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Option Pricing for Log-Symmetric Distributions of Returns

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Abstract We derive an option pricing formula on assets with returns distributed according to a log-symmetric distribution. Our approach is consistent with the no-arbitrage option pricing theory: we propose the natural risk-neutral measure that keeps the distribution of returns in the same log-symmetric family reflecting thus the specificity of the stock's returns. Our approach also provides insights into the Black–Scholes formula and shows that the symmetry is the key property: if distribution of returns X is log-symmetric then $1/X$ is also log-symmetric from the same family. The proposed options pricing formula can be seen as a generalization of the Black–Scholes formula valid for lognormal returns. We treat an important case of log returns being a mixture of symmetric distributions with the particular case of mixtures of normals and show that options on such assets are underpriced by the Black–Scholes formula. For the log-mixture of normal distributions comparisons with the classical formula are given.

Keywords Martingale measure · Option price · Returns · Log-symmetric distribution · Mixture of normal distributions

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1 Introduction

The famous Black–Scholes formula for pricing options has paved the way for economic valuations in many areas and is used for more efficient risk management in society. Option theory is used not only for stocks, bonds and other traded financial papers but also for evaluation of various government guarantees, such as in underwriting price of wheat, wool and other commodities, e.g. Bardsley and Cashin (1990); Turvey (1992). In this wider context the Black–Scholes formula is used even if some of the basic assumptions on its derivations are not satisfied. It is therefore important to broaden the set of assumptions and obtain a generalized Black–Scholes formula.

In the classical, Black–Scholes approach to option pricing it is assumed "a priori" that the daily returns on assets have a lognormal distribution, i.e. their logarithms are normally distributed. In this paper we examine relaxation of this assumption and replace it by a wider log-symmetry assumption. Empirical evidence shows that log daily returns for some assets have symmetric distributions with tails different to normal. We give a modification of the Black–Scholes formula for such cases.

Log-symmetric distributions belong to the log-elliptical family of distributions considered by Fang et al. (1990). It includes important classes of log-Student-t, log-exponential power family (log-EPF) and log-mixtures.

In McDonald (1996, 3.23a–b) it was suggested to use the distribution function of the underlying distribution of returns instead of the standard normal in the Black–Scholes formula for option pricing. However, such an approach is not consistent with no arbitrage theory of option pricing. Here we give a model for prices based on log-symmetric daily returns, and then present an option pricing formula on such assets that firstly, is consistent with the no-arbitrage option pricing theory and secondly, reflects the specificity of the stock's returns by keeping the risk-neutral distribution of the returns in the same family as the observed returns.

For pricing options by no-arbitrage principle, one needs to use an equivalent martingale/ risk-neutral measure, namely, such a probability distribution Q that makes the discounted stock price process $S_n e^{-rn}$ into a Q-martingale. Here S_n is the price of stock at time n and e^r is risk-free return over one period. It turns out that any Q satisfying the equation

$$E_Q(X_1) = e^r, \tag{1.1}$$

where X_1 denotes the return on stock over a single period, and keeps returns independent is an equivalent martingale measure. This equation is also justified by the economic argument that in the risk-neutral world all assets have expected return equal to the risk-free return e^r. Mathematically, there are many distributions, equivalent to the original one, which shift the mean of X to e^r. However, it seems only natural to keep the distribution of X in the same log-symmetric family and change only the location parameter. The proposed choice of Q (we call it *natural*) is unambiguous and satisfies the economic requirements of keeping returns independent. It also recovers the Black–Scholes formula if the original distribution of returns is lognormal. Using the change of numeraire technique, the option price is given by

$$C = S_0 Q_1(S_N > K) - Ke^{-rN} Q(S_N > K),$$

where Q is the equivalent measure under which the process S_n/e^{rn} is a martingale, and Q_1 is the equivalent measure under which e^{rn}/S_n is a martingale. It is an important observation that the symmetry in a log-symmetric distribution guarantees that the reciprocal variable is also log-symmetric. In models we consider variances exist, therefore the normal approximation to the above probabilities for $Q_1(S_N > K)$ and $Q(S_N > K)$ yields a formula similar to the Black–Scholes formula. As a result we find conditions when the assets with log-symmetric distributions are underpriced by the classical Black–Scholes formula. We specify the formula for the important case of log-mixture of normals and demonstrate that if the distribution contains contaminate data this underpricing becomes significant.

2 Log-Symmetric Distributions

In this section we introduce notations and survey results on log-symmetric distributions needed in the sequel.

The class of log-symmetric distributions is a generalization of the lognormal distribution. X has a log-symmetric distribution if $Y = \log X$ has a symmetric distribution, namely there is a number μ, called the location parameter, such that

$$-(Y-\mu) \stackrel{D}{=} (Y-\mu).$$

The definition in terms of characteristic functions is given below by Eq. 2.1. This is the univariate analogue of the multivariate version of the log-elliptic distribution (Fang et al. 1990, Section 2.8). Log-symmetric family has a long history of modelling security returns. It was observed that their empirical distributions have more kurtosis than predicted by the normal distribution and this phenomenon often results in heavier tails. Fama (1965) made the first detailed study of stock returns in the context of symmetric stable distributions. Blattberg and Gonedes (1974) alternatively suggested using a log-Student-t distribution. Hürlimann (1995, 2001) shows good fit for daily index returns and non-life insurance data with log-Laplace and log-double Weibull distributions. In these papers he gives interesting theoretical and empirical arguments in favor of considering above data as belonging to some transform $T(\cdot)$ [where the ubiquitous one is logarithm, $T(\cdot) = \ln(\cdot)$] of a symmetric family of distributions. He gives estimation methods for the location and scale parameters. He also gives goodness-of-fit statistics, where together with classical chi-square and Andersen–Darling statistics (Hürlimann calls it Cramer–von-Mises statistics, cf. D'Agostino and Stephens 1986, Sections 4.2.2, 4.2.3) he also suggested *mean excess* distance statistics and *limited expected value* distance statistics, more familiar to actuarial audience. As the goal of this paper lies beyond a deep discussion of estimating technics, we refer the reader to cited above papers and point out that these techniques are well documented in statistical and finance literature. Notice also that the volatility σ in our approach is simply the scale parameter of the corresponding symmetric distribution and can be estimated with the methods mentioned above.

The characteristic function $\varphi(t)$ of a univariate distribution symmetric around 0 has the form

$$\varphi(t) = \psi\left(\frac{1}{2}t^2\right),$$

where function $\psi(u) : [0, \infty) \to R$ is called the characteristic generator of the symmetric family. The random variable Y is symmetric with location μ and scale σ, denoted $Y \sim S(\mu, \sigma^2, \psi)$, if it's characteristic function can be expressed as

$$\varphi_Y(t) = e^{it\mu}\psi\left(\frac{\sigma^2}{2}t^2\right). \tag{2.1}$$

Clearly, for the normal family the characteristic generator is $\psi(u) = e^{-u}$. In general, a member of the log-symmetric family need not have a density, but if the density exists it has the following form

$$f_Y(y, \mu, \sigma, g) = c\frac{1}{\sigma}g\left(\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}\right). \tag{2.2}$$

The function $g(u)$ is known as the density generator of the symmetric family. The condition

$$\int_0^\infty x^{-1/2}g(x)dx < \infty \tag{2.3}$$

guarantees that $g(x)$ is a density generator (Fang et al. 1990, Chap. 2.2). The normalizing constant c can be explicitly determined,

$$c = \frac{1}{\sqrt{2\pi}}\left[\int_0^\infty x^{-1/2}g(x)dx\right]^{-1}. \tag{2.4}$$

For a normal family the density generator is $g(u) = e^{-u}$.

For a random variable Y from a symmetric family we denote $Y \sim S(\mu, \sigma^2, \psi)$ when the distribution is specified by the characteristic generator, and $Y \sim S(\mu, \sigma^2, g)$ when it is specified by the density generator. It should be noted that the condition 2.3 does not imply existence of the mean and variance of Y. It can be shown by a simple transformation in the equation $EY = \int yf_Y(y)dy$ that the condition

$$\int_0^\infty g(x)dx < \infty \tag{2.5}$$

guarantees the existence of the mean. It then follows that for $Y \sim S(\mu, \sigma^2, \psi)$ ($Y \sim S(\mu, \sigma^2, g)$) the parameter μ is the mean of Y, $EY = \mu$. If in addition

$$\left|\psi'(0)\right| < \infty, \tag{2.6}$$

then the variance exists and is equal to

$$V(Y) = -\psi'(0)\,\sigma^2. \tag{2.7}$$

The characteristic generator can be chosen in such a way that

$$\psi'(0) = -1, \tag{2.8}$$

so that the variance becomes equal to the parameter σ^2,

$$V(Y) = \sigma^2.$$

Notice that the condition 2.6 is equivalent to the condition

$$\int_0^\infty \sqrt{x}g(x)dx < \infty. \tag{2.9}$$

For details on symmetric (elliptical) distributions, examples and application in evaluation of some risk measures see Landsman and Valdez (2003).

We say that a random variable X has a log-symmetric distribution, denoted $X \sim LS(\mu, \sigma^2, \psi)$ (or $X \sim LS(\mu, \sigma^2, g)$), if $Y = \log(X)$ has a symmetric distribution $Y \sim S(\mu, \sigma^2, \psi)$ (or $S(\mu, \sigma^2, g)$). If the condition 2.4 holds for $X \sim LS(\mu, \sigma^2, g)$ we can write the density function of X

$$f_X(x, \mu, \sigma, g) = c\frac{1}{\sigma}\frac{1}{x}g\left(\frac{1}{2}\frac{(\log x - \mu)^2}{\sigma^2}\right). \tag{2.10}$$

This formula is a generalization of the lognormal distribution.

The expectation of a log-symmetric distribution may fail to exist (as, for example, the log-Student distribution), because the condition

$$EX = Ee^Y < \infty, \tag{2.11}$$

requires at least exponential decay of the density of the corresponding symmetric distribution. Notice that the condition 2.11 is equivalent to the extension of the characteristic generator $\psi(t)$, defined on non-negative part of the line R^+, to the negative part R^- (or at least to its subset), so that the moment generating function of Y is given by (see Eq. 2.1)

$$M_Y(t) = Ee^{tY} = e^{t\mu}\psi\left(-\frac{\sigma^2}{2}t^2\right). \tag{2.12}$$

If the moment generating function of Y exists for $t \leq 1$ then mean of a log-symmetric distribution is given by

$$E(X) = e^{\mu}\psi\left(-\frac{\sigma^2}{2}\right). \tag{2.13}$$

In the following subsection we consider a log-symmetric family with the characteristic generator that can be extended to the negative part R^-.

Log-exponential power family (LEPF)

The log-exponential power family is a log-symmetric family with the density generator of the form

$$g(u) = \exp(-b\,u^{\delta/2}), u > 0, b > 0.$$

Parameter $\delta > 0$ is called the power parameter. Choosing constant

$$b = \left(\frac{\Gamma\left(\frac{3}{\delta}\right)2}{\Gamma\left(\frac{1}{\delta}\right)}\right)^{\delta/2},$$

we get the property that the variable $Y \sim S(\mu, \sigma, g)$ has variance $V(Y) = \sigma^2$ because Eq. 2.8 holds. The constant c is then equal

$$c = \frac{\delta\Gamma\left(\frac{3}{\delta}\right)^{1/2}}{2\Gamma\left(\frac{1}{\delta}\right)^{3/2}}.$$

The density of LEPF can be written from Eq. 2.10

$$f_X(x,\mu,\sigma,g) = c\frac{1}{\sigma}\frac{1}{x}\exp\left(-b\left(\frac{1}{2}\frac{(\log x-\mu)^2}{\sigma^2}\right)^{\delta/2}\right).$$

From the table of Fourier transforms (see Oberhettinger 1973, Table 1:83) we obtain straightforwardly that characteristic generator of EPF equals

$$\psi(u) = \frac{1}{\Gamma(\frac{1}{\delta})}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{2n+1}{\delta})}{(2n)!}\left(-\frac{2\Gamma(\frac{1}{\delta})}{\Gamma(\frac{3}{\delta})}u\right)^n. \tag{2.14}$$

For $\delta = 2$ (normal family), we get, taking into account that

$$\Gamma\left(n+\frac{1}{2}\right) = (2n-1)!!\frac{\sqrt{\pi}}{2^n},$$

(here $n!! = 1\cdot 3\cdot 5\cdots\ n$ if n is odd and $n!! = 2\cdot 4\cdot 6\cdots n$ if n is even) we recover the normal generator given in the previous section

$$\psi(u) = \sum_{n=0}^{\infty}\frac{1}{n!}(-u)^n = \exp(-u). \tag{2.15}$$

For $\delta = 1$ we have the double Laplace distribution. It follows from Eq. 2.14

$$\psi(u) = \sum_{n=0}^{\infty}(-u)^n = \frac{1}{1+u},\quad 0 < u < 1, \tag{2.16}$$

3 Generalized Black–Scholes Model in Discrete Time

The main model for options pricing in finance is the Black–Scholes model. This is a continuous time model, which is obtained when the constant continuously compounding rate of return is perturbed by white noise. This implies that stock prices follow a geometric Brownian motion and that log returns have a normal distribution. Empirically observed distributions, however, are often not lognormal, but rather log-symmetric. In the proposed model we replace the assumption of log-normality by a more general assumption of log-symmetry. Although continuous time models are more flexible because they can accommodate trading at any times and more convenient for mathematical analysis, here we develop a model in discrete time. This is mainly due to following reasons. Firstly, in practice stock prices are necessarily observed at discrete times, therefore it is reasonable to model prices at such times. Secondly, there is a problem of embedding an arbitrary marginal distribution of returns (such as a given log-symmetric distribution) into a continuous time process, because for many distributions continuous time processes with given marginals do not have suitable properties, for example their trajectories may have jumps rather than being continuous. Some research has been done recently on models of Levy processes in finance, e.g. Schoutens (2003), and a continuous time model with given log-symmetric marginals will be developed elsewhere. Finally, a model in discrete time is simpler and can serve as an approximation to the continuous time model, cf. the well-known binomial model as an approximation to the Black–Scholes.

In our model we keep two out of three assumptions of the Black–Scholes model. Namely, we assume independence of returns and that they come from the same distribution. We do not assume that this distribution is lognormal but a more general log-symmetric distribution. The assumptions of independence of returns and time invariance of their distribution allows one to obtain the empirical distribution and choose the most suitable theoretical model.

We assume that the price process S_n is observed at times $n = 0, 1, 2 \dots, N$. Let X_n denote the returns

$$X_n = \frac{S_n}{S_{n-1}}. \tag{3.1}$$

Clearly, the stock price at time N is given by the product of returns

$$S_N = S_0 \prod_{n=1}^{N} X_n. \tag{3.2}$$

We assume that X_n's are strictly positive (prices are positive), independent and identically distributed random variables with a log-symmetric distribution. Our task is to price an option on S_N, such as a call option paying $(S_N - K)^+ = \max(S_N - K, 0)$ at time N. We take the approach of pricing by no arbitrage theory. In this theory, the price of an option is such a value (price) that does not allow for existence of arbitrage trading strategies. An arbitrage strategy is a self-financing portfolio with initial zero value and a non-negative final value. It turns out that in many models the arbitrage-free price exists and is unique. By the first fundamental theorem of asset pricing (Harrison and Pliska (1981); Shiryaev (1999), p. 413, Chap. 5 2b, Klebaner (2005) Section 11.2) the model does not have arbitrage opportunities if and only if there exists an equivalent martingale measure (EMM), i.e. a measure Q with the same null sets as the original measure P , such that the discounted by the risk-free interest rate stock price process $S_n e^{-rn}$, $n \le N$, is a martingale under the probability Q. Here r stands for the continuously compounded riskless rate of return over one period.

Remark here that the risk-neutral approach to pricing of options necessarily demands existence of the first moments of the stock price process, because it is a prerequisite condition for a process to be a martingale.

For option pricing we change the original probability P to Q, and the related properties and quantities referring to Q are prefaced by Q, for example Q-martingale, Q-distribution, Q-mean, meaning that the calculations are done by using the probability Q.

It is easy to see that under the probability Q self-financing replicating portfolios discounted by the risk-free rate are also martingales under Q. Due to this property the approach to pricing of options by using an equivalent martingale measure Q is also known as risk-neutral. Thus any self-financing portfolio has the mean under Q discounted by the risk-free rate equal to its initial value. Equivalence of measures implies that it is impossible to create wealth starting from zero, thus eliminating arbitrage strategies. For an option paying C_N at time N we propose to take its price at time zero

$$C_0 = e^{-rN} E_Q(C_N).$$

Such choice does not lead to arbitrage opportunities: this is well known when a perfect hedge exists, but it is also true when a perfect hedge for C_N does not exist, (see Shiryaev 1999, p. 398).

Next we represent suggested price in terms of the Q-distribution of the stock and its reciprocal. The arbitrage-free price of a call option with time to expiration N and strike K is given by

$$\begin{aligned} C &= e^{-rN} E_Q(S_N - K)^+ \\ &= e^{-rN} E_Q(S_N - K) I(S_N > K) \\ &= e^{-rN} E_Q(S_N I(S_N > K)) - e^{-rN} K E_Q(I(S_N > K)) \\ &= e^{-rN} E_Q(S_N I(S_N > K)) - e^{-rN} K Q(S_N > K) \\ &= S_0 Q_1(S_N > K) - e^{-rN} K Q(S_N > K), \end{aligned} \tag{3.3}$$

where Q_1 is the measure under which e^{rn}/S_n is a martingale. The last step in the above formula is justified as follows. $S_N e^{-rN}$ is positive and because $S_n e^{-rn}$ is a Q-martingale $E_Q(S_N e^{-rN}) = S_0$. Thus $\Lambda = S_N e^{-rN}/S_0$ is a positive random variable with $E_Q(\Lambda) = 1$. Define Q_1 by $dQ_1/dQ = \Lambda$. Then Q_1 is also a probability measure, and expectations under these probabilities are related by the formula: for any random variable U,

$$E_{Q_1}(U) = E_Q(\Lambda U) = E_Q((S_n e^{-rn}/S_0)U).$$

Thus

$$\begin{aligned} e^{-rN} E_Q(S_N I(S_N > K)) &= S_0 E_Q(\Lambda I(S_N > K)) \\ &= S_0 E_{Q_1}(I(S_N > K)) = S_0 Q_1(S_N > K). \end{aligned}$$

This is the change of numeraire formula (see Geman et al. 1995, or e.g. Klebaner 2005, Sect. 11.5). The above calculations are valid for any equivalent martingale measure Q.

It turns out that in the models under consideration there are infinitely many equivalent martingale measures $Q \sim P$, in other words such models are incomplete. For pricing in incomplete markets a number of choices for Q have been suggested in the literature. For example, Gerber and Shiu (1994) (see also Kallsen and Shiryaev 2002, and references therein) suggested the use of the Esscher's change of measure. The Esscher transform is attractive, because in the lognormal setup the group of Esscher transforms of the normal distribution coincides with the group of locations of the normal distribution and as such it does not take us outside the normal family. In fact, all symmetric families preserve the location invariance property, because for any symmetric $Y \sim S(\mu, \sigma^2, \psi)$ and any constant c, $Y' = Y + c$ has characteristic function (see Eq. 2.1)

$$\varphi_{Y'}(t) = e^{it(\mu+c)} \psi\left(\frac{\sigma^2}{2} t^2\right),$$

i.e., $Y' \sim S(\mu + c, \sigma^2, \psi)$. In that sense we consider the group of locations of distribution from the symmetric family as a natural basis for the appropriate change of measure instead of the "literal" use of the group of the Esscher transforms, which seems less natural for the non normal setup and takes us outside the underlying family, producing even nonsymmetric measures. For option pricing on log-symmetric models the following result is important.

Proposition 3.1 *Let X be log-symmetric $X \sim LS(\mu, \sigma^2, \psi)$. Then $1/X$ is also log-symmetric $LS(-\mu, \sigma^2, \psi)$.*

Proof Let $X = e^Y$, where $Y \sim S(\mu, \sigma^2, \psi)$, then $(Y - \mu) \sim S(0, \sigma^2, \psi)$. By definition of a symmetric distribution $-(Y - \mu) \overset{D}{=} (Y - \mu) \sim S(0, \sigma^2, \psi)$, implying that $-Y \sim S(-\mu, \sigma^2, \psi)$. Since $1/X = e^{-Y}$ the statement follows. □

The distribution of returns is log-symmetric, by assumption, $X_n \sim LS(\mu, \sigma^2, \psi)$, i.e. $X_n = e^{Y_n}$, with $Y_n \sim S(\mu, \sigma^2, \psi), n = 1, \ldots, N$. If variance $V(Y_n) < \infty$, we normalize the characteristic generator ψ such that Eq. 2.8 holds, i.e., $V(Y_n) = \sigma^2$.

It is clear that we can talk about returns in terms of log-returns, which is more convenient because of the definition of log-symmetry.

Denote by f_μ the density of the log-returns, $Y_n \sim S(\mu, \sigma^2, \psi)$. We index it only by the location parameter μ since we consider it as a one-parameter family with σ and ψ fixed. Note that this density is either chosen from empirical studies or can be supported by such. It makes economic sense that the independent log-returns Y_n remain independent in the risk-neutral world. It also makes economical sense, rather than being a mathematical requirement, that the new density of Y_n in the risk-neutral world belongs to the same family of distributions as the empirically observed. This is because the risk-neutrality requirement merely states that $E_Q(X) = e^r$ and can be achieved by shifting the mean of the distribution, rather changing it entirely. This is certainly the case in models where the risk-neutral measure is unique, eg. Binomial or Black–Scholes. These two requirements lead to the following change of measure

$$\frac{dQ}{dP} = \Lambda_N = \prod_{n=1}^{N} \frac{f_{\mu^*}(Y_n)}{f_\mu(Y_n)}. \tag{3.4}$$

It turns out that Q so defined is equivalent to the original one P, preserves independence and preserves the log-symmetric family of marginals, as the next result shows.

Theorem 3.1 *Let Q be defined by Eq. 3.4. Then the returns $X_1, X_2, \ldots, X_N$ remain independent and identically distributed under Q with the Q-density function of returns in the same log-symmetric family with location parameter μ replaced by μ^* and density function f_{μ^*}.*

Proof it is technical and given in the Appendix. □

It now remains to choose such μ^* in Eq. 3.4 to make Q a risk-neutral measure, so that the Q-expected returns over one period are the same as the risk-free return e^r. Since the mean of a log-symmetric distribution is given by Eq. 2.13 $E(X) = e^\mu \psi(-\frac{\sigma^2}{2})$ we immediately obtain the following corollary.

Corollary 3.1 *The measure Q in Eq. 3.4 is risk-neutral if and only if*

$$\mu^* = r - \log \psi(-\sigma^2/2). \tag{3.5}$$

We call Q in Eq. 3.4 with μ^* given by Eq. 3.5 the *natural* risk-neutral measure, because it is the only choice of the equivalent measure which preserves independence, the log-symmetric family of marginals and gives the risk-free expected return.

Remark that in the lognormal case $\psi(u) = e^{-u}$ and we recover the known risk-neutral measure with mean $\mu^* = r - \sigma^2/2$ appearing in the Black–Scholes formula.

Next result shows that the discounted stock price process is a martingale under the natural risk-neutral measure. A similar result also holds for the reciprocal process used in option pricing (Eq. 3.3).

Theorem 3.2 *Let Q be defined by Eq. 3.4. For the process $S_n e^{-rn}$, $n \leq N$, to be a martingale it is necessary and sufficient that Q is risk-neutral, i.e.*

$$\mu^* = r - \log \psi(-\sigma^2/2).$$

For the process e^{rn}/S_n, $n \leq N$, to be a martingale it is necessary and sufficient that

$$\mu_1^* = r + \log \psi(-\sigma^2/2). \tag{3.6}$$

Proof By using properties of conditional expectation and Eq. 3.1, we have

$$E_Q(S_{n+1}e^{-r(n+1)}|S_0, \ldots, S_n) = S_n e^{-rn} E_Q(e^{-r} X_{n+1}) = S_n e^{-rn} E_Q(e^{-r} X_1),$$

where the last equality is due to returns being Q-identically distributed. For $S_n e^{-rn}$ to be a Q-martingale the equation $E_Q(S_{n+1}e^{-r(n+1)}|S_0, \ldots, S_n) = S_n e^{-rn}$ must hold implying Eq. 1.1

$$E_Q(X_1) = e^r.$$

Since $E_Q(X_1) = e^{\mu^*} \psi(-\sigma^2/2)$ the claim follows. On the other hand if Eq. 1.1 holds, then the above equation shows that $S_n e^{-rn}$ is a Q-martingale.

Similarly we obtain that for the process e^{rn}/S_n to be a martingale it is necessary and sufficient that the condition

$$E_{Q_1}\left(\frac{1}{X}\right) = e^{-r}$$

is satisfied. Using that the reciprocal of a log-symmetric distribution is again log-symmetric (Proposition 3.1) and the expression for the mean of a log-symmetric distribution (Eq. 2.13) we obtain

$$-\mu_1^* + \log \psi\left(-\frac{\sigma^2}{2}\right) = -r,$$

and Eq. 3.6 follows. □

Consider now the pricing formula 3.3 and write

$$\log S_N = \log S_0 + \sum_{n=1}^{N} \log X_n = \log S_0 + \sum_{n=1}^{N} Y_n.$$

By Theorem 3.2 $\log S_N$ has the Q-distribution as a sum of i.i.d. random variables $S(\mu^*, \sigma^2, \psi)$ and has Q_1 -distribution as a sum of i.i.d. random variables $S(\mu_1^*, \sigma^2, \psi)$. Applying the central limit theorem applied to $\log S_N$ for Q and Q_1 we obtain the following result.

Theorem 3.3 *Let $X = e^Y \sim LS(\mu, \sigma^2, \psi)$ with $EX < \infty$. Then the arbitrage-free price of a call option with N periods to expiration is given by*

$$C(N) = S_0 Q_1(S_N > K) - Ke^{-rN} Q(S_N > K), \tag{3.7}$$

where the Q distribution of Y is $S(\mu^*, \sigma^2, \psi)$ with μ^* given in Eq. 3.5 and the Q_1 distribution of Y is $S(\mu_1^*, \sigma^2, \psi)$ with μ_1^* given in Eq. 3.6. For large N

$$C(N) \approx S_0 \Phi\left(d_1\right) - e^{-rN} K \Phi\left(d_2\right), \tag{3.8}$$

$$d_1 = \frac{\ln(S_0/K) + (r + \log\psi(-\sigma^2/2))N}{\sigma\sqrt{N}}, \tag{3.9}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \log\psi(-\sigma^2/2))N}{\sigma\sqrt{N}}.$$

Note that for a lognormal distribution this approximate formula is the exact Black–Scholes formula for any N.

An application of the formula 3.8 to the log-Laplace distribution, which is Laplace distribution of log returns, gives the following option pricing formula.

Corollary 3.2 *Let returns follow a log-Laplace distribution, then the generalized option pricing formula 3.8 is given by*

$$C(N) \approx S_0 \Phi\left(\frac{\ln(S_0/K) + \left(r - \log\left(1 - \frac{\sigma^2}{2}\right)\right)N}{\sigma\sqrt{N}}\right) - e^{-rN} K \Phi\left(\frac{\ln(S_0/K) + \left(r + \log\left(1 - \frac{\sigma^2}{2}\right)\right)N}{\sigma\sqrt{N}}\right).$$

Proof From Eq. 2.16 we have

$$\psi(-u) = \frac{1}{1-u}, \quad 0 < u < 1, \tag{3.10}$$

from which the result follows. □

Introduce a measure of deviation from normality

$$\Delta(\psi, \sigma^2) = \log\psi(-\sigma^2/2) - \sigma^2/2. \tag{3.11}$$

Then the generalized option pricing formula 3.8 becomes

$$C(N) \approx S_0 \Phi\left(\frac{\ln(S_0/K) + (r + \sigma^2/2)N + N\Delta(\psi, \sigma^2)}{\sigma\sqrt{N}}\right) - e^{-rN} K \Phi\left(\frac{\ln(S_0/K) + (r - \sigma^2/2)N - N\Delta(\psi, \sigma^2)}{\sigma\sqrt{N}}\right). \tag{3.12}$$

Corollary 3.3 *If the generator of a symmetric family satisfies*

$$\psi(-u) > e^u \tag{3.13}$$

then the modified option price is greater than the Black–Scholes price.

Proof Condition 3.13 is equivalent to $\Delta(\psi, \sigma^2) > 0$, and the statement follows from the formula 3.12 by monotonicity of the probability distribution function. □

Theorem 3.4 *If the equation*

$$\frac{d}{du}\psi(-u) = \psi(-u), u \geq 0 \tag{3.14}$$

has no other solution but the point $u = 0$ and

$$\psi''(0) > 1 \tag{3.15}$$

then the modified option price is greater then the Black–Scholes price. In particular, these conditions hold if the function $\log \psi(-u)$, $u \geq 0$ is strictly convex.

Proof For function $D_\psi(u) = \log \psi(-u) - u$, we have

$$D_\psi(0) = 0, \tag{3.16}$$

and taking into account Eq. 2.8

$$D'_\psi(0) = -\frac{\psi'(-u)}{\psi(-u)}_{|u=0} - 1 = 0. \tag{3.17}$$

From the second condition of the Theorem we have, taking into account, Eq. 2.8

$$D''_\psi(0) = \psi''(0) - 1 > 0,$$

i.e., $u = 0$ is minimum point of $D_\psi(u)$, $u \geq 0$, and from the first condition it is unique minimum point and Eq. 3.13 follows.

If $\log \psi(-u)$ is strictly convex we have $D''_\psi(u) = \frac{d^2}{du^2} \log \psi(-u) > 0$, hence $D'_\psi(u)$ is increasing for $u \geq 0$, and from Eq. 3.17 we have $D'_\psi(u) > 0$, i.e. both conditions of the Theorem hold. □

Corollary 3.4 *For log-Laplace distribution of returns the modified option price is greater than the Black–Scholes price.*

Proof The statement follows from Eq. 3.10 and Theorem 3.4 because the equation

$$\frac{1}{(1-u)^2} = \frac{1}{1-u}, \quad 0 < u < 1,$$

has the only solution $u = 0$ and $\psi''(0) = 2$. □

Corollary 3.5 *For log-EPF distribution of returns with the parameter δ in the range $1 \leq \delta < 2$, the modified option price is greater than the Black–Scholes price.*

Table 1 Option prices and percentage differences obtained by MBS and BS formulae for log-Laplace distributed weekly returns, $S_0 = \$50, \ \ K = \$54$

Number weeks	1	11	21	31	41	51	61
MBS formula	2.9	13.31	18.77	22.76	25.95	28.59	30.83
BS formula	2.85	13.14	18.54	22.51	25.68	28.3	30.54
Percentage differences(%)	1.75	1.29	1.24	1.11	1.05	1.02	0.95

Proof See Appendix □

Let us notice that because

$$\frac{d}{du}\log\psi(-u) = -\frac{\psi'(-u)}{\psi(-u)} = u + O(u^2), u \to 0$$

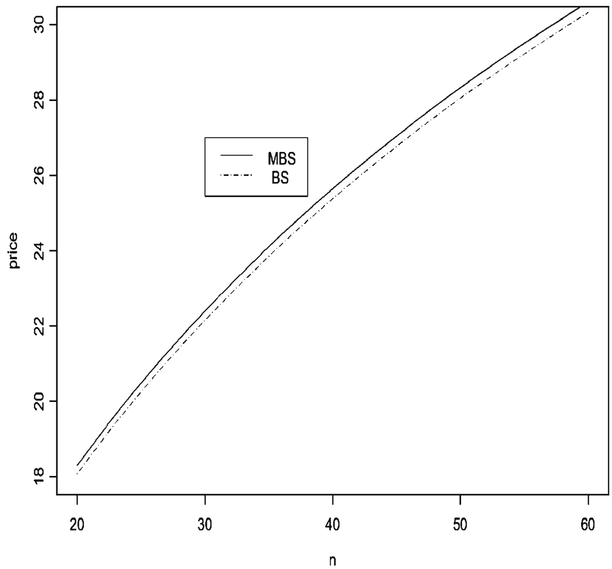
(having in mind Eq. 2.8) the deviation from normality is of order

$$\Delta(\psi, \sigma^2) = O(\sigma^4), \sigma \to 0,$$

and the classical Black–Scholes (BS) formula is considerably robust in the sense that for small σ it coincides with modified Black–Scholes (MBS) pricing formula. However, for weekly returns, σ can reach the range $0.2 - 0.25$, and then the distinction between MBS and BS for large n is noticeable (see Table 1).

In Fig. 1 we show the continuous graphs of MBS and BS for the log-Laplace weekly returns varying with n. In the following section, where log returns are modelled with mixture of symmetric distribution, such difference can be quite significant.

Fig. 1 MBS and BS prices for log-Laplace weekly returns varying in n; $\sigma_1 = 0.23$, $S_0 = 50$, $K = 54$



4 Mixtures of Symmetric Distributions

In this section we show how the important case of mixture of two or more symmetric distributions with the same means and characteristic generators but different variances fits into our set up.

Let φ_1 and φ_2 be two characteristic functions of distributions on R. By definition, a mixture distribution with the contamination parameter ε, $0 < \varepsilon < 1$, is the distribution with the characteristic function

$$\varphi_\varepsilon = (1-\varepsilon)\varphi_1 + \varepsilon\varphi_2.$$

(It is easy to see that such a combination of characteristic functions is indeed a characteristic function.) It follows immediately that a similar relation holds for the densities when they exist

$$f_\varepsilon = (1-\varepsilon) f_1 + \varepsilon f_2.$$

Let now the two distributions be from the symmetric family with the same mean, the same characteristic generator ψ but different variances σ_1^2 and σ_2^2. It is clear that such mixture is also symmetric with the characteristic generator being also of the mixture form

$$\psi_\varepsilon(u) = (1-\varepsilon)\psi\left(\sigma_1^2 u\right) + \varepsilon\psi\left(\sigma_2^2 u\right)$$

Condition 2.11 for the components of the mixture guarantees finite variance for the corresponding symmetric distribution, i.e, the variance of mixture, σ_ε^2, exists and of the form

$$\sigma_\varepsilon^2 = (1-\varepsilon)\sigma_1^2 + \varepsilon\sigma_2^2. \tag{4.1}$$

The characteristic generator of the mixture, ψ_ε, can be normalized so that $\psi_\varepsilon'(0) = -1$, i.e.

$$\psi_\varepsilon(u) = (1-\varepsilon)\psi\left(\frac{\sigma_1^2}{\sigma_\varepsilon^2}u\right) + \varepsilon\psi\left(\frac{\sigma_2^2}{\sigma_\varepsilon^2}u\right), \tag{4.2}$$

which guarantees that σ_ε^2 is the variance of the mixture. One can use the pricing formula 3.8 also for returns distributed $LS(\mu, \sigma_\varepsilon^2, \psi_\varepsilon)$, i.e. the logarithm of which is the mixture of symmetric distributions $S(\mu, \sigma_\varepsilon^2, \psi_\varepsilon)$. In what follows we show that the price for log-mixture data is greater than the corresponding price for homogeneous data with the variance of the mixture.

Theorem 4.1 *Let $X \sim LS(\mu, \sigma_\varepsilon^2, \psi_\varepsilon)$ with $EX < \infty$ and a strictly convex characteristic generator $\psi(-u)$. Then the option price given by Eq. 3.8 is greater than the option on the corresponding log-symmetric distribution $LS(\mu, \sigma_\varepsilon^2, \psi)$ with volatility of the mixture σ_ε.*

Proof By Jensen's inequality and Eq. 4.2

$$\begin{aligned}\psi_\varepsilon\left(\frac{-\sigma_\varepsilon^2}{2}\right) &= (1-\varepsilon)\psi\left(-\frac{\sigma_1^2}{2}\right) + \varepsilon\psi\left(-\frac{\sigma_2^2}{2}\right)\\ &> \psi\left(-(1-\varepsilon)\frac{\sigma_1^2}{2} - \varepsilon\frac{\sigma_2^2}{2}\right) = \psi\left(-\frac{\sigma_\varepsilon^2}{2}\right),\end{aligned}$$

where the inequality above is by convexity of $\psi(-u)$. Thus from Eq. 3.11

$$\Delta(\psi_\varepsilon, \sigma_\varepsilon^2) = \log \psi_\varepsilon\left(\frac{-\sigma_\varepsilon^2}{2}\right) - \frac{\sigma_\varepsilon^2}{2} > \psi\left(-\frac{\sigma_\varepsilon^2}{2}\right) - \frac{\sigma_\varepsilon^2}{2} = \Delta(\psi, \sigma_\varepsilon^2).$$

Thus

$$\Phi\left(\frac{\ln(S_0/K) + N(r+\sigma_\varepsilon^2/2) + N\Delta(\psi_\varepsilon, \sigma_\varepsilon^2)}{\sigma\sqrt{N}}\right) > \Phi\left(\frac{\ln(S_0/K) + N(r+\sigma_\varepsilon^2/2) + \Delta(\psi, \sigma_\varepsilon^2)}{\sigma\sqrt{N}}\right)$$

and

$$\Phi\left(\frac{\ln(S_0/K) + N(r-\sigma_\varepsilon^2/2) - N\Delta(\psi_\varepsilon, \sigma_\varepsilon^2)}{\sigma\sqrt{N}}\right) < \Phi\left(\frac{\ln(S_0/K) + N(r-\sigma_\varepsilon^2/2) - \Delta(\psi, \sigma_\varepsilon^2)}{\sigma\sqrt{N}}\right).$$

Now the pricing formula 3.12 implies that the option price on assets with log mixture is greater than the corresponding log-symmetric distribution with volatility of the mixture σ_ε. □

Corollary 4.1 *If X belongs to the log-mixture of EPF distributions then the conclusion of Theorem* 4.1 *holds.*

Proof If $\log X$ belongs to the mixture of EPF distributions then $\psi(-u)$ is given by the series in Eq. 2.14. In this series all the terms are nonnegative. Differentiation of

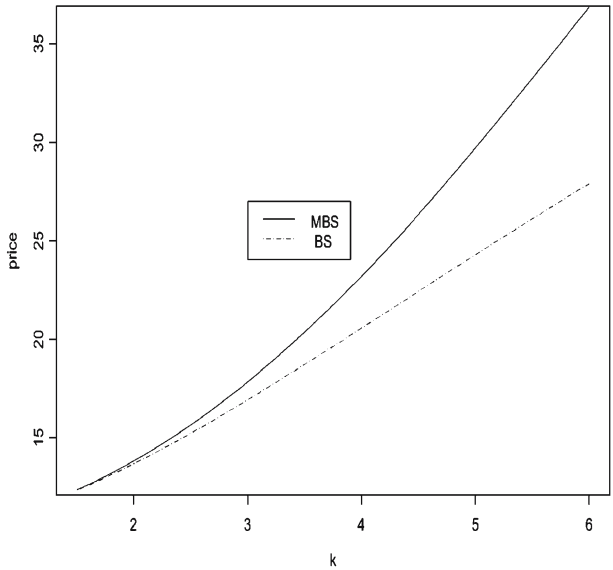


Fig. 2 MBS and BS prices for log of normal mixtures varying in rate $k = \sigma_2/\sigma_1$; $\varepsilon = 0.12$, $n = 10$, $S0 = 50$, $K = 60$

Table 2 Option prices and percentage differences obtained by MBS and BS formulae for log mixture of normally distributed weekly returns, $S_0 = \$50$, $K = \$60$, $\varepsilon = 0.12$

Number weeks	1	11	21	31	41	51	61
Ratio $k = \sigma_2/\sigma_1$	1.5	2.25	3.0	3.75	4.5	5.251	6.0
MBS formula	12.33	14.69	17.84	21.74	26.34	31.49	36.88
BS formula	12.36	14.42	16.92	19.64	22.44	25.22	27.91
Percentage differences(%)	0.24	1.89	5.4	10.7	17.39	24.88	32.14

the series term by term gives again a series with nonnegative terms. This shows that characteristic generator $\psi(-u)$ is convex. □

Applying Theorem 4.1 to the log-mixture of normal distributions we obtain a well-known in folklore result.

Corollary 4.2 *If X belongs to the log-mixture of normal distributions, then the option price is greater than the one given by Black–Scholes formula with the volatility of the mixture.*

Here we specify the modified option pricing formula for the log-mixture of normals using Eq. 4.2

$$C(N) \approx S_0\Phi(d_1) - e^{-rN} K\Phi(d_2). \tag{4.3}$$

$$d_1 = \frac{\ln(S_0/K) + \left(r + \log\left((1-\varepsilon)e^{\sigma_1^2/2} + \varepsilon e^{\sigma_2^2/2}\right)\right)N}{\sqrt{(1-\varepsilon)\sigma_1^2 + \varepsilon\sigma_2^2}\sqrt{N}},$$

$$d_2 = \frac{\ln(S_0/K) + \left(r - \log\left((1-\varepsilon)e^{\sigma_1^2/2} + \varepsilon e^{\sigma_2^2/2}\right)\right)N}{\sqrt{(1-\varepsilon)\sigma_1^2 + \varepsilon\sigma_2^2}\sqrt{N}}.$$

In the Fig. 2 we compare the graphs of modified Black–Scholes (MBS) and classical Black–Scholes (BS) price formulae in dependence of ratio $k = \sigma_2/\sigma_1$ and fixed ε. One can see in the graphs that even for small percent of contamination (12%) MBS departure from BS for values $k > 2$ is significant. The exact prices and the percentage differences are represented in Table 2.

Note that this special case of log-mixture of normal distributions was considered in Ritchey (1990). But the method and price suggested in that paper really works only for one period to expiration. For more than one period to expiration, his approach leads to a formula with a huge number of components with awfully complicated coefficients (see Ritchey 1990, Eqs. 8–11). Moreover, the well-known problem of label switching, which always appears in n-times repeated mixtures, also makes applications of the result problematic.

5 Conclusion

We have considered option pricing on assets with log-symmetric distributions of returns, that is when the log of returns have a symmetric distribution. In the framework of pricing by no-arbitrage/risk-free pricing we have suggested the natural risk-neutral measure that keeps the distribution of returns in the same log-symmetric family and obtained a generalization of the Black–Scholes formula. An important application of our method is for log-mixture of normal distributions. We have shown that Black–Scholes formula always underprices options on such assets, and demonstrated that this difference can be significant.

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Appendix

Proof of Theorem 3.1 For some fixed $y_1, y_2, \ldots, y_n \leq \infty$ denote by A the following set in R^N $A = \{u_1, \ldots, u_n : u_1 \leq y_1, \ldots u_N \leq y_N\}$ and by $A_n = \{u_n : u_n \leq y_n\}$, and consider $Q(Y_1 \leq y_1, \ldots, Y_N \leq y_N) = Q((Y_1, \ldots, Y_N) \in A)$. The indicator function of A, $I_A(\mathbf{u})$ equals to 1 if $\mathbf{u} \in A$ and 0 otherwise, has the property $I_A(\mathbf{u}) = \prod_{n=1}^N I_{A_n}(u_n)$, and we have

$$\begin{aligned} Q(Y_1 \leq y_1, \ldots, Y_N \leq y_N) &= E_Q I_A((Y_1, \ldots, Y_N)) \\ &= E_P(\Lambda I_A((Y_1, \ldots, Y_N))) \\ &= E_P\left(\prod_{n=1}^N \frac{f_{\mu^*}(Y_n)}{f_\mu(Y_n)} I_{A_n}(Y_n)\right). \end{aligned}$$

Using that the expectation of a function of a random vector is the integral of that function with respect to the joint density, and that the joint P-density of $Y_1, \ldots, Y_N$ equals to $\prod_{n=1}^N f_\mu(u_n)$ by independence, we obtain further

$$\begin{aligned} Q(Y_1 \leq y_1, \ldots, Y_N \leq y_N) &= \int \ldots \int \prod_{n=1}^N \frac{f_{\mu^*}(u_n)}{f_\mu(u_n)} I_{A_n}(u_n) \prod_{n=1}^N f_\mu(u_n) du_1 \ldots du_N \\ &= \int \ldots \int \prod_{n=1}^N f_{\mu^*}(u_n) I_{A_n}(u_n) du_1 \ldots du_N \\ &= \prod_{n=1}^N \int_{A_n} f_{\mu^*}(u_n) du_n. \end{aligned}$$

Taking all the $y_i = \infty$ for $i \neq n$ we obtain that Q-marginals have the density f_{μ^*} and Y_n's are all identically Q-distributed

$$Q(Y_n \leq y_n) = \int_{A_n} f_{\mu^*}(u_n) du_n.$$

Now putting this expression into the equation above, we obtain Q-independence

$$Q(Y_1 \leq y_1, \ldots, Y_N \leq y_N) = \prod_{n=1}^N Q(Y_n \leq y_n),$$

and the theorem is proved.

Proof of Corollary 3.5 Compare two series (see Eq. 2.14)

$$\psi(-u) = \frac{1}{\Gamma\left(\frac{1}{\delta}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{\delta}\right)}{(2n)!} \left(\frac{2\Gamma\left(\frac{1}{\delta}\right)}{\Gamma\left(\frac{3}{\delta}\right)} u\right)^n =: \sum_{n=0}^{\infty} a_n$$

$$\frac{d}{du}\psi(-u) = \frac{1}{\Gamma\left(\frac{1}{\delta}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{\delta}\right)}{\Gamma\left(\frac{3}{\delta}\right)} \frac{\Gamma\left(\frac{2n+3}{\delta}\right)}{(2n+1)!} \left(\frac{2\Gamma\left(\frac{1}{\delta}\right)}{\Gamma\left(\frac{3}{\delta}\right)} u\right)^n =: \sum_{n=0}^{\infty} b_n.$$

Denote by $t = 1/\delta$. It is straightforward

$$\frac{b_n}{a_n} = h(t) = \frac{\Gamma(t)}{\Gamma(3t)} \frac{\Gamma((2n+3)t)}{\Gamma((2n+1)t)} \frac{1}{(2n+1)}, n = 0, 1, .., . \tag{5.4}$$

and $1/2 \le t \le 1$ when $1 \le \delta \le 2$. Using representation for $\Gamma(3t)$ (see Gradshteyn and Ryzhik 2000, 8.335(2)) we can write after differentiation of logarithm of $h(t)$

$$\begin{aligned}\frac{d}{dt}\log(h(t)) = &-3\log 3 - \Psi(t+1/3) - \Psi(t+2/3) \\ &+ (2n+1)(\Psi((2n+3)t) - \Psi((2n+1)t)) \\ &+ 2\Psi((2n+3)t),\end{aligned} \tag{5.5}$$

where $\Psi(t) = \frac{d}{dt}\log\Gamma(t)$ is well-known digamma or psi-function. As function $\Psi(t)$ is strictly monotone on $(0, \infty)$, for $t \ge 1/2$, we have the following inequality

$$\begin{aligned}\Psi((2n+3)t) - \Psi((2n+1)t) &\ge \Psi((2n+1)t+1) - \Psi((2n+1)t) \\ &= \frac{1}{(2n+1)},\end{aligned} \tag{5.6}$$

using the well-known recursion $\Psi(t+1) - \Psi(t) = 1/t$. Using again monotone property of psi-function we obtain from Eqs. 5.5 and 5.6 for $1/2 \le t \le 1$

$$\frac{d}{dt}\log(h(t)) \ge -3\log 3 - \Psi(4/3) - \Psi(5/3) + 1 + 2\Psi(n+3/2).$$

Then for $n \ge 3$

$$\begin{aligned}\frac{d}{dt}\log(h(t)) &\ge -3\log 3 - \Psi(4/3) - \Psi(5/3) + 1 + 2\Psi(3+3/2) \\ &= 0.432\,17 > 0, \;\; 1/2 \le t \le 1\end{aligned}$$

and function $\log(h(t))$ and together with it $h(t)$ are strictly monotone increasing functions for $n \ge 3$ and $1/2 \le t \le 1$ and from Eq. 5.4 we have

$$\frac{b_n}{a_n} > h\left(\frac{1}{2}\right) = 1, 1/2 \le t \le 1, n \ge 3. \tag{5.7}$$

For $n = 1, 2$ the inequality 5.7 holds straightforwardly. It follows from Eq. 5.7 that $u = 0$ is the only solution of Eq. 3.14. The Corollary follows from Theorem 3.4 because $\psi''(0) > 0$, when $\delta \in [1, 2)$.

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