y(t), \sqrt{h(\alpha^*(s))}z(t)) \in L^2_{\mathcal{F}^W_{\eta^*(T+M)}}(\Omega^*; [0, T + M]; \mathbb{R}^k)\).

Consider the following BSDE

\[dY(t) = f(t, y(t), \sqrt{h(\alpha^*(s))}z(t))dt + Z(t)dW^*(\eta^*(t)) \quad (44)\]

with \(Y(T + M) = Y^*\).

Let \(M(t) = E[Y^* - \int_0^{T+M} f(s, y(s), \sqrt{h(\alpha^*(s))}z(s))ds|\mathcal{F}^W_{\eta^*(t)}]\). Then using Lemma 3.5, there exists a process \(Z^*(t) \in L^2_{\mathcal{F}^W_{\eta^*(T+M)}}(\Omega^*; [0, T + M]; \mathbb{R}^{k \times m})\), such that

\[M(t) = M(0) + \int_0^t Z^*(s)dW^*(\eta^*(t)). \quad (45)\]

Now define

\[Y^*(t) = M(t) + \int_0^t f(s, y(s), \sqrt{h(\alpha^*(s))}z(s))ds. \quad (46)\]

It is easily seen that the pair \(\{(Y^*(t), Z^*(t))\}\) solves equation (44).

Moreover, from equation (45) it follows that \(Y^*(t) \in L^2_{\mathcal{F}^W_{\eta^*(T+M)}}(\Omega^*, C([0, T + M]; \mathbb{R}^k))\).

Having shown this first step, we can now follow the proof of Theorem 2.1 in [3] to obtain the existence and the uniqueness of the BSDE (43).
Corollary 4.2 On the probability space \((\Omega, \mathcal{G}, Q, \{\mathcal{F}_t\})\), let us assume that the function 
\[ g : \Omega \times [0, T + M] \times \mathbb{R}^k \times \mathbb{R}^{k \times m} \rightarrow \mathbb{R}^k \text{ is } \mathcal{N} \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^{k \times m}) \text{-measurable where } \mathcal{N} \text{ denotes the progressive } \sigma\text{-field.} \]

where \(\mathcal{N}\) denotes the progressive \(\sigma\text{-field when the probability space } (\Omega, \mathcal{G}, Q)\) is equipped with the filtration \(\{\mathcal{F}_t\}\) and satisfies

i) \(g(\cdot, 0, 0) \in L^2_{\mathcal{H}}(\Omega; [0, T + M]; \mathbb{R}^k)\).

ii) There exists a constant \(L > 0\) such that \((\forall (y, \bar{y}, z, \bar{z}) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{k \times m} \times \mathbb{R}^{k \times m})\)

\[ |g(t, y, z) - g(t, \bar{y}, \bar{z})| \leq L |y - \bar{y}| + |z - \bar{z}| \quad Q \otimes \lambda \text{-a.s.} \]

Then for any given \(Y \in L^2_{\mathcal{H}^{T+M}}(\Omega; \mathbb{R}^k)\), the following backward stochastic differential equation

\[ Y(t) = Y - \int_t^{T+M} g(s, Y(s), \sqrt{h(\alpha(s))}X(s))ds - \int_t^{T+M} X(s)d\mathcal{N}_t \quad (47) \]

admits a unique solution \((Y(t), X(t)) \in L^2_{\mathcal{H}}(\Omega; C([0, T + M]; \mathbb{R}^k)) \times L^2_{\mathcal{H}}(\Omega; [0, T + M]; \mathbb{R}^{k \times m})\)

Moreover, the process \(\{X(t)\}\) can be written in the following form

\[ X(t) = \sqrt{h(\alpha(t))}Z(t), \quad (48) \]

where \(Z(t) \in L^{2, h(\alpha)}_{\mathcal{H}}(\Omega; [0, T + M]; \mathbb{R}^{k \times m}).\)

Proof: For \((y, z)\) fixed in \(\mathbb{R}^k \times \mathbb{R}^{k \times m}\), the process \(\{g(t, x, y)\}\) defined on \((\Omega, \mathcal{G}, Q)\) is \(\{\mathcal{H}_t\}\)-progressively measurable. Since \(\mathcal{H}_t = \mathcal{F}^{W^*}_t \otimes \{0, \Omega\}, \)

\[ (\forall \omega^* \in \Omega^*), \quad (\forall (\bar{\omega}_1, \bar{\omega}_2) \in \bar{\Omega} \times \bar{\Omega}), \quad g(\omega^*, \bar{\omega}_1, t, x, y) = g(\omega^*, \bar{\omega}_2, t, x, y). \]

For \(\bar{\omega}_1\) fixed in \(\bar{\Omega}\), let us define the function \(f : \Omega^* \times [0, T + M] \times \mathbb{R}^k \times \mathbb{R}^{k \times m} \rightarrow \mathbb{R}^k\) such that \(f(\omega^*, t, x, y) = g(\omega^*, \bar{\omega}_1, t, x, y).\) Then \(f\) is \(\mathcal{M} \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^{k \times m})\)-measurable since \(\mathcal{N}\) is isomorphic to \(\mathcal{M} \otimes \{0, \Omega\}.\)

Clearly, \(f\) so defined satisfies items i) and ii) of Proposition 4.1. Therefore, we can claim that there exists a unique solution to the following equation

\[ Y^*(t) = Y^* - \int_t^{T+M} f(s, Y^*(s), \sqrt{h(\alpha^*(s))}Z^*(s))ds - \int_t^{T+M} Z^*(s)dW^*(\eta^*(s)). \]

It is easy to show that

\[ \int_t^{T+M} Z^*(s)dW^*(\eta^*(s)) = \int_t^{T+M} Z^*(s)\sqrt{h(\eta^*(s))}h^+(\eta^*(s))dW^*(\eta^*(s)). \]

Now define

\[ Y(\omega^*, \bar{\omega}, t) = Y^*(\omega^*, t), \quad Z(\omega^*, \bar{\omega}, t) = Z^*(\omega^*, t). \]

Since, \((Y^*(t), Z^*(t)) \in L^2_{\mathcal{H}^{T+M}}(\Omega^*; C([0, T + M]; \mathbb{R}^k)) \times L^{2,h(\alpha^*)}_{\mathcal{H}^{T+M}}(\Omega^*; [0, T + M]; \mathbb{R}^{k \times m}),\)

then \(\{(Y(t), X(t)) \in L^2_{\mathcal{H}}(\Omega; C([0, T + M]; \mathbb{R}^k)) \times L^2_{\mathcal{H}}(\Omega; [0, T + M]; \mathbb{R}^{k \times m})\}\)

where

\[ X(t) = Z(t)\sqrt{h(\eta^*(s))} \text{ and } Z(t) \in L^{2, h(\alpha)}(\Omega; [0, T + M]; \mathbb{R}^{k \times m}). \]

Following the same arguments as in the proof of Proposition 3.2, we obtain that the process \(\{(Y(t), X(t))\}\) is solution of the BSDE (47). Then we have shown the existence of a solution with the desired property (see equation (48)). The uniqueness follows exactly as in [3]. \qed
For $p \in \mathbb{R}^{1+n+r}$, let us define $p^n \in \mathbb{R}$, $p^\xi \in \mathbb{R}^n$ and $p^\mu \in \mathbb{R}^r$ such that $p = \begin{pmatrix} p^n \\ p^\xi \\ p^\mu \end{pmatrix}$.

For $q \in \mathbb{R}^{(1+n+r)\times m}$, let us denote $q^n \in \mathbb{R}^{1\times m}$, $q^\xi \in \mathbb{R}^{n\times m}$ and $q^\mu \in \mathbb{R}^{r\times m}$ such that $q = \begin{pmatrix} q^n \\ q^\xi \\ q^\mu \end{pmatrix}$.

Let us introduce the following map: $(\forall (\eta, \xi, p^\xi) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$

$$H(\eta, \xi, p^\xi) = A(\eta, \xi)^\top p^\xi.$$ 

**Theorem 4.3** There exists a unique solution to the following backward stochastic differential equation

\begin{align*}
dp^n(t) &= -A^n(\bar{\eta}(t), \bar{\xi}(t))^\top \pn(t) + [B^n(\bar{\eta}(t))\diag(\bar{\theta}(t))]^\top \pn(t) + \sum_{j=1}^m D_{jn}(\bar{\eta}(t), \bar{\xi}(t))^\top q^n_j(t) \sqrt{h(\bar{\theta}(t))} dt + \pn(t)d\bar{V}(t) \\
dp^\xi(t) &= -A^\xi(\bar{\eta}(t), \bar{\xi}(t))^\top \p^\xi(t) h(\bar{\theta}(t)) dt - \sum_{j=1}^m D_{j\xi}(\bar{\eta}(t), \bar{\xi}(t))^\top q^\xi_j(t) \sqrt{h(\bar{\theta}(t))} dt + q^\xi(t)d\bar{V}(t) \\
dp^\mu(t) &= q^\mu(t)d\bar{V}(t)
\end{align*}

with

\begin{align*}
p^n(T + M) &= -\rho_0 N_\eta(\bar{\eta}(T + M), \bar{\xi}(T + M)) \\
p^\xi(T + M) &= -\rho F_\xi(\bar{\xi}(T + M), \bar{\mu}(T + M)) \\
p^\mu(T + M) &= -\rho_0 N_\mu(\bar{\xi}(T + M), \bar{\mu}(T + M)) - \rho F_\mu(\bar{\xi}(T + M), \bar{\mu}(T + M))
\end{align*}

and

\begin{align*}
dP(t) &= -A_\xi(\bar{\eta}(t), \bar{\xi}(t))^\top P(t) h(\bar{\theta}(t)) dt - P(t) A_\xi(\bar{\eta}(t), \bar{\xi}(t)) h(\bar{\theta}(t)) dt - \sum_{j=1}^m [D_{j\xi}(\bar{\eta}(t), \bar{\xi}(t))^\top P(t) [D_{j\xi}(\bar{\eta}(t), \bar{\xi}(t))] h(\bar{\theta}(t)) dt \\
&\quad - \sum_{j=1}^m \left([D_{j\xi}(\bar{\eta}(t), \bar{\xi}(t))^\top Q_j(t) + Q_j(t) [D_{j\xi}(\bar{\eta}(t), \bar{\xi}(t))]\right) \sqrt{h(\bar{\theta}(t))} dt + H_{\xi\xi}(\bar{\eta}(t), \bar{\xi}(t), p^\xi(t)) h(\bar{\theta}(t)) dt + \sum_{j=1}^m Q_j(t) d[\nabla(t)]_j
\end{align*}

with

\begin{align*}
P(T + M) &= -\rho_0 F_{\xi\xi}(\bar{\xi}(T + M), \bar{\mu}(T + M)), \\
such that \{p(t)\} \in L^2_H(\Omega; [0, T + M]; \mathbb{R}^{1+n+r}), \{q(t)\} \in L^2_H(\Omega; [0, T + M]; \mathbb{R}^{(1+n+r)\times m}), \{P(t)\} \in L^2_H(\Omega; [0, T + M]; \mathbb{S}^n) and \{Q_j(t)\} \in L^2_H(\Omega; [0, T + M]; \mathbb{S}^n), for all j \in \mathbb{N}_m
\end{align*}
where \( p(t) \equiv \begin{pmatrix} p^n(t) \\ p^\xi(t) \\ p^\mu(t) \end{pmatrix} \) and \( q(t) \equiv \begin{pmatrix} q^n(t) \\ q^\xi(t) \\ q^\mu(t) \end{pmatrix} \). Moreover, there exist
\[
\{k(t)\} \in L^2_{\eta\xi}(\Omega; [0, T+M]; \mathbb{R}^{(1+n+r)\times m}), \quad \text{and} \quad \{Q_j(t)\} \in L^2_{\eta\xi}(\Omega; [0, T+M]; \mathbb{S}^n),
\]
such that
\[
q(t) = \sqrt{h(\alpha(t))}k(t)
\]
\[
Q_j(t) = \sqrt{h(\alpha(t))}K_j(t)
\]
for all \( j \in \mathbb{N}_m \).

**Proof:** This result is a straightforward consequence of the previous Theorem. \( \square \)

Let us introduce the Hamiltonian:
\[
\mathcal{H}(\eta, \xi, \alpha, \theta, p, \bar{q}, P) \equiv H(\eta, \xi, p^\xi)h(\alpha) + \theta^\top \text{diag}[\alpha]B(\eta)^\top p^\xi + p^n h(\alpha) + l(\alpha, \theta)^\top p^\mu
\]
\[
+ \frac{1}{2} \text{tr}[D(\eta, \xi)^\top PD(\eta, \xi)]h(\alpha) + \text{tr}[D(\eta, \xi)^\top \bar{q}]\sqrt{h(\alpha)}.
\]

**Theorem 4.4** There exist \((\rho_0, \rho) \in \mathbb{R} \times \mathbb{R} \) satisfying \( \rho_0 \geq 0 \) and \( \rho_0^2 + \rho^2 = 1 \) and
\[
\{p(t)\} \in L^2_{\eta\xi}(\Omega; [0, T+M]; \mathbb{R}^{1+n+r}), \quad \{q(t)\} \in L^2_{\eta\xi}(\Omega; [0, T+M]; \mathbb{R}^{1+n+r})
\]
\[
\{P(t)\} \in L^2_{\eta\xi}(\Omega; [0, T+M]; \mathbb{S}^n), \quad \text{and} \quad \{Q_j(t)\} \in L^2_{\eta\xi}(\Omega; [0, T+M]; \mathbb{S}^n),
\]
for all \( j \in \mathbb{N}_m \) solutions of equations (49)-(56) such that \((\forall (\alpha, \theta) \in \mathbb{R}^r \times \mathbb{R}^r)\)
\[
\mathcal{H}(\tilde{\eta}(t), \tilde{\xi}(t), \tilde{\alpha}(t), \tilde{\theta}(t), p(t), r(t), P(t)) \geq \mathcal{H}(\eta(t), \xi(t), \alpha, \theta, p(t), r(t), P(t)),
\]
where \( r(t) \equiv q^\xi(t) - P(t)D(\eta, \xi)\sqrt{h(\alpha(t))} \).

**References**


