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## ON THE ASYMPTOTIC DISTRIBUTION OF THE MAXIMUM NUMBER OF INFECTIVES IN EPIDEMIC MODELS WITH IMMIGRATION

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### Abstract

This paper considers the asymptotic distribution of the maximum number of infectives in an epidemic model by showing that, as the initial number of susceptibles converges to infinity, the process of infectives converges almost surely to a birth and death process. The model studied here is more general than usual (see e.g. Bailey (1975), Bharucha-Reid (1960), Keilson (1979)) in that it incorporates immigration and the limiting birth and death process is non-linear. The main novelty of the present paper is the martingale approach used to prove the above-mentioned convergence.

MAXIMUM NUMBER OF INFECTIVES; GAMBLER'S RUIN PROBLEM; MARTINGALE; BIRTH AND DEATH PROCESS

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 92D30

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### 1. Introduction

Consider a population of individuals which consists of  $n$  susceptibles and  $m$  infectives. Suppose that at time  $t > 0$  there are  $N(t)$  susceptibles and  $M(t)$  infectives ( $N(0) = n$ ,  $M(0) = m$ ), and in a small time interval  $\Delta t$  there is the probability  $\Lambda[N(t), M(t)]\Delta t + o(\Delta t)$  that a susceptible becomes an infective, and the probability  $\mu M(t)\Delta t + o(\Delta t)$  that an infective leaves the population through death, recovery or isolation. The function  $\Lambda(x, y)$  is non-decreasing and positive in its variables  $x$  and  $y$  and  $\Lambda^*(y) = \lim_{x \rightarrow \infty} \Lambda(x, y)$  exists for all  $y$ . The population is also assumed to replenish by immigration, i.e. in a small time interval  $\Delta t$  there is the probability  $\nu\Delta t + o(\Delta t)$  that a new individual arrives into the population. Moreover, the probability that an arriving individual is infective is equal to  $p$ . Denote the epidemic model thus described by  $\Pi_1$ .

In our further consideration we shall also deal with the birth and death process of a more simple epidemic model without immigration in which the sum  $N(t) + M(t)$  is assumed to remain at the constant value  $n + m$  for all time moments  $t$  (there is no departure from the population). This is a model for the spread of a non-dangerous infection (see for example Bharucha-Reid (1960)). The absence of immigration in this

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model will be compensated for by the value  $pv$  which will be added to the function  $\Lambda(x, y)$ . This model will be denoted by  $\Pi_2$ .

Let  $\kappa_m = \inf\{t: M(t) = 0\}$ . The aim of this paper is to obtain the asymptotic distributions of  $\sup_{0 < t \leq \kappa_m} M(t)$  for the model  $\Pi_1$  under various assumptions on the behavior of the variables  $n, v$  and  $p$ . For this purpose the method of Abramov (1991), which was applied there to Markovian queueing systems, is used. Note that a closely related problem has been considered for the usual epidemic model in Keilson (1979), p. 84, with the aid of the Green's function.

It is shown in the paper that  $M(t)$  converges almost surely to a birth and death process. The birth and death process approximations of epidemics have a long history, going back to Bartlett (1955) and Kendall (1956). Convergence of a sequence of epidemic processes to a limiting branching process is considered by, for example, Metz (1978) and Ball (1983).

**2. Some properties of the epidemic model**

For the sake of simplicity let us suppose that  $m = 1$ , i.e. at the initial time moment there is only one infective amongst the population. The extension to the case of arbitrary  $m$  is not difficult. Let  $f(1) = 1$ , and  $f(r), r \geq 2$ , be the number of situations during the time interval  $(0, \kappa_1)$  when either at the moment of a susceptible becoming infected or at the moment of an immigration of an infective the number of infectives in the population becomes  $r$ . Let  $t_{r,1}, t_{r,2}, \dots, t_{r,f(r)}$  be these ascending time moments, and  $s_{r,1}, s_{r,2}, \dots, s_{r,f(r)}$  be the ascending time moments when at the moment of an infective leaving the population there are  $r - 1$  infectives among the population. For  $r \geq 2$ , let  $\mathcal{F}_r = \sigma(f(1), f(2), \dots, f(r))$ . In the case when the initial number of susceptibles increases infinitely, as well as in other extremal situations, we shall use the notation  $\{f^*(r)\}$  for the respective limiting processes,  $\mathcal{F}_r^* = \sigma(f^*(1), f^*(2), \dots, f^*(r))$ .

Let us consider the time intervals

$$(1) \quad (t_{r,1}, s_{r,1}), (t_{r,2}, s_{r,2}), \dots, (t_{r,f(r)}, s_{r,f(r)}).$$

If within each of the intervals in (1) there occur  $r + 1$  infectives, then *either* a susceptible must become an infective *or* at the moment of immigration an infective must join the population. There are  $f(r + 1)$  of these situations. It is clear that the semi-intervals

$$(2) \quad [t_{r+1,1}, s_{r+1,1}), [t_{r+1,2}, s_{r+1,2}), \dots, [t_{r+1,f(r+1)}, s_{r+1,f(r+1)})$$

are contained in the intervals in (1). Let us delete the semi-intervals (2) from the intervals (1) and connect the ends. Then the number of infectives within residual intervals is equal to  $r$ .

Denote the  $\mathcal{F}_r$ -measurable random variables

$$(3) \quad N_r^- = \inf_{\mathcal{I}_r} N(t),$$

$$(4) \quad N_r^+ = \sup_{\mathcal{I}_r} N(t),$$

where  $\mathfrak{F}_r = \{t_{r,1}, t_{r,2}, \dots, t_{r,f(r)}, s_{r,1}, s_{r,2}, \dots, s_{r,f(r)}\}$ .

Take into consideration the following inequality:

$$(5) \quad E \left\{ \frac{r\mu f(r+1)}{\Lambda(N_r^+, r) + pv} \mid f(r) \right\} \leq f(r) \leq E \left\{ \frac{r\mu f(r+1)}{\Lambda(N_r^-, r) + pv} \mid f(r) \right\}.$$

In order to explain this inequality, consider first the epidemic model  $\Pi_2$  for which all of the suggested notation remains in force. Let  $\Lambda(r) = \Lambda(n - r + 1, r) + pv$ . Then

$$\begin{aligned} E\{f(r+1) \mid f(r)\} &= f(r) \sum_{k=1}^{\infty} k \int_0^{\infty} \exp(-\Lambda(r)t) \frac{(\Lambda(r)t)^k}{k!} r\mu \exp(-r\mu t) dt \\ &= f(r)\Lambda(r)/(r\mu). \end{aligned}$$

Therefore,

$$E \left\{ \frac{r\mu f(r+1)}{\Lambda(r)} \mid f(r) \right\} = f(r).$$

This relation remains in force if the initial number of susceptibles in model  $\Pi_2$  is a random variable. Assume that the value  $n$  for this model is replaced by  $N_r^- + r - 1$ . Then considering both models  $\Pi_1$  and  $\Pi_2$  over the same probability space reveals that the number of susceptibles in model  $\Pi_2$  is always not greater than the number of susceptibles in model  $\Pi_1$ . Since in addition the function  $\Lambda(x, y)$  is non-decreasing we obtain the right side of inequality (5). The left side of inequality (5) is explained analogously by replacing the value  $n$  by  $N_r^+ + r - 1$  in model  $\Pi_2$ .

Let us return to model  $\Pi_1$ . By (5) the stochastic process

$$(6) \quad \left[ \frac{r! \mu^r f(r+1)}{\prod_{k=1}^r [\Lambda(N_k^+, k) + pv]}, \mathfrak{F}_{r+1} \right]$$

is a supermartingale and the stochastic process

$$(7) \quad \left[ \frac{r! \mu^r f(r+1)}{\prod_{k=1}^r [\Lambda(N_k^-, k) + pv]}, \mathfrak{F}_{r+1} \right]$$

is a submartingale. Using the martingale properties from (6), (7) we have

$$\begin{aligned} (8) \quad \frac{\prod_{k=1}^r [\Lambda(N_k^-, k) + pv]}{r! \mu^r} &\leq E\{f(r+1) \mid f(1)\} = E f(r+1) \\ &\leq \frac{\prod_{k=1}^r [\Lambda(N_k^+, k) + pv]}{r! \mu^r} \leq \frac{\prod_{k=1}^r [\Lambda^*(k) + pv]}{r! \mu^r}, \quad f(1) = 1. \end{aligned}$$

The inequalities (8) permit us to prove the following statements.

*Lemma 1.* Let us assume

$$(9) \quad \sum_{r=1}^{\infty} \frac{\prod_{k=1}^r [\Lambda^*(k) + pv]}{r! \mu^r} < \infty.$$

If  $n \rightarrow \infty$  then

$$P \left\{ \lim_{n \rightarrow \infty} \Lambda(N_r^-, r) = \lim_{n \rightarrow \infty} \Lambda(N_r^+, r) = \Lambda^*(r) \right\} = 1,$$

and the limiting stochastic process

$$\left[ \frac{r! \mu^r f^*(r+1)}{\prod_{k=1}^r [\Lambda^*(k) + pv]}, \mathcal{F}_{r+1}^* \right]$$

is a martingale.

*Lemma 2.* Let us assume that  $pv \rightarrow C \geq 0$  as soon as  $v \rightarrow \infty, p \rightarrow 0$  and

$$(10) \quad \sum_{r=1}^{\infty} \frac{\prod_{k=1}^r [\Lambda^*(k) + C]}{r! \mu^r} < \infty.$$

Then

$$P \left\{ \lim_{\substack{v \rightarrow \infty \\ p \rightarrow 0}} \Lambda(N_r^-, r) = \lim_{\substack{v \rightarrow \infty \\ p \rightarrow 0}} \Lambda(N_r^+, r) = \Lambda^*(r) \right\} = 1,$$

and the limiting stochastic process

$$(11) \quad \left[ \frac{r! \mu^r f^*(r+1)}{\prod_{k=1}^r [\Lambda^*(k) + C]}, \mathcal{F}_{r+1}^* \right]$$

is a martingale.

*Lemma 3.* Let us assume that both  $pv \rightarrow C \geq 0$  as soon as  $v \rightarrow \infty, p \rightarrow 0$  and  $n \rightarrow \infty$ . If the convergence (10) holds then

$$P \left\{ \lim_{\substack{v \rightarrow \infty \\ p \rightarrow 0 \\ n \rightarrow \infty}} \Lambda(N_r^-, r) = \lim_{\substack{v \rightarrow \infty \\ p \rightarrow 0 \\ n \rightarrow \infty}} \Lambda(N_r^+, r) = \Lambda^*(r) \right\} = 1,$$

and the limiting stochastic process (11) is a martingale.

*Note.* The conditions (9) and (10) imply that the limiting epidemic processes are subcritical. Under these conditions the limiting distributions of  $\sup_{0 < t \leq \kappa_n} M(t)$  do exist.

If these conditions are not fulfilled the aforementioned limiting distribution can exist and need not be defective. The finding of the limiting distribution in the supercritical case is expected to be a very difficult problem.

*Proof of Lemma 1.* Indeed, for every  $r \geq 1$  the following inequality holds:

$$(12) \quad n - N_r^- \leq \sum_{k=1}^{\infty} f(k).$$

Because of (8) and (9) we have:

$$(13) \quad \sum_{k=1}^{\infty} \mathbf{E} f(k) \leq \sum_{r=1}^{\infty} \frac{\prod_{k=1}^r [\Lambda^*(k) + pv]}{r! \mu^r} < \infty.$$

Therefore

$$\mathbf{P} \left\{ \lim_{n \rightarrow \infty} (n - N_r^-) < \infty \right\} = 1,$$

or

$$(14) \quad \mathbf{P} \left\{ \lim_{n \rightarrow \infty} \frac{N_r^-}{n} = 1 \right\} = 1.$$

Next, taking into consideration the relation (14) we obtain:

$$\mathbf{P} \left\{ \lim_{n \rightarrow \infty} \Lambda(N_r^-, r) = \lim_{n \rightarrow \infty} \Lambda(N_r^+, r) = \Lambda^*(r) \right\} = 1.$$

Now, taking the limit in (6) and (7) we can see that the limiting stochastic process

$$\left[ \frac{r! \mu^r f^*(r+1)}{\prod_{k=1}^r [\Lambda^*(k) + pv]}, \mathcal{F}_{r+1}^* \right]$$

is both a super- and submartingale simultaneously and therefore it is a martingale. This ends the proof of Lemma 1.

*Proof of Lemma 2.* For small  $\varepsilon > 0$  denote

$$(15) \quad N_r^-(\varepsilon) = \inf_{t_{r,1} + \varepsilon \leq t \leq s_{r,f(r)}} N(t),$$

provided that  $t_{r,1} + \varepsilon \leq s_{r,f(r)}$ . If  $t_{r,1} + \varepsilon > s_{r,f(r)}$  let us set  $N_r^-(\varepsilon) = N(\varepsilon)$ . Then

$$(16) \quad N(\varepsilon) - N_r^-(\varepsilon) \leq \sum_{k=1}^{\infty} f(k).$$

Let us assume that  $\varepsilon v \rightarrow \infty$  as soon as  $\varepsilon \rightarrow 0, v \rightarrow \infty$ . Then  $N(\varepsilon)$  infinitely increases with probability 1 and in view of (16) and (13)

$$N_r^-(\varepsilon) \rightarrow \infty \quad \text{as } \varepsilon \rightarrow 0$$

with probability 1. Therefore, both  $\Lambda(N_r^-(\varepsilon), r) \rightarrow \Lambda^*(r)$  and  $\Lambda(N_r^+, r) \rightarrow \Lambda^*(r)$  with probability 1. Taking the limit in (6) and (7) we can see that the limiting process

$$\left[ \frac{r! \mu^r f^*(r+1)}{\prod_{k=1}^r [\Lambda^*(k) + C]}, \mathcal{F}_{r+1}^* \right]$$

is both a super- and submartingale simultaneously and therefore it is a martingale. The proof of this lemma is completed.

*Proof of Lemma 3.* The method of proof is analogous to that of Lemma 1.

### 3. Limit theorems

Let us find the asymptotic distributions of the maximum number of infectives during the epidemic period.

*Theorem 1.* Suppose that condition (9) is fulfilled. Then

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ \sup_{0 < t \leq \kappa_m} M(t) \leq r \right\} = \frac{\alpha_{r,m}}{\alpha_{r,0}} \quad (m = 1, 2, \dots; r = m, m + 1, \dots),$$

where the values  $\alpha_{r,i}$  are calculated by the formula:

$$\alpha_{r,i} = \sum_{k=i}^r \frac{\prod_{l=1}^r [\Lambda^*(l) + pv]}{k! \mu^k} \quad (i = 0, 1, \dots; r = i, i + 1, \dots).$$

(The empty product is assumed to be 1.)

*Theorem 2.* Let us assume that  $pv \rightarrow C \geq 0$  as soon as  $v \rightarrow \infty, p \rightarrow 0$  and condition (10) is fulfilled. Then

$$\lim_{\substack{v \rightarrow \infty \\ p \rightarrow 0}} \mathbf{P} \left\{ \sup_{0 < t \leq \kappa_m} M(t) \leq r \right\} = \frac{\alpha_{r,m}}{\alpha_{r,0}} \quad (m = 1, 2, \dots; r = m, m + 1, \dots),$$

where the values  $\alpha_{r,i}$  are calculated by the formula:

$$(17) \quad \alpha_{r,i} = \sum_{k=i}^r \frac{\prod_{l=1}^k [\Lambda^*(l) + C]}{k! \mu^k} \quad (i = 0, 1, \dots; r = i, i + 1, \dots).$$

*Theorem 3.* Let us assume that both  $pv \rightarrow C \geq 0$  as soon as  $v \rightarrow \infty, p \rightarrow 0$  and  $n \rightarrow \infty$ . If the condition (10) is fulfilled then

$$\lim_{\substack{v \rightarrow \infty \\ p \rightarrow 0 \\ n \rightarrow \infty}} \mathbf{P} \left\{ \sup_{0 < t \leq \kappa_m} M(t) \leq r \right\} = \frac{\alpha_{r,m}}{\alpha_{r,0}} \quad (m = 1, 2, \dots; r = m, m + 1, \dots),$$

where the values  $\alpha_{r,i}$  are calculated from the formula (17).

*Proof of Theorem 1.* Using Lemma 1 under the assumption  $n \rightarrow \infty$  we obtain that the limiting value of the function  $\Lambda[N(t), M(t)]$  is equal to  $\Lambda^*[M(t)]$  for every  $t \in (0, \kappa_1]$  with probability 1, and in addition the limiting stochastic process

$$\left[ \frac{r! \mu^r f^*(r+1)}{\prod_{k=1}^r [\Lambda^*(k) + pv]}, \mathcal{F}_{r+1}^* \right]$$

is a martingale. Therefore we deal with the birth and death process which is generated by an epidemic model  $\Pi_2$  with an infinite initial number of susceptibles. Therefore, the problem is reduced to consideration of the epidemic model without immigration, and under the assumption that in a small time interval  $\Delta t$  there is the limiting probability  $\{\Lambda^*[M(t)] + pv\} \Delta t + o(\Delta t)$  that a susceptible becomes an infective.

In order to find the asymptotic distribution of the maximum number of infectives within the  $(0, \kappa_1]$ , let us consider one version of the gambler's ruin problem.

Two gamblers  $A$  and  $B$  gamble until one of them is completely ruined. The gamblers  $A$  and  $B$  have  $l$  and  $k$  dollars respectively. One or other gambler wins 1 dollar in a game. If the gambler  $A$  has  $j$  dollars before the current game ( $1 \leq j \leq k + l - 1$ ), he wins the game with probability  $u_j$  and loses with probability  $v_j = 1 - u_j$ . The ruin probability of gambler  $A$  is the following (see for example Shirayev (1984), p. 124, Abramov (1991), p. 37):

$$p_A = \frac{\sum_{i=l}^{l+k-1} \prod_{j=1}^i \rho_j}{\sum_{i=0}^{l+k-1} \prod_{j=1}^i \rho_j}, \quad \rho_j = \frac{v_j}{u_j}.$$

For the given problem we should equate the gain of 1 dollar by gambler  $A$  with a susceptible becoming an infective, and the loss of 1 dollar by gambler  $A$  with an infective leaving the population. Then there is probability  $[\Lambda^*(l) + pv]/[\Lambda^*(l) + pv + l\mu]$  that gambler  $A$  wins in a game provided that there are  $l$  infectives among the population. Therefore,

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ \sup_{0 < t \leq \kappa_1} M(t) \leq r \right\} = \frac{\alpha_{r,1}}{\alpha_{r,0}}.$$

The generalization to an arbitrary value  $m$  is evident. The theorem is proved.

*Proofs of Theorems 2 and 3.* The proofs repeat the proof of Theorem 1, replacing  $pv$  by its limiting value  $C$ .

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## References

- ABRAMOV, V. M. (1991) *Investigation of the Queueing System with Service Depending on Queue Length*. Donish, Dushanbe (in Russian).
- BAILEY, N. T. J. (1975) *The Mathematical Theory of Infectious Diseases and its Applications*, 2nd edn. Griffin, London.
- BALL, F. G. (1983) The threshold behavior of epidemic models. *J. Appl. Prob.* **20**, 227–241.
- BARTLETT, M. S. (1955) *An Introduction to Stochastic Processes*. Cambridge University Press.
- BHARUCHA-REID, A. T. (1960) *Elements of the Theory of Markov Processes and their Applications*. McGraw-Hill, New York.
- KEILSON, J. (1979) *Markov Chain Models — Rarity and Exponentiality*. Springer-Verlag, New York.
- KENDALL, D. G. (1956) Deterministic and stochastic epidemics in closed populations. *Proc 3rd Berkeley Symp. Math. Statist. Prob.* **4**, 149–165.
- METZ, J. A. J. (1978) The epidemic in a closed population with all susceptibles equally vulnerable: some results for large susceptible populations and small initial infections. *Acta Biotheoretica* **27**, 75–123.
- SHIRYAYEV, A. N. (1984) *Probability*. Springer-Verlag, New York.