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ON A PROPERTY OF A REFUSALS STREAM

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Abstract

This paper consists of two parts. The first part provides a more elementary proof of the asymptotic theorem of the refusals stream for an $M/GI/1/n$ queueing system discussed in Abramov (1991a). The central property of the refusals stream discussed in the second part of this paper is that, if the expectations of interarrival and service time of an $M/GI/1/n$ queueing system are equal to each other, then the expectation of the number of refusals during a busy period is equal to 1. This property is extended for a wide family of single-server queueing systems with refusals including, for example, queueing systems with bounded waiting time.

REFUSALS; $M/GI/1/n$ QUEUEING SYSTEM; BUSY PERIOD; RENEWAL PROCESS

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0. Introduction

Consider an $M/GI/1/n$ queueing system. Let L_n be the number of lost customers during a busy period. Abramov (1984), (1991a) studied the asymptotic behavior of EL_n as $n \rightarrow \infty$. In the case where the expectations of interarrival and service times are equal to each other, the interesting property that $EL_n = 1$ for all $n \geq 0$ was proved. In Abramov (1991b) a similar property was obtained for the more general $M/GI/1/\infty$ queueing system under the following setting. It was assumed that if an arriving customer met more than n waiting places busy, he abandoned the system with probability $p > 0$ and joined the queue with probability $q = 1 - p$. Under mutually equal expectations of interarrival and service times it was shown that $EL_n = 1$, i.e. in this case the expectation of the number of refusals during a busy period is independent of the values n and p .

However, the proofs suggested in Abramov (1991a) are not elementary. They are based on a consideration of the busy period characteristics over the same probability space for both an $M/GI/1/n$ queueing system and an $M/GI/1/\infty$ queueing system with service depending on queue length and on an application of the coupling method. Using that approach a number of identities relating the expectations of the various characteristics of these two queueing systems during a given busy period were analytically deduced. The analytical approach led to complicated algebraic transformations. The method of

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the first section of this paper is essentially simpler. It is based on the well-known probabilistic idea of considering the stationary process and renewal arguments.

The recurrent equations for the expectation of a busy period for an $M/GI/1/n$ queueing system were obtained analytically by Tomko (1967). He used the well-known Takács representation for a busy period of single-server queueing systems with Poisson input with the aid of imbedded subperiods (see Takács (1962)). Tomko (1967) and Cooper and Tilt (1976) showed that the expectation of a busy period ET_n for an $M/GI/1/n$ queueing system satisfies the following representation:

$$(1) \quad \begin{aligned} ET_n &= \sum_{i=0}^n \pi_i ET_{n-i+1}, & ET_0 &= b, \\ \pi_i &= \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x), \end{aligned}$$

where λ is the Poisson input parameter, b is the expectation of service time, and $B(x)$ is the probability distribution function of service time. According to Wald's identity, the analogous representation holds for the expectation of the number of served customers during a busy period $E\nu_n$:

$$(2) \quad E\nu_n = \sum_{i=0}^n \pi_i E\nu_{n-i+1}, \quad E\nu_0 = 1.$$

The method of the first part of the paper permits us to obtain representations analogous to (1) and (2) for the expectations EL_n :

$$(3) \quad EL_n - 1 = \sum_{i=0}^n \pi_i (EL_{n-i+1} - 1), \quad EL_0 = \lambda b = \rho_1,$$

where ρ_1 is the traffic intensity. This allows us to apply the Takács theorem on the asymptotic behavior of this type of recurrent relation (Takács 1967).

In the second section of this paper we prove that the property 'an expectation of the number of refusals is equal to 1' is valid for a wide family of single-server queueing systems with refusals in which the expectations of interarrival and service times are equal to each other.

1. Properties of a refusals stream for an $M/GI/1/n$ queueing system

Consider an $M/GI/1/n$ queueing system, the input stream of which is Poissonian with rate λ and the distribution function of the service time is $B(t)$, $\rho_j = \lambda^j \int_0^\infty t^j dB(t)$. Let T_n, I_n, L_n denote respectively a stationary busy period, an idle period and the number of lost customers during a busy period. Let $p(i), i = 0, 1, \dots, n + 1$, be the average fraction of time that there are i customers in the system.

For the rate of putting out customers we have the following equation:

$$(4) \quad b^{-1}[1 - p(0)] = \lambda[1 - p(n + 1)].$$

According to the renewal reward theorem (see for example Karlin and Taylor (1975)) and because of $EL_n = \lambda^{-1}$ we have

$$(5) \quad p(0) = \mathbf{E}L_n / (\mathbf{E}L_n + \mathbf{E}T_n) = \lambda^{-1} / (\lambda^{-1} + \mathbf{E}T_n).$$

Using the renewal theorem once again and the PASTA property (see Asmussen (1987)) we obtain

$$(6) \quad \mathbf{E}L_n / (\lambda^{-1} + \mathbf{E}T_n) = \lambda p(n+1)$$

(i.e. the rate of lost customers is computed as an average rate of reward on the left-hand side, and is equal to $\lambda p(n+1)$ by the PASTA property).

Inserting (5) and (6) into (4), after a little algebra we obtain

$$(7) \quad b^{-1}(\rho_1 - 1)\mathbf{E}T_n = \mathbf{E}L_n - 1.$$

Therefore (3) follows immediately from (7) and (1).

Now, in order to obtain the asymptotic theorem for $\mathbf{E}L_n$ as $n \rightarrow \infty$ let us apply the following result by Takács (1967).

Lemma. Let π_0, π_1, \dots be a sequence of non-negative numbers, $\pi_0 + \pi_1 + \dots = 1$. Denote $\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j$ for $|z| \leq 1$, $\gamma_m = \pi^{(m)}(1-0)$, and $Q_k = \sum_{j=0}^k \pi_j Q_{k-j+1}$, where Q_0 is an arbitrary non-zero real number. If $\gamma_1 < 1$ then

$$\lim_{k \rightarrow \infty} Q_k = Q_0 / (1 - \gamma_1).$$

If $\gamma_1 = 1, \gamma_2 < \infty$ then

$$\lim_{k \rightarrow \infty} Q_k / k = 2Q_0 / \gamma_2.$$

If $\gamma_1 > 1$ then

$$\lim_{k \rightarrow \infty} (Q_k - Q_0 / \{\delta^k [1 - \pi'(\delta)]\}) = Q_0 / (1 - \gamma_1),$$

where δ is the least (absolute) root of the equation $z = \pi(z)$.

Taking into account that $\mathbf{E}L_0 = \rho_1$ and denoting $\beta(s) = \int_0^{\infty} e^{-sx} dB(x), s \geq 0$, we obtain the following theorem.

Theorem 1. If $\rho_1 < 1$ then $\lim_{n \rightarrow \infty} \mathbf{E}L_n = 0$. If $\rho_1 = 1$ then $\mathbf{E}L_n = 1$ for all $n \geq 0$. If $\rho_1 > 1$ then

$$(8) \quad \lim_{n \rightarrow \infty} \left(\mathbf{E}L_n - \frac{(\rho_1 - 1)\varphi^{-n}}{1 + \lambda\beta'(\lambda - \lambda\varphi)} \right) = 0,$$

where φ is the least (absolute) root of the functional equation

$$(9) \quad z = \beta(\lambda - \lambda z).$$

Let us consider the case when $\rho_1 = 1 + \varepsilon$, where $\varepsilon > 0$ is a sufficiently small value. It was shown on p. 326 of Subhankulov (1976) that, as $\varepsilon \rightarrow 0$, $\varphi = 1 - 2\rho_2^{-1}\varepsilon + O(\varepsilon^2)$.

Applying this estimation, we obtain the following result.

Theorem 2. Let $\rho_1 = 1 + \varepsilon, \varepsilon > 0, \rho_2 < \infty$ and $\varepsilon n \rightarrow C \geq 0$ as $\varepsilon \rightarrow 0$ and $n \rightarrow \infty$. Then

$$\mathbf{E}L_n = e^{2C/\rho_2} [1 + O(\varepsilon)].$$

2. Extension of the property of the refusals stream

For a more detailed study of the properties of the refusals stream let us consider the following generalization of an $M/GI/1/n$ queueing system.

Consider an $M/GI/1/\infty$ queueing system, the input stream of which is Poissonian with parameter λ and the probability distribution function of the service time $B(x)$ satisfies the condition $\lambda \int_0^\infty x dB(x) = 1$. Suppose that each arriving customer is given a specific length which will be called the *customer length*. This characteristic need not coincide with a service time. Let us assume that the customer lengths are positive independent identically distributed random variables (this assumption is technical). Along with the assumption that the expectations of interarrival and service times are equal, let us also assume that if, at the moment of arrival, the cumulative customer length exceeds the value K (which can be assumed to be a positive random variable with some given probability distribution) then the arriving customer abandons the system; otherwise he joins the queue. The residual customer length in service is assumed to be some positive random variable depending on the residual service time only.

Theorem 3. *The expectation of the number of refusals during a busy period is equal to 1.*

Proof. Let us consider first the case when the residual customer length in service is always equal to zero. Then, denoting the customer lengths in line by l_1, l_2, \dots let us define the following random variable η :

$$\eta = \max \left\{ r \geq 0 : \sum_{m=1}^r l_m \leq K \right\}$$

(the empty sum is assumed to be zero). The random variable η is the queue length before refusal. Therefore, repeating the proof of Theorem 1 with this random number of waiting places we obtain $EL_\eta = 1$ by using the formula for the total probability.

Now let us consider the case when the residual length of every customer in a service is a random variable. Denote the customer lengths in service before refusals by a_1, a_2, \dots, a_τ , where τ is the number of refusals during a busy period (if $\tau = 0$, the empty sequence is assumed to be null). We must prove that $E\tau = 1$. In the given case we have the following family of a random number of random variables:

$$\eta_i = \max \left\{ r \geq 0 : \sum_{m=1}^r l_m \leq \max(0, K_i - a_i) \right\}, \quad i = 1, \dots, \tau,$$

where K_1, K_2, \dots are independent random variables having the same distribution as the random variable K , and η_1, η_2, \dots is the corresponding sequence of the random queue lengths before refusals. Let $\xi_k = \sum_{j=1}^{\tau} \mathbf{1}_{\{\eta_j = k\}}$. Hence there is a random variable η denoting the random number of waiting places and $L_\eta = \xi_0 + \xi_1 + \dots$ is the number of refusals during a busy period. Repeating the proof of Theorem 1 in this case we also obtain $EL_\eta = 1$ by the formula for the total probability. On the other hand, according to Wald's identity

$$E \sum_{k=0}^{\infty} \zeta_k = E \sum_{j=1}^{\tau} \sum_{k=0}^{\infty} \mathbf{1}_{\{\eta_j=k\}} = E\tau$$

and therefore $E\tau = 1$.

Let us consider three simple examples for applications of Theorem 3.

Example 1. Consider a single-server queueing system $M/GI/1/\infty$. Assume that the expectations of interarrival and service times are equal. If an arriving customer meets n customers in the system he abandons the queue with probability $p_n > 0$ and joins the queue with probability $q_n = 1 - p_n$.

Example 2. Consider a single-server queueing system $M/GI/1/\infty$ under the same assumption: the expectations of interarrival and service times are equal. Suppose that the waiting (or sojourn) time of each customer is restricted by some positive random variable K .

Proposition 1. For the queueing systems described in Examples 1 and 2 the expectation of the number of refusals during a busy period is equal to 1.

Note. This statement for the queueing system described in Example 1 is a generalization of the respective statement in Abramov (1991b). The queueing process considered there was a special case of this one.

Example 3. Consider a single-server queueing system in which the customers arrive by groups according to Poisson input with rate λ and are served one-by-one. Let l be the expectation of the group size, and b be the expectation of the service time. Suppose that a busy period begins when the first group of customers arrives at the server. The number of waiting places is restricted by the value $n \geq 1$ and if, after the arrival of a group of customers, the total number of customers exceeds the value n , the arrived group abandons the system and otherwise joins it.

Proposition 2. Let us assume that $\lambda bl = 1$. Then the expectation of the number of lost groups of customers during a busy period is equal to 1.

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