ESTIMATING AND PREDICTING VOLATILITY BY USING IMPLIED AND HISTORICAL VOLATILITIES

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Abstract. If volatility parameter σ^2, involved in Black-Sholes formula, is time dependent, the parameter σ^2 is replaced by its average value $(T-t)^{-1}\int_t^T \sigma^2(s)ds$. We propose a method of evaluation of this averaged parameter via others $(T_i-t_i)^{-1}\int_{t_i}^{T_i}\sigma^2(s)ds$ corresponding to implied and historical volatilities. We also consider the case of stochastic volatility.

1. Introduction

It is well known that a problem with the empirical implementation of the Black-Scholes model is that the variance assumption is inconsistent with the data. Estimating and forecasting the spot volatility of a traded asset has been studied for some time in the literature, see e.g. Elliott (1993) and Lamoureux and Lastrapes (1993). Options models with stochastic volatility were considered in Wiggins (1987), Hull and White (1987), Scott (1987), Johnson and Shanno (1987), and Stein and Stein (1991), Heston (1993), Duan (1995).

Day and Lewis (1993) use the Black-Scholes model for implied volatilities and compare them with Garch historic volatilities. Their results show that implied volatilities provide forecasts better than the Garch model.

We consider two broad models for spot volatility $\sigma^2(t)$. The first one is when $\sigma^2(t)$ is an unknown non-random function, and the second is when $\sigma^2(t)$ is a random process, independent of the driving Brownian motion motion in the stock. For these models we estimate/predict volatility by using the orthogonal projection method. We use real data on AUD/USD implied vols on Sept. 24 2001 to illustrate the theoretical results by a specific modelling examples.

2. Historic and implied volatilities

Denote by S_t the price of an asset at time t, $0\le t\le T^\bullet$. Assume that in the risk-neutral world (with risk-free rate r) the price of stock satisfies the following stochastic differential equation

$$dS_t = rS_tdt + \sigma_tS_tdB_t, \tag{1}$$

where B_t is the Brownian motion. Then the process σ_t is known as the volatility of the stock.

Define historic volatility, also known as realized volatility, on time interval $[t_0, t_1]$ by

$$H_{t_0}(t_1) = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \sigma_s^2 ds, \tag{2}$$

(for our purpose it is convenient to take the averaged version).

It is well known that the theoretical price (no-arbitrage price) of a European call option with maturity T and strike K is given by the expression

$$C_t(K,T) = e^{r(T-t)} E(S_T - K)^+ | S_t). \tag{3}$$

Next write the Black-Scholes formula as a function of the parameter σ^2

$$\mathrm{BS}(\sigma^2) = S\Phi(h_t) - Ke^{-r(T-t)}\Phi(h_t - \sqrt{\sigma^2(T-t)}), \tag{4}$$

where

$$h_t = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}},$$

and Φ is the standard normal distribution function.

It is well known that in the case of constant volatility $\sigma_t \equiv \sigma$, $0 \le t \le T^\bullet$, in the equation (1) the theoretical price (3) is given by the Black-Scholes formula (4) with $S = S_t$.

Denote by $C_t^m(K,T)$ the market price of the option. Implied volatility $I_t(T,K)$ is defined as the solution of the following equation

$$\mathrm{BS}(I_t(K,T)) = C_t^m(K,T). \tag{5}$$

Mathematically, this means that market prices of options translate into implied volatilities,

$$I_t(K,T) = \mathrm{BS}^{-1}(C_t^m(K,T)). \tag{6}$$

One can show that given t, T, S and K, for any market price C^m, the implied volatility (6), exists, is unique and nonnegative (e.g. [9]). If now we also assume that market prices of options are the same as theoretical prices $C_t^m(K,T) = C_t(K,T)$, then

$$I_t(K,T) = \mathrm{BS}^{-1}(C_t(K,T)). \tag{7}$$

Suppose now that from the market prices of spot and options we know m implied volatilities $I_t(T_i)$ with maturities T_i and strikes K_i, $i = 1, \dots, m$ (strikes are suppressed in notations), and k historic volatilities $H(t_j)$, $j = 1, \dots, k$, and we want to estimate the implied volatility $I_t(T^\bullet)$, needed for options with maturity $T^\bullet$, $T^\bullet \ne T_i$'s.

We propose the following estimate of $I_t(T^\bullet)$ (denoted by $\widehat{\cdot}$)

$$\widehat{I}_t(T^\bullet) = \alpha_0 + \sum_{i=k+1}^{m+k} \alpha_i I_t(T_i) + \sum_{j=1}^{k} \alpha_j H(t_j), \tag{8}$$

where α_i's are chosen to make $\widehat{I}_t(T^\bullet)$ the best linear estimator (predictor) based on historic and implied volatilities.

We illustrate with estimation of volatility in the FX AUD/USD using data on 24 Sept. 2001 shown in Figure 1. The observed implied volatilities are at the money: $18.6\%, 16.4\%, 15.6\%, 15.0\%, 14.\%, 13.5\%$ with maturities 1 day, 1 month, 2 months, 3 months, 6 months and 1 year respectively.

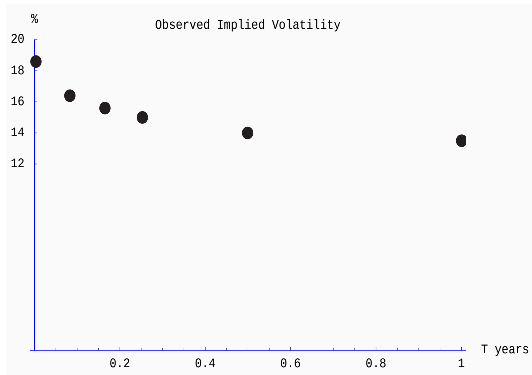

FIGURE 1. AUD/USD Implied Volatilities observed on 24th Sept. 2001.

3. VOLATILITY AS AN UNKNOWN DETERMINISTIC FUNCTION

In the case when the volatility is unknown deterministic function implied volatilities are given by (e.g. [25])

$$I_t(T) = \frac{1}{T-t}\int_t^T \sigma^2(s)ds. \tag{9}$$

Knowledge of m implied and k historic volatilities means that we know $m+k$ integrals of $\sigma^2(t)$ over various time intervals, k over past $[t_0, t_j]$ and m over future $[t, T_i]$:

$$\begin{aligned} &\int_t^{T_i} \sigma^2(s)ds = I_t(T_i)(T_i - t),\ i = 1, \dots, m \\ &\int_{t_0}^{t_j} \sigma^2(s)ds = H(t_j)(t_j - t_0),\ j = 1, \dots, k. \end{aligned} \tag{10}$$

Similarly, $I_t(T^\bullet)$, which needs to be estimated, is the integrated $\sigma^2(t)$ over $[t, T^\bullet]$:

$$\int_t^{T^\bullet} \sigma^2(s)ds := I_t(T^\bullet)(T^\bullet - t).$$

Suppose that $\sigma^2(t)$ can be approximated in a suitable basis of functions ϕ_ℓ by a finite sum

$$\sigma^2(t) = \sum_{\ell=1}^{n} c_\ell \phi_\ell(t), \tag{11}$$

then the coefficients c_ℓ's are obtained by using the values $I_t(T_i)$ and $H(t_j)$, and by using these coefficients we obtain $\widehat{I}_t(T^\bullet)$, the estimated/predicted volatility over $[t, T^\bullet]$.

Introduce

$$a_{i\ell} = \frac{1}{T_i - t}\int_t^{T_i} \phi_\ell(s)ds \quad \text{and} \quad b_{j\ell} = \frac{1}{t_j - t_0}\int_{t_0}^{t_j} \phi_\ell(s)ds.$$

Then $I_t(T_i) = \sum_{\ell=1}^n c_\ell a_{i\ell}$, and $H(t_j) = \sum_{\ell=1}^n c_\ell b_{j\ell}$

or, in the vector-matrix form with vectors $C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ and $U = \begin{pmatrix} H_t(t_1) \\ \vdots \\ H_t(t_k) \\ I_t(T_1) \\ \vdots \\ I_t(T_m) \end{pmatrix}$

and matrix

$$A = \begin{pmatrix} b_{11} & \dots & \dots & \dots & b_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ b_{k1} & \dots & \dots & \dots & b_{kn} \\ a_{11} & \dots & \dots & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & \dots & \dots & b_{mn} \end{pmatrix},$$

$$U = AC.$$

Then, using the result on ill-determined linear systems of equations (Theorem 3), we obtain

Theorem 1. *The best linear estimator/predictor $\widehat{I}_t(T^\bullet)$ based on implied and historical volatilities is given by*

$$\widehat{I}_t(T^\bullet) = \frac{1}{T^\bullet - t} \sum_{\ell=1}^n \widehat{c}_\ell \int_t^{T*} \phi_\ell(s) ds,$$

where $\widehat{C} = (A^*A)^\oplus A^*U$ is the vector of estimated coefficients in the expansion of $\sigma^2(t)$, $(A^*A)^\oplus$ is the pseudoinverse matrix of A^*A and A^* is the transpose matrix of A.

It is not hard to see that when the choice of basis is successful and parameters $\widehat{c}_\ell$'s provide a good approximation for historic and implied volatilities, then approximation $\widehat{I}_t(T^\bullet)$ of $I_t(T^\bullet)$ is also reasonable.

In our example, the chosen basis is: $\phi_1(t) = 1$, $\phi_j(t) = \sin(w_j t)$ with $w_j = j\pi/3$, $j = 1, 2, ..., 20$, that is the first basis function $\sin(w_1 t)$ has its period of 6, the second $\sin(w_2 t)$ of 3, and so on. Figure 3 shows the estimated implied volatilities (by varying $T^\bullet$) plotted with the observed implied volatilities.

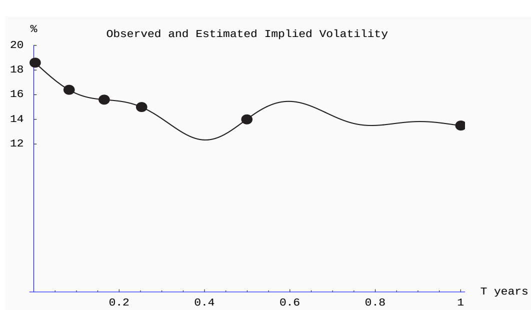

FIGURE 2. Estimated volatility in a deterministic volatility model

4. Volatility as a random process

In the case when the volatility is a stochastic process independent of Brownian motion B_t in (1), conditioning on the volatility and using [25], we obtain

$$C_t(K,T) = E\Big(\mathrm{BS}(\frac{1}{T-t}\int_t^T \sigma^2(s)ds)\Big) \tag{12}$$

Therefore theoretical implied volatilities are given by

$$I_t(T) = \mathrm{BS}^{-1}\left(E\Big(\mathrm{BS}(\frac{1}{T-t}\int_t^T \sigma^2(s)ds)\Big)\right). \tag{13}$$

Since this expression is too involved for analytical computations, we replace theoretical implied volatilities by their (method of moments) estimates

$$\bar{I}_t(T) = \frac{1}{T-t}\int_t^T \sigma^2(s)ds. \tag{14}$$

Using the orthogonal projection result given in Appendix, we arrive at

Theorem 2. *The best linear estimator/predictor $\widehat{I}_t(T^\bullet)$ based on implied and historical volatilities is given by*

$$\widehat{I}_t(T^\bullet) = r_* + \Gamma_*\Gamma^\oplus \begin{pmatrix} H(t_1)-\alpha_1 \\ \vdots \\ H(t_k)-\alpha_k \\ I_t(T_1)-r_1 \\ \vdots \\ I_t(T_m)-r_m \end{pmatrix} \tag{15}$$

where

$$\begin{aligned} r_* &:= E\bar{I}_t(T^\bullet) = (T^\bullet - t)^{-1}\int_t^{T^\bullet} E\sigma^2(s)ds \\ \alpha_i &:= EH(t_i) = t_i^{-1}\int_0^{t_i} E\sigma^2(s)ds \\ r_i &:= E\bar{I}_t(T_i) = (T_i - t)^{-1}\int_t^{T_i} E\sigma^2(s)ds. \end{aligned}$$

and Γ_* and Γ are matrices, with entries $\gamma_*(i)$ and $\gamma(ij)$, of sizes $1\times(k+m)$ and $(k+m)\times(k+m)$ respectively given by (with $v(s)=\sigma^2(s)$)

$$\gamma_*(i) = (T^\bullet - t)^{-1}t_i^{-1}\int_t^{T*}\int_0^{t_i}\mathrm{Cov}(v(s_1),v(s_2))ds_1ds_2 \ _{\{i=1,\dots,k\}}$$

$$\gamma_*(i) = (T^\bullet - t)^{-1}(T_i - t)^{-1}\int_t^{T*}\int_t^{T_i}\mathrm{Cov}(v(s_1),v(s_2))ds_1ds_2, \ _{\{i=1,\dots,m\}}$$

and

$$\gamma(i,j) = t_i^{-1}t_j^{-1}\int_0^{t_i}\int_0^{t_j}\mathrm{Cov}(v(s_1),v(s_2))ds_1ds_2,\ {}_{\{i,j=1,\dots,k\}}$$

$$\gamma(i,j) = t_i^{-1}(T_j-t)^{-1}\int_0^{t_i}\int_t^{T_j}\mathrm{Cov}(v(s_1),v(s_2))ds_1ds_2,\ {}_{\left\{\substack{i=1,\dots,k\\ j=1,\dots,m}\right\}}$$

$$\gamma(i,j) = (T_i-t)_i^{-1}(T_j-t)^{-1}\int_t^{T_i}\int_t^{T_j}\mathrm{Cov}(v(s_1),v(s_2))ds_1ds_2,\ {}_{\{i,j=1,\dots,m\}}.$$

4.1. **Heston's model.** Heston (1993) stochastic volatility model is described by the following equations

$$\begin{aligned} dS_t &= rS_tdt + \sqrt{v_t}S_tdB_t \\ dv_t &= \alpha(\mu - v_t)dt + \delta\sqrt{v_t}\,dW_t, \end{aligned} \tag{16}$$

where we take Brownian motions W_t and B_t to be uncorrelated (independent), although in general they are allowed to be correlated.

Calculation of expectations and correlations required in the above theorem are given in the Appendix.

The estimated spot volatility can also be given by the estimation formula (15). In this example we assume the initial volatility $v_0 = 0.186^2$. The parameters of the spot volatility model are taken to be $\alpha = 1.2, \mu = 0.12^2$ and $\delta = 0.05$. Figure 3 shows the future spot volatility $\sqrt{v(t)}$ estimated from one known implied volatility $I_0(1) = 0.12^2$. On the right of Figure 3, the estimated volatility is the same as the observed implied volatility 0.12^2 for a large α.

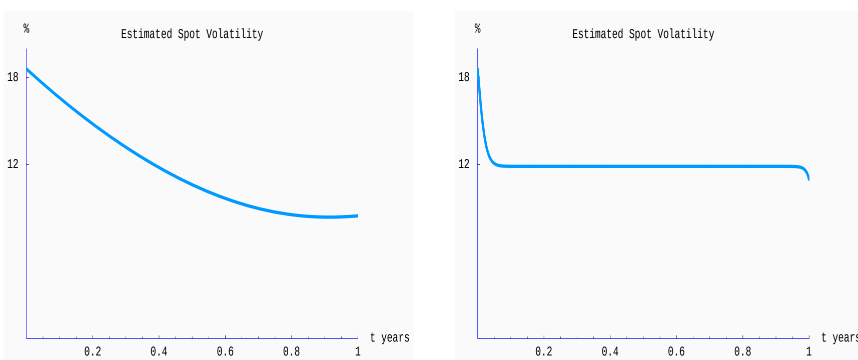

FIGURE 3. Estimated future spot volatility.

Using the estimation formula (15) we can of course obtain the estimated future spot volatility and implied volatility given three known implied volatilities $I_0(0.25) = 0.165^2,\ I_0(0.5) = 0.145^2,\ I_0(1) = 0.135^2$ as shown in Figures 4.

5. CONCLUSION

We have formulated and found expressions for estimation/prediction of implied volatilities in a number of models for spot volatility by providing a unified approach utilizing the projection method.

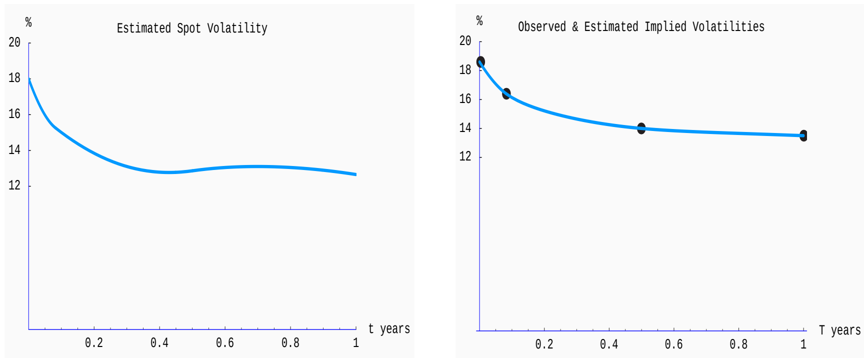

FIGURE 4. Estimated spot and implied volatilities - given three known implied volatilities.

REFERENCES

[1] Albert, A. (1972): Regression and the Moore–Penrose Pseudoinverse. Academic, New York London

[2] Albert, A. and Sittler, R.W. (1965): A method of computing least squares estimators that keep up with the data. SIAM J. Control Optimization, **3**, 384–417

[3] Anderson, T. (1958): Introduction to Multivariate Analysis. Wiley, New York

[4] Black, F. (1976).The Pricing of Commodity Contracts. Journal of Financial Economics. 9, 167-179.

[5] Black, F. (1973). The Pricing of Options and Coporate Liabilities. Journal of Political Economics. 81, 637-659.

[6] Day, T.E. and C.M. Lewis (1993). Forecasting Futures Market Volatility. Journal of Derivatives, Winter.

[7] Duan, J. C. (1995). The GARCH Option Pricing Model. Mathematical Finance. 5 (1) 13-32.

[8] Elliott Robert J. (1993). Estimating The Volatility of An Exchange Rate. Applied Stochastic Models and Data Analysis. 131-135.

[9] Fielding M., Klebaner F., and Landsman Z. (2001) Implied Volatility and Option Prices with the Generalized Student-t distribution. Working paper.

[10] Fouque,-Jean-Pierre; Papanicolaou,-George; Sircar,-K.-Ronnie 2000. Derivatives in Financial Markets with Stochastic Volatility. Cambridge - Univeristy Prress.

[11] Grunbichler, A and F. A. Longstaff 1996. Valuing futures and options on volatility. Journal of Banking and Finance. 20, 985-1001.

[12] Heston, S. (1993). A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. The Review-of-Financial-Studies. 13 (3), 585-625.

[13] Heston, S. and S Nandi (2000). A Closed-Form GARCH Option Valuation Model. The Review-of-Financial-Studies. 13 (3) 585-625.

[14] Hull J. and A. White (1987). The Pricing of Options on Assets with Stochastic Volatilities. The Journal of Finance. XLII No. 2 June, 281-285.

[15] Lamoureux C.G. and W. D. Lastrapes (1993). Forecasting Stock Return Variance: Toward an Understanding of Stochastic Implied Volatilities. The Review of Financial Studies. 6 No. 2.

[16] Liptser, R. and Shiryaev, A.N¿ (2001). Statistics of Random Processes (II Applications). Springer

[17] Marsaglia, G. (1964): Conditional means and covariance of normal variables with singular covariance matrix. J. Am. Stat. Assoc., **308**, 59, 1203–4

[18] Mao, Xuerong. Stochastic Differential Equations and their Applications. Horwood Series in Mathematics and Applications, 1997.

[19] Merton, R. C. (1973). Theory of Rational Option Pricing. Bell Journal of Economics and Management Science. 4, 141-183.
[20] Oksendal, Bernt (2000). Stochastic Differential Equations: an introduction with applications. Springer.
[21] Parzen, Emanuel. Stochastic Processes. Holden-Day, 1962.
[22] Renault, E. and N. Touzi (1996). Option Hedging and Implied Volatilities in a Stochastic Volatility Model. Mathematical Finance. 6, 279-302.
[23] Scott, L. O. (1987). Option pricing when the variance changes randomly: Theory, estimation and an application. Journal of Financial and Quantitative Analysis, 22, 419-438.
[24] Scott, L. (1991). Random Variance Option Pricing. Advances in Futures and Options Research. 5 , 113-135.
[25] Stein, E.M. & J.C. Stein (1991). Stock Price Distributions with Stochastic Volatility: An Analytic Approach. The Review of Financial Studies Vol. 4, (4), 727-752.
[26] Wiggins, J. B. (1987). Option values under stochastic volatility: Theory and empirical estimates. Journal of Financial Economics, 19, 351-372.

6. Appendix

The method, known as the orthogonal projection, is given by the following theorem (see e.g. [1], [2], [3], [16] (§13), [17]). Below $A^{\oplus}$ and * denote the Moore-Penrose pseudoinverse matrix of A (see [1]) and the transposition symbol respectively.

Theorem 3. *Let $U = AC$, where U and C are vectors, and A is a matrix. U and A are known and C is to be determined. Then $\widehat{C} = (A^*A)^{\oplus}A^*U$*

Theorem 4. *Let $X = (X_1, X_2, \dots, X_k)$ and $Y = (Y_1, \dots, Y_l)$ be random vectors, $C_{XX} = \mathrm{Cov}(X, X)$, $C_{XY} = \mathrm{Cov}(X, Y)$, $C_{YY} = \mathrm{Cov}(Y, Y)$. The orthogonal projection $\widehat{E}(X|Y)$ of X on a linear space generated by Y is given by are given by*

$$\widehat{E}(X|Y) = E(X) + C_{XY}C^{\oplus}_{XX}(Y - E(Y)). \tag{17}$$

The mean square projection error is given by

$$P = E(X - \widehat{E}(X|Y))(X - \widehat{E}(X|Y))^* = C_{XX} - C_{XY}C^{\oplus}_{YY}C^*_{XY}$$

Comment here that in the particular case of jointly normal distributions, the orthogonal projection turns out to be also the best possible estimator (predictor) in the mean-square error sense, given by the conditional expectation $E(X|Y)$.

Calculations of Expectation and Correlation of Implied Volatilities

As defined in Heston model (16), $v_s,\ 0 \le t \le s < \infty$, can be written in integral form

$$v_s = v_t + \alpha \int_t^s (\mu - v_u)du + \delta \int_t^s \sqrt{v_u}dW_u \tag{18}$$

A random process v_s is homogeneous diffusion and possesses stationary version in the wide sense. For application point of view, it make sense to compute the expectation, variance and correlation function for its stationary version.

Taking the expectation from both sides of (18) we find

$$Ev_s = Ev_t + \alpha \int_t^s (\mu - Ev_u) du.$$

In view of $Ev_s \equiv Ev_t$, we have

$$Ev_s \equiv \mu. \tag{19}$$

To find the variance of v_s, set $v_s^\circ = v_s - \mu$. Then (18) and (19) provide

$$v_s^\circ = v_t^\circ - \alpha \int_t^s v_u^\circ du + \delta \int_t^s \sqrt{v_u} dW_u \tag{20}$$

and by Itô's formula

$$(v_s^\circ)^2 = (v_t^\circ)^2 - 2\alpha \int_t^s (v_u^\circ)^2 du + 2\delta \int_t^s v_u^\circ \sqrt{v_u} dW_u + \delta^2 \int_t^s v_u du. \tag{21}$$

Taking the expectation from both sides of (21) we find

$$\mathrm{Var}(v_s) = \mathrm{Var}(v_t) - 2\alpha \int_t^s \mathrm{Var}(v_u) du + \delta^2 \int_t^s \mu du$$

and, since $\mathrm{Var}(v_s) \equiv \mathrm{Var}(v_t)$, it holds

$$\mathrm{Var}(v_s) \equiv \frac{\delta^2 \mu}{2\alpha}. \tag{22}$$

To find $\mathrm{Cov}(v_s, v_t) = Ev_s^\circ v_t^\circ$, notice that from (20) it follows

$$v_s^\circ v_t^\circ = (v_t^\circ)^2 - \alpha \int_t^s v_u^\circ v_t^\circ du + \delta v_t^\circ \int_t^s \sqrt{v_u} dW_u. \tag{23}$$

Since, obviously $Ev_t^\circ \int_t^s \sqrt{v_u} dW_u \equiv 0$, taking the expectation from both sides of (23), we get $\mathrm{Cov}(v_s, v_t) = \mathrm{Var}(v_t) - \alpha \int_t^s \mathrm{Cov}(v_u, v_t) du$, i.e. $\frac{d\,\mathrm{Cov}(v_s,v_t)}{ds} = -\alpha\, \mathrm{Cov}(v_s, v_t)$. Hence, for $s > t$ we have $\mathrm{Cov}(v_s, v_t) = \mathrm{Var}\, v_t e^{-\alpha(s-t)}$.

Generally,

$$\mathrm{Cov}(v_s, v_t) = \frac{\delta^2 \mu}{2\alpha} e^{-\alpha|s-t|}. \tag{24}$$

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