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FCLT and MDP for Stochastic Lotka–Volterra Model

F.C. Klebaner · A. Lim · R. Liptser

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Abstract In this paper, we continue an asymptotic analysis of a stochastic version of the Lotka–Volterra model for predator–prey interactions. While the fluid approximation and large deviations were shown in Klebaner and Liptser (*Ann. Appl. Probab.* **11**, 1263–1291, 2001) here we establish the diffusion approximation and moderate deviations.

Keywords Predator–prey models · Functional central limit theorem · Moderate deviations

1 Introduction

1.1 Deterministic Lotka–Volterra System

The Lotka–Volterra system of ordinary differential equations (Lotka [4] and Volterra [8])

$$\begin{aligned} \dot{x}_t &= \mathfrak{a} x_t - \mathfrak{b} x_t y_t, \\ \dot{y}_t &= \mathfrak{c} x_t y_t - \mathfrak{d} y_t, \end{aligned} \tag{1.1}$$

with positive x_0, y_0 describes the behavior of a predator–prey system in terms of the prey and predator population densities x_t, y_t. The positive parameters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$ define the birth and death rates in these populations (see, e.g., [5]).

The above model is formulated in terms of population densities and not population sizes. This is reasonable when population sizes are large and densities are expressed relatively

F.C. Klebaner (✉) · A. Lim
School of Mathematical Sciences, Building 28M, Monash University, 3800, Clayton Campus, Victoria, Australia
e-mail: fima.klebaner@sci.monash.edu.au

A. Lim
e-mail: ashley.lim@sci.monash.edu.au

R. Liptser
Electrical Engineering Systems, Tel Aviv University, 69978 Ramat Aviv, Tel Aviv, Israel
e-mail: liptser@eng.tau.ac.il

to some large threshold parameter K. A model for population sizes, which are integers, taking into account effects of random events of birth and death of the predator and prey was introduced in [2], where it was shown that the deterministic model (1.1) appears as an approximation (fluid approximation). This stochastic model is described below.

1.2 Stochastic Lotka–Volterra System

Let X_t and Y_t be population sizes, i.e., the numbers of preys and predators at time t. They satisfy simple balance equations that track changes in populations due to births and deaths. These are modeled by counting point processes, precisely,

$$\begin{aligned} X_t &= X_0 + \pi'_t - \pi''_t, \\ Y_t &= Y_0 + \widehat{\pi}'_t - \widehat{\pi}''_t, \end{aligned} \tag{1.2}$$

where

X_0 and Y_0 are the initial population sizes of preys and predators, respectively,
π'_t is the number of preys born up to time t,
π''_t is the number of preys killed by time t,
$\widehat{\pi}'_t$ is the number of predators born up to time t,
$\widehat{\pi}''_t$ is the number of predators died up to time t.

This model also involves a large positive parameter K, which can be interpreted as a threshold and with respect to which the population densities are calculated. This parameter appears in the definition of the intensities of the counting processes.

We refer to the counting processes $\pi'_t, \pi''_t, \widehat{\pi}'_t, \widehat{\pi}''_t$ as "double Poisson processes", since they have jumps of size one and their intensities are random. They are defined on some stochastic basis

$$(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\ge 0}, P)$$

satisfying the general conditions, that is, $\pi'_t, \pi''_t, \widehat{\pi}'_t, \widehat{\pi}''_t$, as well as X_t and Y_t, are adapted to the filtration $\mathbf{F}$. We make the following assumption:

Assumption A *The counting processes $\pi'_t, \pi''_t, \widehat{\pi}'_t, \widehat{\pi}''_t$ are right-continuous with left limits, and their jumps are disjoint. In other words, all events related to birth, killing, and dying in a short time period are disjoint.*

Intensities depend on X_t, Y_t, K and parameters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ appearing in (1.1):

$\mathfrak{a}X_t$ *is the intensity of* π'_t,
$\frac{\mathfrak{b}}{K}X_tY_t$ *is the intensity of* π''_t,
$\frac{\mathfrak{c}}{K}X_tY_t$ *is the intensity of* $\widehat{\pi}'_t$,
$\mathfrak{d}Y_t$ *is the intensity of* $\widehat{\pi}''_t$.

The processes

$$\begin{aligned} M'_t &= \pi'_t - \int_0^t \mathfrak{a}X_s ds, \\ M''_t &= \pi''_t - \int_0^t \frac{\mathfrak{b}}{K}X_sY_s ds, \end{aligned} \tag{1.3}$$

$$\widehat{M}'_t = \widehat{\pi}'_t - \int_0^t \frac{\mathfrak{c}}{K} X_s Y_s ds,$$

$$\widehat{M}''_t = \widehat{\pi}''_t - \int_0^t \mathfrak{d} Y_s ds$$

are **F**-local martingales. The existence of the above processes was proved in [2] but, for ease of reference, here we briefly mention the construction.

Take four independent sequences of Poisson processes with constant parameters $\mathfrak{a}$, $\mathfrak{b}/K$, $\mathfrak{c}/K$, and $\mathfrak{d}$, respectively: $[(\Pi_t^{\mathfrak{a}}(i))_{t\ge 0}]_{i\ge 1}$, $[(\Pi_t^{\mathfrak{b}/K}(i))_{t\ge 0}]_{i\ge 1}$, $[(\Pi_t^{\mathfrak{c}/K}(i))_{t\ge 0}]_{i\ge 1}$, and $[(\Pi_t^{\mathfrak{d}}(i))_{t\ge 0}]_{i\ge 1}$. In [2], it was shown that the system of Itô equations

$$\begin{aligned} X_t &= X_0 + \int_0^t \sum_{i\ge 1} I(X_{s-}\ge i) d\Pi_s^{\mathfrak{a}}(i) - \int_0^t \sum_{i\ge 1} I(X_{s-}Y_{s-}\ge i) d\Pi_s^{\mathfrak{b}/K}(i), \\ Y_t &= Y_0 + \int_0^t \sum_{i\ge 1} I(X_{s-}Y_{s-}\ge n) d\Pi_s^{\mathfrak{c}/K}(i) - \int_0^t \sum_{i\ge 1} I(Y_{s-}\ge i) d\Pi_s^{\mathfrak{d}}(i) \end{aligned} \tag{1.4}$$

with $X_0, Y_0 > 0$, has a unique solution on $[0,\infty)$. Taking $\pi'_t = \int_0^t \sum_{i\ge 1} I(X_{s-}\ge i) d\Pi_s^{\mathfrak{a}}(i)$ and $A'_t = \int_0^t \sum_{i\ge 1} I(X_s \ge i)\mathfrak{a} ds$ implies that $M'_t = \pi'_t - A'_t$ is a local martingale. Since X_s takes values in the set $\{0,1,2,\dots,\}$, $\sum_{i\ge 1} I(X_s\ge i) = X_s$, that is, $A'_t = \int_0^t \mathfrak{a} X_s ds$. In the same way, the counting processes π''_t, $\widehat{\pi}'_t$, $\widehat{\pi}''_t$ and the local martingales $M''_t = \pi''_t - A''_t$, $\widehat{M}' = \widehat{\pi}'_t - \widehat{A}'_t$, $\widehat{M}'' = \widehat{\pi}''_t - \widehat{A}''_t$ are defined, where $A''_t = \int_0^t \frac{\mathfrak{b}}{K} X_s Y_s ds$, $\widehat{A}'_t = \int_0^t \frac{\mathfrak{c}}{K} X_s Y_s ds$, $\widehat{A}''_t = \int_0^t \mathfrak{d} Y_s ds$. We can now rewrite (1.4) as

$$\begin{aligned} X_t &= X_0 + \int_0^t \left[\mathfrak{a} X_s - \frac{\mathfrak{b}}{K} X_s Y_s\right] ds + [M'_t - M''_t], \\ Y_t &= Y_0 + \int_0^t \left[\frac{\mathfrak{c}}{K} X_s Y_s - \mathfrak{d} Y_s\right] ds + [\widehat{M}'_t - \widehat{M}''_t]. \end{aligned} \tag{1.5}$$

The same equations in terms of the population densities $x_t^K = \frac{X_t}{K}$ and $y_t^K = \frac{Y_t}{K}$ become

$$\begin{aligned} x_t^K &= x_0 + \int_0^t [\mathfrak{a} x_s^K - \mathfrak{b} x_s^K y_s^K] ds + m_t^K, \\ y_t^K &= y_0 + \int_0^t [\mathfrak{c} x_s^K y_s^K - \mathfrak{d} y_s^k] ds + \widehat{m}_t^K, \end{aligned} \tag{1.6}$$

where $m_t^K = \frac{M'_t - M''_t}{K}$ and $\widehat{m}_t^K = \frac{\widehat{M}'_t - \widehat{M}''_t}{K}$ are pure jump martingales with predictable quadratic variation processes

$$\begin{aligned} \langle m^K\rangle_t &= \frac{1}{K}\int_0^t (\mathfrak{a} x_s^K + \mathfrak{b} x_s^K y_s^K) ds, \\ \langle \widehat{m}^K\rangle_t &= \frac{1}{K}\int_0^t (\mathfrak{c} x_s^K y_s^k + \mathfrak{d} y_s^K) ds. \end{aligned} \tag{1.7}$$

The choice of the stochastic model (1.2) as a prelimit one for the deterministic system (1.1) is justified by the following approximation result proved in [2].

Theorem A (Fluid approximation Theorem 2 of [2]). *Let x^K_t, y^K_t satisfy* (1.6) *with $x^K_0 = x_0$, $y^K_0 = y_0$, and let x_t, y_t solve* (1.1) *with the same initial conditions x_0, y_0. Fix any $T > 0$. Then, for all $\varepsilon > 0$,*

$$\lim_{K\to\infty} P\Big(\sup_{t\le T}(|x^K_t - x_t| + |y^K_t - y_t|) > \varepsilon\Big) = 0.$$

The next asymptotic for the family $[(x^K_t, y^K_t)_{t\ge 0}]_{K\to\infty}$ is the large-deviation principle (LDP) in the Skorokhod space of nonnegative two-dimensional vector-valued functions $(\mathbb{D}_{[0,\infty)}(\mathbb{R}^2_+)$ equipped with the Lindvall–Skorokhod metric ϱ. The same result holds for the locally uniform metric ρ. For the definition of LDP, see e.g., [1, 9].

Theorem B (LDP Theorem 3 of [2]). *The LDP holds with the rate of speed $\frac{1}{K}$ and the* (*good*) *rate function*

$$J(\phi,\psi) = \begin{cases} \int_0^\infty \sup_{\lambda,\mu}(\lambda\dot{\phi}_t + \mu\dot{\psi}_t - G(\lambda,\mu;\phi_t,\psi_t))dt, & (\phi,\psi)\in\mathcal{F}, \\ \infty, & \textit{otherwise}, \end{cases}$$

where $\mathcal{F} = \{(\phi,\psi)\in\mathbb{D}^{2,+}_{[0,\infty)} : \phi_0 = 0, d\phi_t = \dot{\phi}_t dt;\ \ \psi_0 = 0, d\psi_t = \dot{\psi}_t dt\}$ and

$$G(\lambda,\mu;u,v) = G_\phi(\lambda;u,v) + G_\psi(\mu;u,v)$$

with

$$G_\phi(\lambda;u,v) = \lambda(\mathfrak{a} - \mathfrak{b}v)u + (e^\lambda - 1 - \lambda)\mathfrak{a}u + (e^{-\lambda} - 1 + \lambda)\mathfrak{b}uv,$$
$$G_\psi(\mu;u,v) = \mu(\mathfrak{c}u - \mathfrak{d})v + (e^\mu - 1 - \mu)\mathfrak{c}uv + (e^{-\mu} - 1 + \mu)\mathfrak{d}v.$$

2 Results

Let $\xi^K_t = \sqrt{K}(x^K_t - x_t),\ \ \zeta^K_t = \sqrt{K}(y^K_t - y_t)$ denote the difference between the stochastic model and its deterministic limit on the CLT scale, and write them as the vector $\begin{pmatrix}\xi^K_t\\ \zeta^K_t\end{pmatrix}$. The first result concerns the functional central limit theorem (FCLT) or so-called diffusion approximation for the family $[(Z^K_t)_{t\ge 0}]_{K\to\infty}$.

Theorem 2.1 *The family $[(Z^K_t)_{t\ge 0}]_{K\to\infty}$ possesses a weak limit $(Z_t)_{t\ge 0}$ in the Lindvall–Skorokhod topology, which is a Gaussian diffusion process solving the Itô integral equation*

$$Z_t = \int_0^t A(s)Z_s ds + M_t, \tag{2.1}$$

where

$$A(t) = \begin{pmatrix} \mathfrak{a} - \mathfrak{b}y_t & -\mathfrak{b}x_t \\ \mathfrak{c}y_t & \mathfrak{c}x_t - \mathfrak{d} \end{pmatrix} \tag{2.2}$$

and $M_t = \begin{pmatrix} \int_0^t \sqrt{\mathfrak{a}x_s + \mathfrak{b}x_s y_s}\, dW_s \\ \int_0^t \sqrt{\mathfrak{c}x_s y_s + \mathfrak{d}x_s}\, d\widehat{W}_s \end{pmatrix}$ with independent Wiener processes W_t and $\widehat{W}_t$.

The next result studies the difference on the moderate deviation scale:

$$\alpha \in \left(0, \frac{1}{2}\right), \qquad \text{“}\alpha = \frac{1}{2} \text{ is FCLT scale”},$$

$$\left.\begin{array}{l} \xi_t^{K,\alpha} = K^\alpha (x_t^K - x_t) \\ \zeta_t^{K,\alpha} = K^\alpha (y_t^K - y_t) \end{array}\right\} = Z_t^{K,\alpha} \equiv K^{\alpha-0.5} Z_t^K.$$

We prove the moderate deviation principle (MDP) for the family $[(Z_t^{K,\alpha})_{t\geq 0}]_{K\to\infty}$ in the metric space $(\mathbb{D}_{[0,\infty)}(\mathbb{R}_+^2),\ \varrho)$, the Skorokhod space of nonnegative two-dimensional vector-valued functions with the Lindvall–Skorokhod metric ϱ.

Theorem 2.2 *The family $[(Z_t^{K,\alpha})_{t\geq 0}]_{K\to\infty}$ obeys the MDP with the rate of speed $K^{-2\alpha}$ and the rate function*

$$J(\Psi) = \begin{cases} \frac{1}{2}\int_0^\infty \|\dot{\Psi}_t - A(t)\Psi_t\|^2_{B^{-1}(t)} dt, & \Psi_0 = 0;\ d\Psi_t = \dot{\Psi}_t dt, \\ \infty, & \textit{otherwise}, \end{cases}$$

where $A(t)$ is as in (2.2), and

$$B(t) = \begin{pmatrix} \mathfrak{a}x_t + \mathfrak{b}x_t y_t & 0 \\ 0 & \mathfrak{c}x_t y_t + \mathfrak{d}y_t \end{pmatrix}. \tag{2.3}$$

3 Proofs

We start with the notation used in the sequel:

$$A^K(t) = \begin{pmatrix} \mathfrak{a} - \mathfrak{b}y_t^K & -\mathfrak{b}x_t \\ \mathfrak{c}y_t^K & \mathfrak{c}x_t - \mathfrak{d} \end{pmatrix},$$

$$B^K(t) = \begin{pmatrix} \mathfrak{a}x_t^K + \mathfrak{b}x_t^K y_t^K & 0 \\ 0 & \mathfrak{c}x_t^K y_t^K + \mathfrak{d}y_t^K \end{pmatrix},$$

$$M_t^K = \begin{pmatrix} \sqrt{K} m_t \\ \sqrt{K}\widehat{m}_t \end{pmatrix},$$

$$M_t^{K,\alpha} = \begin{pmatrix} K^{0.5-\alpha} m_t \\ K^{0.5-\alpha}\widehat{m}_t \end{pmatrix},$$

$$Z_t^K = \begin{pmatrix} \xi_t^K \\ \zeta_t^K \end{pmatrix},$$

$C, C_1, C_2, \ldots$ are generic positive constants.

3.1 Proof of Theorem 2.1

Proof From (1.6) and the definition of Z_t^K it follows that

$$Z_t^K = \int_0^t A^K(s) Z_s^K ds + M_t^K. \tag{3.1}$$

The jumps of the martingale M_t^K are of size $\frac{1}{\sqrt{K}}$, and its predictable quadratic variation process is

$$\langle M^K\rangle_t = \int_0^t B^K(s)ds.$$

The Gaussian diffusion process Z_t is unique, since it is defined by the linear Itô integral equation (2.1), where the martingale M_t has the predictable quadratic variation process $\langle M\rangle_t = \int_0^t B(s)ds$.

Due to the above remarks, the conditions of Theorem 8.3.1 of [3] are satisfied, since, for all $T > 0$,

$$\sup_{t\le T}\left|\int_0^t [A^K(s) - A(s)]Z_s^K ds\right| \xrightarrow[K\to\infty]{\text{prob.}} 0 \tag{3.2}$$

and

$$\sup_{t\le T}\left|\int_0^t [B^K(s) - B(s)]ds\right| \xrightarrow[K\to\infty]{\text{prob.}} 0. \tag{3.3}$$

By the definition of $A^K(s)$ and $A(s)$, (3.2) holds, provided that, for all $T > 0$,

$$\int_0^T |y_s^K - y_s||\xi_s^K|ds \xrightarrow[K\to\infty]{\text{prob.}} 0. \tag{3.4}$$

Also, by the definition of $B^K(s)$ and $B(s)$,

$$\int_0^T \big(|x_s^K - x_s| + |y_s^K - y_s| + |x_s^K - x_s||y_s^K - y_s|\big)ds \xrightarrow[K\to\infty]{\text{prob.}} 0 \tag{3.5}$$

implies (3.3).

Note that Theorem A gives (3.5) and, thus, (3.3). Theorem A also gives (3.4) under the condition

$$\lim_{n\to\infty}\overline{\lim_{K\to\infty}}\, P\Big(\sup_{t\le T}|\xi_t^K| > n\Big) = 0. \tag{3.6}$$

Thus, to prove (2.1) it only remains to verify (3.6).

We will instead prove that

$$\lim_{n\to\infty}\overline{\lim_{K\to\infty}}\, P\Big(\sup_{t\le T}(|\xi_t^K| + |\zeta_t^K|) > n\Big) = 0. \tag{3.7}$$

First, set $T_n^K = \inf\{n : x_t^K \vee y_t^K \ge n\}$. By Corollary 1 to Lemma 1 of [2], we find that, for all $T > 0$,

$$\lim_{n\to\infty} P(T_n \le T) = 0.$$

Thus, to prove (3.7) it suffices to show that, for all $n \ge 1$,

$$\lim_{a\to\infty}\limsup_{K\to\infty} P\Big(\sup_{t\le T_n^K\wedge T}(|\xi_t^K| + |\zeta_t^K|) \ge a\Big) = 0.$$

Since

$$\sup_{t\le T_n^K\wedge T}(x_t^K\vee y_t^K)\le n+1,$$

there exists a constant L_n depending on n and T such that, for all $t\le T_n^K\wedge T$,

$$\sqrt{K}\big|(\mathfrak{a}x_s^K-\mathfrak{b}x_s^Ky_s^K)-(\mathfrak{a}x_s-\mathfrak{b}x_sy_s)\big|\le L_n(|\xi_t^K|+|\zeta_t^K|)$$

and

$$\sqrt{K}\big|(\mathfrak{c}x_s^Ky_s^K-\mathfrak{d}y_s^K)-(\mathfrak{c}x_sy_s-\mathfrak{d}y_s)\big|\le L_n(|\xi_t^K|+|\zeta_t^K|).$$

Thus, using the semimartingale representations for ξ and ζ, we obtain

$$|\xi^K_{t\wedge T_n^K}|+|\zeta^K_{t\wedge T_n^K}|\le 2L_n\int_0^t|\xi^K_{s\wedge T_n^K}|+|\zeta^K_{s\wedge T_n^K}|ds+\sup_{t\wedge T_n^K}\big|\sqrt{K}m_t^K\big|+\sup_{t\wedge T_n^K}\big|\sqrt{K}\widehat{m}_t^K\big|$$

and by the Gronwall–Bellman inequality we find that

$$\sup_{t\le T_n^K\wedge T}|\xi_t^K|+|\zeta_t^K|\le e^{2L_nT}\Big(\sup_{t\wedge T_n^K}|\sqrt{K}m_t^K|+\sup_{t\wedge T_n^K}|\sqrt{K}\widehat{m}_t^K|\Big).$$

Thus, the desired result holds if

$$\lim_{a\to\infty}\varlimsup_{K\to\infty}P\Big(\sup_{t\wedge T_n^K}\big|\sqrt{K}m_t^K\big|\ge a\Big)=0, \tag{3.8}$$

$$\lim_{a\to\infty}\varlimsup_{K\to\infty}P\Big(\sup_{t\wedge T_n^K}\big|\sqrt{K}\widehat{m}_t^K\big|\ge a\Big)=0.$$

In order to prove the latter, we apply the Doob inequality:

$$P\Big(\sup_{t\wedge T_n^K}|\sqrt{K}m_t^K|\ge a\Big)\le\frac{1}{a^2}E\big\langle\sqrt{K}m^K\big\rangle_{T_n^K},$$

$$P\Big(\sup_{t\wedge T_n^K}|\sqrt{K}\widehat{m}_t^K|\ge a\Big)\le\frac{1}{a^2}E\big\langle\sqrt{K}\widehat{m}^K\big\rangle_{T_n^K}.$$

Note that

$$\big\langle\sqrt{K}m^K\big\rangle_{T_n^K}=\int_0^{T_n^K}(\mathfrak{a}x_s^K+\mathfrak{b}x_s^Ky_s^K)ds\le C(n,T),$$

$$\big\langle\sqrt{K}\widehat{m}^K\big\rangle_{T_n^K}=\int_0^{T_n^K}(\mathfrak{c}x_s^Ky_s^k+\mathfrak{d}y_s^K)ds\le C(n,T)$$

with $\lim_{a\to\infty}C(n,T)/a^2=0$. Therefore, (3.8) holds, and the result is proved. □

3.2 Proof of Theorem 2.2

3.2.1 Preliminaries

Recall that $Z_t^{K,\alpha} = K^{-\alpha} Z_t^K$ with Z_t^K defined by (3.1). Write

$$Z_t^{K,\alpha} = \int_0^t A^K(s) Z_s^{K,\alpha} ds + \frac{1}{K^\alpha} M_t^K. \tag{3.9}$$

On the other hand, $Z_t^{K,\alpha} = K^\alpha \binom{x_t^K}{y_t^K} - K^\alpha \binom{x_t}{y_t}$ is the difference of a pure jump process and a deterministic continuous process.

Let $\mu^{K,\alpha}(ds, dz)$ be the jump measure of $K^\alpha \binom{x_t^K}{y_t^K}$, and let $\nu^{K,\alpha}$ denote the compensator of $\mu^{K,\alpha}$. Since the jump size of the counting processes $\pi_t', \pi_t'', \widehat{\pi}_t', \widehat{\pi}_t''$ is 1, the random process $Z_t^{K,\alpha}$ has jumps of size $K^{-(0.5+\alpha)}$. By Assumption A, the jumps of $Z_t^{K,\alpha}$ take values in the set

$$\mathfrak{A} = K^{-(0.5+\alpha)}\big\{(1,0), (-1,0), (0,1), (0,-1)\big\}.$$

Equation (3.9) and the above fact enable us to conclude that

$$M_t^{K,\alpha} = \frac{1}{K^\alpha} M_t^K = \int_0^t \int_{\mathfrak{A}} z(\mu_s^{K,\alpha} - \nu_s^{K,\alpha})(ds, dz).$$

Hence, $Z_t^{K,\alpha}$ possesses the following semimartingale representation:

$$Z_t^{K,\alpha} = \int_0^t A^K(s) Z_s^{K,\alpha} ds + \int_0^t \int_{\mathfrak{A}} z(\mu^{K,\alpha} - \nu^{K,\alpha})(ds, dz).$$

Now, we give a description for the compensator $\nu^{K,\alpha}$. Let $\Gamma \subseteq \mathfrak{A}$. Then

$$\mu^{K,\alpha}\big((0,t] \times \Gamma\big) = \sum_{0<s\le t} I(\Delta Z_s^{K,\alpha} \in \Gamma).$$

We have

$$\begin{aligned}
\mu^{K,\alpha}((0,t] \times K^{-(0.5+\alpha)}(1,0)) &= \pi_t', \\
\mu^{K,\alpha}((0,t] \times K^{-(0.5+\alpha)}(-1,0)) &= \pi_t'', \\
\mu^{K,\alpha}((0,t] \times K^{-(0.5+\alpha)}(0,1)) &= \widehat{\pi}_t', \\
\mu^{K,\alpha}((0,t] \times K^{-(0.5+\alpha)}(0,-1)) &= \widehat{\pi}_t'',
\end{aligned}$$

and by Assumption A we obtain

$$\begin{aligned}
\nu^{K,\alpha}((0,t] \times \Gamma_{(1,0)}) &= \int_0^t \mathfrak{a} X_s ds, \\
\nu^{K,\alpha}((0,t] \times \Gamma_{(-1,0)}) &= \int_0^t \frac{\mathfrak{b}}{K} X_s Y_s ds, \\
\nu^{K,\alpha}((0,t] \times \Gamma_{(0,1)}) &= \int_0^t \frac{\mathfrak{c}}{K} X_s Y_s ds, \\
\nu^{K,\alpha}((0,t] \times \Gamma_{(0,-1)}) &= \int_0^t \mathfrak{d} Y_s ds.
\end{aligned}$$

3.2.2 Stochastic Exponential

Following Puhalskii's proof method for Large Deviations [6, 7, Sects. 5.1, 5.2, and 5.3], we introduce a stochastic exponential suitably adapted to the current problem.

Denote by $\langle\!\langle , \rangle\!\rangle$ the inner product. For a vector-function $\lambda(s)$ with bounded measurable entries $\lambda_i(t)$, $i = 1, 2$, we introduce the semimartingale exponential

$$U_t(\lambda) = \exp\bigg(\int_0^t \int_{\mathfrak{A}} \langle\!\langle \lambda(s), dZ_s^{K,\alpha} \rangle\!\rangle\bigg)$$

and determine its multiplicative decomposition

$$U_t(\lambda) = \mathfrak{z}_t(\lambda)\mathcal{E}_t(\lambda),$$

where $\mathfrak{z}_t(\lambda)$ is a positive local martingale, and $\mathcal{E}_t(\lambda)$ is the compensator of $U_t(\lambda)$ up to a local martingale $\mathfrak{z}_t(\lambda)$ ($\mathcal{E}_t(\lambda)$ is a positive predictable process of locally bounded variation).

Applying Itô's formula, we find that

$$\begin{aligned}
U_t(\lambda) &= 1 + \int_0^t U_{s-}(\lambda)\langle\!\langle \lambda(s), dZ_s^{K,\alpha} \rangle\!\rangle + \sum_{s\le t}\big[(U_s(\lambda) - U_{s-}(\lambda)) - U_{s-}(\lambda)\Delta Z_s^{K,\alpha}\big] \\
&= 1 + \int_0^t U_{s-}(\lambda)\langle\!\langle \lambda(s), A^K(s)Z_s^{K,\alpha} \rangle\!\rangle ds \\
&\quad + \int_0^t \int_{\mathfrak{A}} U_{s-}(\lambda)\langle\!\langle \lambda(s), z \rangle\!\rangle [\mu^{K,\alpha} - \nu^{K,\alpha}](ds, dz) \\
&\quad + \int_0^t \int_{\mathfrak{A}} \big[U_{s-}(\lambda)(e^{\langle\!\langle \lambda(s), z \rangle\!\rangle} - 1) - U_{s-}(\lambda)\langle\!\langle \lambda(s), z \rangle\!\rangle\big]\mu^{K,\alpha}(ds, dz) \\
&= 1 + \int_0^t U_{s-}(\lambda)\langle\!\langle \lambda(s), A^K(s)Z_s^{K,\alpha} \rangle\!\rangle ds \\
&\quad + \int_0^t \int_{\mathfrak{A}} U_{s-}(\lambda)(e^{\langle\!\langle \lambda(s), z \rangle\!\rangle} - 1)[\mu^{K,\alpha} - \nu^{K,\alpha}](ds, dz) \\
&\quad + \int_0^t \int_{\mathfrak{A}} U_{s-}(\lambda)\big(e^{\langle\!\langle \lambda(s), z \rangle\!\rangle} - 1 - \langle\!\langle \lambda(s), z \rangle\!\rangle\big)\nu^{K,\alpha}(ds, dz).
\end{aligned}$$

The choice of $\mathcal{E}_t^{-1}(\lambda)$ is imposed by the following condition:

$$U_t(\lambda)\mathcal{E}_t^{-1}(\lambda) \text{ is a local martingale,}$$

which is easily verified by applying Itô's formula to $U_t(\lambda)\mathcal{E}_t^{-1}(\lambda)$. This way enables us to find that $\mathcal{E}_t^{-1}(\lambda)$ solves the linear equation

$$\begin{aligned}
\mathcal{E}_t^{-1}(\lambda) = 1 &- \int_0^t \mathcal{E}_{s-}^{-1}(\lambda)\langle\!\langle \lambda(s), A^K(s)Z_s^{K,\alpha} \rangle\!\rangle ds \\
&- \int_0^t \int_{\mathfrak{A}} \mathcal{E}_{s-}^{-1}(\lambda)\big(e^{\langle\!\langle \lambda(s), z \rangle\!\rangle} - 1 - \langle\!\langle \lambda(s), z \rangle\!\rangle\big)\nu^{K,\alpha}(ds, dz).
\end{aligned}$$

This equation has the unique solution

$$\mathcal{E}_t^{-1}(\lambda)\exp\biggl(-\int_0^t \langle\!\langle \lambda(s), A^K(s)Z_s^{K,\alpha}\rangle\!\rangle ds \\ - \int_0^t\int_{\mathfrak{A}}\bigl(e^{\langle\!\langle \lambda(s),z\rangle\!\rangle}-1-\langle\!\langle \lambda(s),z\rangle\!\rangle\bigr)\nu^{K,\alpha}(ds,dz)\biggr).$$

Thus,

$$\mathcal{E}_t(\lambda)=\exp\biggl(\int_0^t \langle\!\langle \lambda(s), A^K(s)Z_s^{K,\alpha}\rangle\!\rangle ds \\ + \int_0^t\int_{\mathfrak{A}}\bigl(e^{\langle\!\langle \lambda(s),z\rangle\!\rangle}-1-\langle\!\langle \lambda(s),z\rangle\!\rangle\bigr)\nu^{K,\alpha}(ds,dz)\biggr).$$

3.2.3 Puhalskii's Method

By Theorem 2.1 the family $\{(Z_t^K)\}_{K\to\infty}$ possesses a diffusion approximation for the limit process $Z_t=\int_0^t A(s)Z_s ds+\int_0^t B^{1/2}(s)dW_s$. Therefore, the random process $\widetilde{Z}_t^{K,\alpha}:=\frac{1}{K^\alpha}Z_t$,

$$\widetilde{Z}_t^{K,\alpha}=\int_0^t A(s)\widetilde{Z}_s^{K,\alpha}ds+\frac{1}{K^\alpha}\int_0^t B^{1/2}(s)dW_s,$$

is in the framework of Freidlin–Wentzell's LDP with the rate and rate function announced in Theorem 2.2.

In order to prove Theorem 2.2, it suffices to prove that the families $\{(\widetilde{Z}_t^{K,\alpha})\}_{K\to\infty}$ and $\{(Z_t^{K,\alpha})\}_{K\to\infty}$ share the same LDP.

To this end, we introduce the semimartingale exponential

$$\widetilde{U}_t(\lambda)=\exp\biggl(\int_0^t\int_{\mathfrak{A}}\langle\!\langle \lambda(s), d\widetilde{Z}_s^{K,\alpha}\rangle\!\rangle\biggr)$$

and its compensator

$$\widetilde{\mathcal{E}}_t(\lambda)=\exp\biggl(\int_0^t \langle\!\langle \lambda(s), A(s)\widetilde{Z}_s^{K,\alpha}\rangle\!\rangle ds-\frac{1}{2K^{2\alpha}}\int_0^t\langle\!\langle \lambda(s), B(s)\lambda(s)\rangle\!\rangle ds\biggr).$$

Let $\widehat{\mathcal{E}}_t(\lambda)\equiv\widetilde{\mathcal{E}}_t(\lambda)_{\widetilde{Z}^{K,\alpha}=Z^{K,\alpha}}$, that is,

$$\widehat{\mathcal{E}}_t(\lambda)=\exp\biggl(\int_0^t \langle\!\langle \lambda(s), A(s)Z_s^{K,\alpha}\rangle\!\rangle ds-\frac{1}{2K^{2\alpha}}\int_0^t\langle\!\langle \lambda(s), B(s)\lambda(s)\rangle\!\rangle ds\biggr).$$

We are now able to apply Theorem 4.1.2 of Puhalskii [7] which, being adapted to the case considered, states that the above-mentioned family shares the same LDP, provided that

$$\lim_{K\to\infty}\frac{1}{K^{2\alpha}}\log P\biggl(\sup_{t\le T}\biggl|\frac{1}{K^{2\alpha}}\log\mathcal{E}_t(K^{2\alpha}\lambda)-\frac{1}{K^{2\alpha}}\log\widehat{\mathcal{E}}_t(K^{2\alpha}\lambda)\biggr|>\varepsilon\biggr)=-\infty. \tag{3.10}$$

Since

$$\frac{1}{K^{2\alpha}}\log \mathcal{E}_t(K^{2\alpha}\lambda) = \int_0^t \langle\!\langle \lambda(s), A^K(s)Z_s^{K,\alpha}\rangle\!\rangle ds$$
$$+ \int_0^t \int_{\mathfrak{A}} \frac{1}{K^{2\alpha}}\big(e^{K^{2\alpha}\langle\!\langle \lambda(s),z\rangle\!\rangle} - 1 - K^{2\alpha}\langle\!\langle \lambda(s), z\rangle\!\rangle\big)\nu^{K,\alpha}(ds,dz)$$
$$\frac{1}{K^{2\alpha}}\log \widehat{\mathcal{E}}_t(K^{2\alpha}\lambda) = \int_0^t \Big[\langle\!\langle \lambda(s), A(s)Z_s^{K,\alpha}\rangle\!\rangle ds - \frac{1}{2}\langle\!\langle \lambda(s), B(s)\lambda(s)\rangle\!\rangle\Big]ds,$$

the proof of (3.10) is reduced to

$$\lim_{K\to\infty}\frac{1}{K^{2\alpha}}\log P\bigg(\sup_{t\le T}\bigg|\int_0^t\int_{\mathfrak{A}}\frac{1}{K^{2\alpha}}\big(e^{K^{2\alpha}\langle\!\langle \lambda(s),z\rangle\!\rangle} - 1 - K^{2\alpha}\langle\!\langle \lambda(s), z\rangle\!\rangle\big)\nu^{K,\alpha}(ds,dz)$$
$$- \int_0^t \frac{1}{2}\langle\!\langle \lambda(s), B(s)\lambda(s)\rangle\!\rangle ds\bigg|\bigg) = -\infty. \tag{3.11}$$

Further, write

$$\int_0^t\int_{\mathfrak{A}}\frac{1}{K^{2\alpha}}\big(e^{K^{2\alpha}\langle\!\langle \lambda(s),z\rangle\!\rangle} - 1 - K^{2\alpha}\langle\!\langle \lambda(s), z\rangle\!\rangle\big)\nu^{K,\alpha}(ds,dz)$$
$$= \int_0^t \frac{1}{K^{2\alpha}}\big(e^{K^{\alpha-0.5}\langle\!\langle \lambda(s),(1,0)\rangle\!\rangle} - 1 - K^{\alpha-0.5}\langle\!\langle \lambda(s),(1,0)\rangle\!\rangle\big)\mathfrak{a}Kx_s^K ds$$
$$+ \int_0^t \frac{1}{K^{2\alpha}}\big(e^{K^{\alpha-0.5}\langle\!\langle \lambda(s),(-1,0)\rangle\!\rangle} - 1 - K^{\alpha-0.5}\langle\!\langle \lambda(s),(-1,0)\rangle\!\rangle\big)\mathfrak{b}Kx_s^K y_s^K ds$$
$$+ \int_0^t \frac{1}{K^{2\alpha}}\big(e^{K^{\alpha-0.5}\langle\!\langle \lambda(s),(0,1)\rangle\!\rangle} - 1 - K^{\alpha-0.5}\langle\!\langle \lambda(s),(0,1)\rangle\!\rangle\big)\mathfrak{c}Kx_s^K y_s^K ds$$
$$+ \int_0^t \frac{1}{K^{2\alpha}}\big(e^{K^{\alpha-0.5}\langle\!\langle \lambda(s),(0,-1)\rangle\!\rangle} - 1 - K^{\alpha-0.5}\langle\!\langle \lambda(s),(0,-1)\rangle\!\rangle\big)\mathfrak{d}Ky_s^K ds$$

and, due to the definition of $B^K(s)$,

$$\int_0^t \langle\!\langle \lambda(s), B^K(s)\lambda(s)\rangle\!\rangle ds$$
$$= \int_0^t \big[\langle\!\langle \lambda(s),(1,0)\rangle\!\rangle^2\mathfrak{a}x_s^K + \langle\!\langle \lambda(s),(-1,0)\rangle\!\rangle^2\mathfrak{b}x_s^K y_s^K$$
$$+ \langle\!\langle \lambda(s),(0,-1)\rangle\!\rangle^2\mathfrak{c}x_s^K y_s^K + \lambda(s),(0,-1)\rangle\!\rangle^2\mathfrak{d}y_s^K\big]ds.$$

Applying Taylor's formula, we find that there exists a positive constant C such that

$$\bigg|\frac{1}{K^{2\alpha}}\big(e^{K^{\alpha-0.5}\langle\!\langle \lambda(s),(1,0)\rangle\!\rangle} - 1 - K^{\alpha-0.5}\langle\!\langle \lambda(s),(1,0)\rangle\!\rangle\big)K - \frac{1}{2}\langle\!\langle \lambda(s),(1,0)\rangle\!\rangle^2\bigg|\mathfrak{a}x_s^K$$
$$\le \frac{C}{K^{0.5-\alpha}}x_s^K$$
$$\bigg|\frac{1}{K^{2\alpha}}\big(e^{K^{\alpha-0.5}\langle\!\langle \lambda(s),(-1,0)\rangle\!\rangle} - 1 - K^{\alpha-0.5}\langle\!\langle \lambda(s),(-1,0)\rangle\!\rangle\big)K - \frac{1}{2}\langle\!\langle \lambda(s),(-1,0)\rangle\!\rangle^2\bigg|\mathfrak{b}x_s^K y_s^K$$
$$\le \frac{C}{K^{0.5-\alpha}}x_s^K y_s^K$$

$$\left|\frac{1}{K^{2\alpha}}\big(e^{K^{\alpha-0.5}\langle\!\langle\lambda(s),(0,1)\rangle\!\rangle}-1-K^{\alpha-0.5}\langle\!\langle\lambda(s),(0,1)\rangle\!\rangle\big)K-\frac{1}{2}\langle\!\langle\lambda(s),(0,1)\rangle\!\rangle^2\right|\mathfrak{c}x_s^Ky_s^K$$
$$\le\frac{C}{K^{0.5-\alpha}}x_s^Ky_s^K$$
$$\left|\frac{1}{K^{2\alpha}}\big(e^{K^{\alpha-0.5}\langle\!\langle\lambda(s),(1,0)\rangle\!\rangle}-1-K^{\alpha-0.5}\langle\!\langle\lambda(s),(0,-1)\rangle\!\rangle\big)K-\frac{1}{2}\langle\!\langle\lambda(s),(0,-1)\rangle\!\rangle^2\right|\mathfrak{d}x_s^K$$
$$\le\frac{C}{K^{0.5-\alpha}}y_s^K.$$

Now, combining these results, we obtain

$$\begin{aligned}\sup_{t\le T}\bigg|&\int_0^t\int_{\mathfrak{A}}\frac{1}{K^{2\alpha}}\big(e^{K^{2\alpha}\langle\!\langle\lambda(s),z\rangle\!\rangle}-1-K^{2\alpha}\langle\!\langle\lambda(s),z\rangle\!\rangle\big)\nu^{K,\alpha}(ds,dz)\\&-\int_0^t\frac{1}{2}\langle\!\langle\lambda(s),B(s)\lambda(s)\rangle\!\rangle ds\bigg|\\&\le\int_0^T\big|\langle\!\langle\lambda(s),[A^K(s)-A(s)]Z^{K,\alpha}\rangle\!\rangle\big|ds+\frac{1}{2}\int_0^t\big|\langle\!\langle\lambda(s),[B^K(s)-B(s)]\lambda(s)\rangle\!\rangle\big|ds\\&\quad+\frac{C}{K^{0.5-\alpha}}\int_0^T[x_s^K+y_s^K+x_s^Ky_s^K]ds.\end{aligned}$$

Recall that the entries of $A(s)$, $B(s)$, and $\lambda(s)$ are bounded and that the entries of $(A^K(s)-A(s))$ and $(B^K(s)-B(s))$ are in a proportion to $|x_s^K-x_s|$, $|y_x^K-y_s|$, and $|x_s^Ky_s^K-x_sy_s|$. Therefore, (3.11) holds, provided that

(i) $\lim_{K\to\infty}\frac{1}{K^{2\alpha}}\log P\Big(\sup_{t\le T}|x_t^K-x_t|\ge\varepsilon\Big)=-\infty;$
(ii) $\lim_{K\to\infty}\frac{1}{K^{2\alpha}}\log P\Big(\sup_{t\le T}|y_t^K-y_t|ds\ge\varepsilon\Big)=-\infty;$
(iii) $\lim_{K\to\infty}\frac{1}{K^{2\alpha}}\log P\Big(\sup_{t\le T}|x_t^Ky_t^K-x_ty_t|\ge\varepsilon\Big)=-\infty;$
(iv) $\lim_{K\to\infty}\frac{1}{K^{2\alpha}}\log P\Big(\sup_{t\le T}|y_t^K-y_t|\sup_{t\le T}\|Z_t^{K,\alpha}\|\ge\varepsilon\Big)=-\infty;$
(v) $\lim_{K\to\infty}\frac{1}{K^{2\alpha}}\log P\Big(\sup_{t\le T}x_t^K\ge K^{0.5-\alpha}\varepsilon\Big)=-\infty;$
(vi) $\lim_{K\to\infty}\frac{1}{K^{2\alpha}}\log P\Big(\sup_{t\le T}y_t^K\ge K^{0.5-\alpha}\varepsilon\Big)=-\infty;$
(vii) $\lim_{K\to\infty}\frac{1}{K^{2\alpha}}\log P\Big(\sup_{t\le T}x_t^Ky_t^K\ge K^{0.5-\alpha}\varepsilon\Big)=-\infty.$

The proofs of (i)–(vii) are given below in two lemmas.

Lemma 3.1 *Theorem B implies* (i)–(iii) *and* (v)–(vii).

Proof Let us begin with the comment that $J(\phi,\psi)=0$ iff $(\phi_t,\psi_t)\equiv(x_t,y_t)$, where (x_t,y_t) solves (1.1). "If" is obvious. "Only if" is implied by the uniqueness of solution of (1.1).

(i) By Theorem B we have

$$\overline{\lim_{K\to\infty}}\frac{1}{K}\log P\Big(\sup_{t\le T}|x^K_t - x_t|\ge \varepsilon\Big)\le -\inf_{\substack{\phi:\sup_{t\le T}|\phi_t - x_t|\ge\varepsilon\\ \psi_t\equiv y_t}} J(\phi,\psi),$$

where (x_t, y_t) for $(x_, y_t)$ solves (1.1).

The result follows from the facts that

$$\inf_{\substack{\phi:\sup_{t\le T}|\phi_t - x_t|\ge\varepsilon\\ \psi_t\equiv y_t}} J(\phi,\psi)>0 \quad\text{and}\quad \lim_{K\to\infty}\frac{K}{K^{2\alpha}}=\infty.$$

(ii) is proved similarly.

(v) By Theorem B we have

$$\overline{\lim_{K\to\infty}}\frac{1}{K}\log P\Big(\sup_{t\le T}x^K_t\ge K^{0.5-\alpha}\varepsilon\Big)\le \overline{\lim_{K\to\infty}}\frac{1}{K}\log P\Big(\sup_{t\le T}x^K_t\ge C\Big)\xrightarrow[C\to\infty]{}-\infty,$$

and the result is implied by $\lim_{K\to\infty}\frac{K}{K^{2\alpha}}=\infty$.

(vi) is proved similarly.

(vii) follows from (v) and (vi).

(iii) is proved by means of (i), (ii) and (v), (vi). □

Lemma 3.2 *Theorems A and B imply* (iv).

Proof (iv) follows from (ii), provided that

$$\lim_{\ell\to\infty}\overline{\lim_{K\to\infty}}\frac{1}{K^{2\alpha}}\log P\Big(\sup_{t\le T}\|Z^{K,\alpha}_t\|\ge\ell\Big)=-\infty. \tag{3.12}$$

So, it remains to prove (3.12).

Set $\tau_n=\inf\{t\ge 0: x^K_t+y^K_t+x^K_t y^K_t\ge n\},\ n\ge 1$. By (v), (vi), and (vii) we have

$$\lim_{n\to\infty}\overline{\lim_{K\to\infty}}\frac{1}{K^{2\alpha}}\log P(\tau_n\le T)=-\infty. \tag{3.13}$$

Equation (3.13) and the upper bound

$$P\Big(\sup_{t\le T}\|Z^{K,\alpha}_t\|^2\ge\ell^2\Big)\le 2\Big[P\Big(\sup_{t\le T}\|Z^{K,\alpha}_{t\wedge\tau_n}\|^2\ge\ell^2\Big)\vee P(\tau_n\le T)\Big]$$

enable us to reduce the proof of (3.12) to showing that

$$\lim_{\ell\to\infty}\overline{\lim_{K\to\infty}}\frac{1}{K^{2\alpha}}\log P\Big(\sup_{t\le T}\|Z^{K,\alpha}_{t\wedge\tau_n}\|^2\ge\ell^2\Big)=-\infty. \tag{3.14}$$

By Itô's formula we find that

$$\begin{aligned}\|Z^{K,\alpha}_{t\wedge\tau_n}\|^2 &= \int_0^{t\wedge\tau_n}\langle\!\langle Z^{K,\alpha}_s, [(A^K(s))^*+A^K(s)]Z^{K,\alpha}_s\rangle\!\rangle ds\\ &\quad +2\int_0^{t\wedge\tau_n}\int_{\mathfrak{A}}\langle\!\langle Z^{K,\alpha}_{s-}, z\rangle\!\rangle[\mu^{K,\alpha}-\nu^{K,\alpha}](ds,dz)\end{aligned}$$

$$+\int_0^{t\wedge\tau_n}\int_{\mathfrak{A}}\|z\|^2[\mu^{K,\alpha}-\nu^{K,\alpha}](ds,dz)$$
$$+\int_0^{t\wedge\tau_n}\int_{\mathfrak{A}}\|z\|^2\nu^{K,\alpha}(ds,dz). \tag{3.15}$$

Since $\int_0^{t\wedge\tau_n}\int_{\mathfrak{A}}\|z\|^2\nu^{K,\alpha}(ds,dz)=\frac{1}{K^{2\alpha}}\int_0^{t\wedge\tau_n}\operatorname{trace}(B^K(s))ds$, one can rewrite (3.15) as

$$\|Z^{K,\alpha}_{t\wedge\tau_n}\|^2=\int_0^{t\wedge\tau_n}\Big[\big\langle\!\big\langle Z^{K,\alpha}_s,[(A^K(s))^*+A^K(s)]Z^{K,\alpha}_s\big\rangle\!\big\rangle+\frac{1}{K^{2\alpha}}\operatorname{trace}(B^K(s))\Big]ds$$
$$+\int_0^{t\wedge\tau_n}\int_{\mathfrak{A}}\big[2\langle\!\langle Z^{K,\alpha}_{s-},z\rangle\!\rangle+\|z\|^2\big][\mu^{K,\alpha}-\nu^{K,\alpha}](ds,dz).$$

Set

$$u^K(s)=\big\langle\!\big\langle Z^{K,\alpha}_s,[(A^K(s))^*+A^K(s)]Z^{K,\alpha}_s\big\rangle\!\big\rangle+\frac{1}{K^{2\alpha}}\operatorname{trace}(B^K(s)),$$
$$v^K(s)=2\langle\!\langle Z^{K,\alpha}_{s-},z\rangle\!\rangle+\|z\|^2,$$
$$\mathcal{L}^K_t=\int_0^{t\wedge\tau_n}v^K(s)[\mu^{K,\alpha}-\nu^{K,\alpha}](ds,dz)$$

and rewrite the above representation as

$$\|Z^{K,\alpha}_{t\wedge\tau_n}\|^2=\int_0^{t\wedge\tau_n}u^K(s)ds+\mathcal{L}^K_t. \tag{3.16}$$

Hence, the jumps of the processes $\|Z^{K,\alpha}_{t\wedge\tau_n}\|^2$ and $\mathcal{L}^K_t$ coincide. Denote $\Delta\mathcal{L}^K_s=\mathcal{L}^K_s-\mathcal{L}^K_{s-}$. Since $\nu^{K,\alpha}(\{s\},dz)\equiv 0$, from the definition of $\mathcal{L}^K_s$ it follows that

$$\Delta\mathcal{L}^K_s=I(s\le\tau_n)\int_{\mathfrak{A}}[2\langle\!\langle Z^{K,\alpha}_{s-},z\rangle\!\rangle+\|z\|^2]\mu^{K,\alpha}(\{s\},dz).$$

Further, by the Cauchy–Schwarz inequality we have $|\langle\!\langle Z^{K,\alpha}_{s-},z\rangle\!\rangle|\le\|Z^{K,\alpha}_{s-}\|\ \|z\|$. Therefore, $|\Delta\mathcal{L}^K_s|\le I(s\le\tau_n)[2\|Z^{K,\alpha}_{s-}\|K^{-(0.5+\alpha)}+K^{-(1+2\alpha)}]$. The boundedness of the entries of $A^K(t\wedge\tau_n)$ and $B^K(t\wedge\tau_n)$ by a generic constant C_n implies that $\int_0^{t\wedge\tau_n}u^K(s)ds\le\int_0^t C_n(1+\|Z^{K,\alpha}_{s\wedge\tau_n}\|^2)ds$. Now, application of this inequality in (3.16) implies that $\|Z^{K,\alpha}_{t\wedge\tau_n}\|^2\le\int_0^t C_n(1+\|Z^{K,\alpha}_{s\wedge\tau_n}\|^2)ds+(\mathcal{L}^K_t)^+$. This linear integral inequality leads to (here $C(n,T)$ is a positive constant depending on n,T)

$$\|Z^{K,\alpha}_{t\wedge\tau_n}\|^2\le\int_0^t e^{C_n(t-s)}\big[C_n+(\mathcal{L}^K_s)^+\big]ds+(\mathcal{L}^K_t)^+$$
$$\le C(n,T)\Big(1+\sup_{s\le t}(\mathcal{L}^K_s)^+\Big). \tag{3.17}$$

Thus, it suffices to prove that

$$\lim_{\ell\to\infty}\overline{\lim_{K\to\infty}}\frac{1}{K^{2\alpha}}\log P\Big(\sup_{t\le T}(\mathcal{L}^{K,\alpha}_t)^+\ge\ell\Big)=-\infty \tag{3.18}$$

instead of (3.14). We introduce the family of stopping times

$$\sigma^{\ell} = \inf\{t \geq 0 : \mathcal{L}_t^{K,\alpha} \geq \ell\},\ \ell \nearrow \infty,$$

and note that (3.18) holds (consequently, so does (3.14)), provided that

$$\lim_{\ell\to\infty} \overline{\lim_{K\to\infty}} \frac{1}{K^{2\alpha}} \log P(\sigma^{\ell} \leq T) = -\infty. \tag{3.19}$$

The last step of the proof is to show (3.19). The random process $\mathcal{L}_t^K$ is a local martingale. The stochastic exponential

$$\mathcal{E}_t(\lambda) = \exp\bigg(\int_0^{t\wedge\tau_n} \int_{\mathfrak{A}} \big[e^{\lambda v^K(s)} - 1 - \lambda v^K(s)\big] \nu^{K,\alpha}(ds, dz)\bigg), \quad \lambda \in \mathbb{R}, \tag{3.20}$$

is its multiplicative compensator in the sense that

$$\mathfrak{z}_t(\lambda) = \exp(\lambda \mathcal{L}_t^K - \log \mathcal{E}_t(\lambda)) \tag{3.21}$$

is a (positive) local martingale. Hence, for any ℓ, we have $E\mathfrak{z}_{T\wedge\sigma^{\ell}}(\lambda) \leq 1$. Then, particularly, $1 \geq EI(\sigma^{\ell} \leq T)\mathfrak{z}_{T\wedge\sigma^{\ell}}(\lambda)$. By a suitable choice of λ depending on K we intend to find a deterministic lower bound $\mathfrak{z}_*(K)$ for $\mathfrak{z}_{T\wedge\sigma^{\ell}}(\lambda)$ on the set $\{\sigma^{\ell} \leq T\}$. Then we obtain that

$$\frac{1}{K^{2\alpha}} \log P(\sigma^{\ell} \leq T) \leq -\frac{1}{K^{2\alpha}} \log \mathfrak{z}_*(\lambda_K)$$

and, in turn, (3.19), provided that

$$\lim_{\ell\to\infty} \underline{\lim_{K\to\infty}} \frac{1}{K^{2\alpha}} \log \mathfrak{z}_*(K) = \infty. \tag{3.22}$$

Henceforth $\lambda > 0$. On the set $\{\sigma^{\ell} \leq T\}$, we have $\lambda\mathcal{L}^K_{T\wedge\sigma^{\ell}} \geq \lambda\ell$. By the mean-value theorem we have $|e^{\lambda v^K(s)} - 1 - \lambda v^K(s)| = e^{\lambda|v^K(\cdot)|}\frac{\lambda^2(v^K(s))^2}{2}$. Now, we evaluate $v^K(s)$ from above for $s \leq T \wedge \tau_n \wedge \sigma^{\ell}$. Write

$$\begin{aligned}
&\sup_{s\leq T\wedge\tau_n\wedge\sigma^{\ell}} |v^K(s)| \\
&\quad\leq 2 \sup_{s\leq T\wedge\tau_n\wedge\sigma^{\ell}} |\langle\!\langle Z_{s-}^{K,\alpha}, z\rangle\!\rangle| + \|z\|^2 \\
&\quad\leq 2 \sup_{s\leq T\wedge\tau_n\wedge\sigma^{\ell}} \|Z_{s-}^{K,\alpha}\|\|z\| + \|z\|^2 \\
&\quad\leq 2C^{1/2}(n, T)\sqrt{1 + \sup_{s\leq T\wedge\tau_n\wedge\sigma^{\ell}} (\mathcal{L}_{s-}^K)^+}\|z\| + \|z\|^2 \quad \text{(see (3.17))} \\
&\quad\leq 2C^{1/2}(n, T)\sqrt{1+\ell}\|z\| + \|z\|^2.
\end{aligned}$$

For notational convenience, set $r_n = 2C^{1/2}(n, T)\sqrt{\frac{1+\ell}{\ell}}$ and note that r_n is bounded in $\ell > 1$.

All these estimates imply the following upper bound:

$$\log \mathcal{E}_{T\wedge\sigma^\ell}(\lambda) \le \int_0^{T\wedge\tau_n\wedge\sigma^\ell}\int_{\mathfrak{A}} \exp\{\lambda(r_n\sqrt{\ell}\|z\| + \|z\|^2)\}\frac{\lambda^2}{2}(r_n\sqrt{\ell}\|z\| + \|z\|^2)^2\nu^{K,\alpha}(ds,dz)$$

$$\le \int_0^{T\wedge\tau_n\wedge\sigma^\ell}\frac{\lambda^2}{2}\exp\Big\{\lambda\Big[\frac{r_n\sqrt{\ell}}{K^{0.5+\alpha}} + \frac{1}{K^{1+2\alpha}}\Big]\Big\}\Big(\frac{r_n\sqrt{\ell}}{K^{0.5+\alpha}} + \frac{1}{K^{1+2\alpha}}\Big)^2$$

$$\times K\max[\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}](x_s^K + y_s^K + x_s^K y_s^K)ds.$$

Since $(x_s^K + y_s^K + x_s^K y_s^K) \le n$ for $s < \tau_n$, the above inequality is preserved under replacement of $\max[\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}](x_s^K + y_s^K + x_s^K y_s^K)$ by an appropriate constant D_n depending on n. Therefore, one can use the inequality

$$\log \mathcal{E}_{T\wedge\sigma^\ell}(\lambda) \le TD_n\frac{\lambda^2}{2}\exp\Big\{\lambda\Big[\frac{r_n\sqrt{\ell}}{K^{0.5+\alpha}} + \frac{1}{K^{1+2\alpha}}\Big]\Big\}\Big(\frac{r_n\sqrt{\ell}}{K^{0.5+\alpha}} + \frac{1}{K^{1+2\alpha}}\Big)^2 K$$

$$= TD_n\frac{\lambda^2}{2K^{2\alpha}}\exp\Big\{\lambda\Big[\frac{r_n\sqrt{\ell}}{K^{0.5+\alpha}} + \frac{1}{K^{1+2\alpha}}\Big]\Big\}\Big(r_n\sqrt{\ell} + \frac{1}{K^{0.5+2\alpha}}\Big)^2.$$

Applying the above upper bound in (3.21) with $\lambda_K = \frac{K^{2\alpha}}{TD_n r_n^2}$, we find that

$$\log \mathfrak{z}_{T\wedge\sigma^\ell}(\lambda_K) \ge \frac{K^{2\alpha}\ell}{TD_n r_n^2} - \frac{1}{2}\frac{K^{2\alpha}}{TD_n r_n^4}\exp\Big\{TD_n\ell\Big[\frac{r_n\sqrt{\ell}}{K^{0.5-\alpha}} + \frac{1}{K}\Big]\Big\}\Big(r_n\sqrt{\ell} + \frac{1}{K^{0.5+2\alpha}}\Big)^2$$

$$= \log \mathfrak{z}_*(\lambda_K).$$

This lower bound gives (3.22), since

$$\frac{1}{K^{2\alpha}}\log \mathfrak{z}_*(\lambda_K) = \frac{K^{2\alpha}\ell}{TD_n r_n^2} - \frac{1}{2}\frac{K^{2\alpha}}{TD_n r_n^4}\exp\Big\{TD_n\ell\Big[\frac{r_n\sqrt{\ell}}{K^{0.5-\alpha}} + \frac{1}{K}\Big]\Big\}\Big(r_n\sqrt{\ell} + \frac{1}{K^{0.5+2\alpha}}\Big)^2$$

$$\xrightarrow{K\to\infty} \frac{1}{2}\frac{\ell}{TD_n r_n^2} \xrightarrow{\ell\to\infty} \infty,$$

and the proof is completed. □

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