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# INEQUALITIES FOR THE $GI/M/1/n$ LOSS SYSTEM

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**Abstract**

The present paper provides simple inequalities for the number of lost customers during a busy period of a $GI/M/1/n$ queueing system.

*Keywords:* Loss systems; busy period; branching process; number of up- and down-crossings; partial ordering

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## 1. Introduction

Abramov (1991), (1997) and Righter (1999) studied the number of customers lost during a busy period of an $M/GI/1/n$ queueing system. They showed that if the expectations of interarrival and service times in an $M/GI/1/n$ queueing system are equal, then the expected number of lost customers during a busy period is equal to 1 for all $n \geq 0$.

The present paper provides simple inequalities for the number of lost customers during a busy period of the dual $GI/M/1/n$ queueing system. Let $A(t)$ denote the probability distribution function of the interarrival time, let $a(s) = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}A(t),\ s \geq 0$, and let $\mu$ be the reciprocal of the expected service time. The main result of this paper is the following theorem.

**Theorem 1.** *Let $L_n$ denote the number of lost customers during a busy period of a $GI/M/1/n$ queueing system. If $A(t)$ belongs to the NBU class, then*

$$L_n \geq_{\mathrm{st}} Z_{n+1}, \tag{1}$$

*where $\geq_{\mathrm{st}}$ denotes stochastic order and $Z_n$ denotes the number of offspring in the nth generation of the Galton–Watson branching process with $Z_0 = 1$ and the offspring generating function*

$$g(z) = \frac{1 - a(\mu)}{1 - za(\mu)}.$$

*If $A(t)$ belongs to the NWU class, then*

$$L_n \leq_{\mathrm{st}} Z_{n+1}.$$

For the definitions of the NBU and NWU classes, stochastic ordering and related concepts see Stoyan (1983). In the following we will only treat the case where $A(t)$ belongs to the NBU class.

It follows from (1) that if $A(t)$ belongs to the NBU class, then

$$\mathrm{E}L_n \geq \left(\frac{a(\mu)}{1 - a(\mu)}\right)^{n+1}. \tag{2}$$

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The proof of the main result uses the up- and down-crossings approach which was considered by Cohen (1977) and in a number of works by the author (see e.g. Abramov (1991), (1994)). Here we follow Abramov (1991) (see also Abramov (1994)).

## 2. Up- and down-crossings approach for a $GI/M/1/n$ queueing system

Consider a busy period of a $GI/M/1/n$ queueing system and denote by $f_n(j),\ 0 \le j \le n+1$, the number of customers arriving during a busy period who, upon their arrival, meet $j$ customers in the system. It is clear that $f_n(0) = 1$ with probability 1. Let $t^n_{j,1}, t^n_{j,2}, \dots, t^n_{j,f_n(j)}$ be the instants of arrival of these $f_n(j)$ customers, and let $s^n_{j,1}, s^n_{j,2}, \dots, s^n_{j,f_n(j)}$ in turn be the instants of service completions (departures) at which there remain only $j$ customers in the system. Note that $t^n_{n+1,k} = s^n_{n+1,k},\ 1 \le k \le f_n(n+1) = L_n$.

For $0 \le j \le n$ let us consider the intervals

$$(t^n_{j,1}, s^n_{j,1}], (t^n_{j,2}, s^n_{j,2}], \dots, (t^n_{j,f_n(j)}, s^n_{j,f_n(j)}]. \tag{3}$$

Obviously, every interval in (3) is distributed as a busy period of the $GI/M/1/n-j$ queueing system with the same probability distribution function of interarrival and service times as was defined for the initial $GI/M/1/n$ queueing system.

It is clear that the intervals

$$(t^n_{j+1,1}, s^n_{j+1,1}], (t^n_{j+1,2}, s^n_{j+1,2}], \dots, (t^n_{j+1,f_n(j+1)}, s^n_{j+1,f_n(j+1)}] \tag{4}$$

are contained in the intervals (3). Let us delete the intervals in (4) from those in (3) and connect the ends, that is, connect every point $t^n_{j+1,k}$ with the corresponding point $s^n_{j+1,k}$, $1 \le k \le f_n(j+1)$, if the set of intervals (4) is not empty.

Then, under the assumption that $A(t)$ belongs to the NBU class, a residual interarrival time is stochastically not greater than an interarrival time. Let $s^n_{j+1,k}$ and $t^n_{j+1,k+1}$ be two adjacent points within one of the intervals in (3), say in $(t^n_{j,1}, s^n_{j,1}]$. Then $t^n_{j+1,k+1} - s^n_{j+1,k}$ is stochastically not greater than an interarrival time, and therefore, by coupling, $f_n(j+1)$ is stochastically not smaller than $L_j$, the number of losses during a busy period of a $GI/M/1/j$ queueing system with the same probability distribution function of interarrival and service times as was defined for the initial $GI/M/1/n$ queueing system. As a special case, let us consider one of the intervals of (3), $(t^n_{j,i}, s^n_{j,i}]$ say, and let $I_{j,i}$ denote the number of inserted points within this interval after deleting the intervals of (4) and connecting the ends. Then, under the assumption that $A(t)$ belongs to the NBU class, we have

$$I_{j,i} \ge_{\text{st}} L_0 \qquad (0 \le j \le n;\ 1 \le i \le f_n(j)), \tag{5}$$

where $L_0$ denotes the number of lost customers during a busy period of a $GI/M/1/0$ queueing system, which coincides in distribution with the number of arrivals per service time.

Note that the statement of Theorem 1 now follows immediately from (5) because the generating function for the number of arrivals per service time is

$$g(z) = \sum_{k=0}^{\infty} z^k \int_0^{\infty} [1 - A(t)] * A^{*k}(t)\mu e^{-\mu t}\, dt = \frac{1 - a(\mu)}{1 - za(\mu)},$$

where the asterisk denotes convolution and $A^{*k}(t)$ is the $k$-fold convolution of $A$ with itself.

### 3. Busy periods and the number of served customers during a busy period

Next, let $T_n$, $\nu_n$ denote the length of a busy period and the number of served customers during that busy period, respectively. We have:

$$\nu_n = \sum_{i=0}^{n} f_n(i), \tag{6}$$

$$T_n = \sum_{i=1}^{\nu_n} \chi_i, \tag{7}$$

where $\chi_1, \chi_2, \dots$ is the sequence of i.i.d. random variables all having an exponential distribution with parameter $\mu$.

From (5), (6) and (7) we have the following theorem.

**Theorem 2.** *Under the assumption that $A(t)$ belongs to the NBU class,*

$$\nu_n \geq_{\mathrm{st}} Z(n) = \sum_{i=0}^{n} Z_i \tag{8}$$

*and*

$$T_n \geq_{\mathrm{st}} \sum_{i=1}^{Z(n)} \chi_i, \tag{9}$$

*where the branching process $\{Z_i\}$ is as defined in Theorem 1.*

Indeed, (8) follows immediately from (6), and (9) follows from (7) and the following result by Stoyan (1983).

**Lemma 1.** *Let $\{\xi_{n,1}\}$ and $\{\xi_{n,2}\}$ be two sequences of non-negative mutually independent random variables such that*

$$\xi_{n,k} \leq_{\mathrm{st}} \xi_{n+1,k} \qquad (k = 1, 2;\ n = 1, 2, \dots).$$

*If $\kappa_1 \leq_{\mathrm{st}} \kappa_2$, and $\xi_{n,1} \leq_{\mathrm{st}} \xi_{n,2}$, $n = 1, 2, \dots$, it follows that*

$$\sum_{n=1}^{\kappa_1} \xi_{n,1} \leq_{\mathrm{st}} \sum_{n=1}^{\kappa_2} \xi_{n,2}.$$


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