Dynamical Systems With Active Singularities of Elastic Type: A Modeling and Controller Synthesis Framework

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Abstract—A new class of systems characterized by admitting impulsive control action within their singular motion phases, such as dimension changes, state discontinuities, and other irregularities, is introduced. This class, termed dynamical systems with active, or controlled, singularities, encompasses various applications with sensing and/or actuation ultra-fast in comparison with the natural system time scale, including mechanisms with impact-induced motion, power and sensor networks under faults, fast positioning devices, smart skins, and switching electronic circuits. The present work focuses on systems with impact-type singularities induced through system interaction with controlled, or active, state constraints. The latter are assumed to be characterized by parameter-dependent elastic-type constraint violation that vanishes in the limit as the parameter value tends to infinity. A physically well justified representation of this class of systems is proposed, but found to be not well suited for controller synthesis in the singular phase. To address this problem, two equations are derived - one describing controlled rigid impacts in auxiliary stretched time and comparable with the regular control design techniques, termed controlled infinitesimal dynamics equation, and another incorporating controlled impact dynamics in the infinitesimal form in terms of the shift operator along the trajectories of the first equation, termed the limit model. The latter is demonstrated to generate a unique isolated discontinuous system motion, i.e., to provide a tight and well-behaved description of the collision with rigid constraint. It is then shown that the corresponding paths generated by the original physically-based and the limit representations can be made arbitrarily close to each other uniformly except, possibly, in the vicinities of the jump points, by the appropriate choice of the value of the constraint violation parameter. This is shown to permit enforcing, for sufficiently large values of the parameter, the desired limit system behavior onto the original system by simply taking the control signals found through the limit representation, time-scaling them, and inserting the resulting signals directly into the original system. These features show that the procedure developed, in fact, provides a well-posed controller synthesis framework for the class of systems considered. Using this framework, a ball/racket rotationally controlled impact representation is developed, and on its basis, a singular phase control signal in the form of racket rotation velocity is designed, demonstrating that the soft racket provides the specified bounce-off angle increment under a noticeably lower racket rotation velocity and a longer rotation phase than the hard racket.

Index Terms—Control of mechanical systems, controlled discrete transitions, generalized solutions, hybrid systems, impacts, impulsive control, nonsmooth analysis and synthesis, penalty method, singularities, systems with unilateral constraints.

I. INTRODUCTION

SYSTEMS that exhibit singularities in their behavior, such as discontinuities in system motion, jumps in dimension, and lack of continuous dependence on the initial conditions are gaining in technological importance, and interest in their modeling and control, as well as development of new ultra-fast sensing and actuation capabilities during the singular phase is increasing (cf. [18], [17], [27], [33], [75], [59], [3], [4], [78], [61], and [77]). For example, in mechanical systems, collisions between the interacting bodies cause the abrupt change in their velocities and thereby create the discrete-continuous behavior. Specific examples include robotic manipulators [54], [75], vibro-impact mechanisms [2], walking (and, potentially, jumping) biped robots [33], and many others (cf. [19] and [17], with its extensive bibliography). Moreover, in the cases of juggling mechanical systems [16], [76], [73] and micro- and electromechanical systems (MEMS) with thin-film impact microactuators ([8]–[10], [35], and [40]) motion is induced only through organized impacts, hence it could be controlled only through these impacts. Such motion is especially important in MEMS where it is characterized by significantly reduced friction and adhesion, known to be severe problems in this area [68], [69], in armored vehicles [29] where reliable projectile deflection could ensure survivability, in faulted power networks [23], mobile sensor networks [36], and microgrids [41], [42], where stopping of cascading fault propagation and/or “self-healing” is accomplished by the impact-type reconfiguration of the network topology [1], and in positioning applications, such as cutting, where the abrupt direction change of the cutting torch motion is required to eliminate a substrate burn-through at the reverse endpoints. Therefore, singularities play a principal role in modeling and control of these systems and cannot be neglected.

Traditionally, the control effort in such systems has been spent to induce the a priori rigidly specified singular phase that does not admit any corrective actions within it. For example, at...
present, in the case of singularities induced by collision with constraints [17], [75], [33], the latter are viewed as “passive impacters” which, during the phase of the engagement of the system with the constraint, enter passively into the determination of the post-impact system motion and do not substantially alter the motions induced by the interaction from impact to impact. In this sense, the fault clearing switching in power systems [39], [1] is, at present, passive as well, since it does not admit any corrective actions within the switching phases.

Examining the impact games, such as ping-pong and air hockey, it becomes obvious, however, that the performance of the systems considered above could be improved. Indeed, in air hockey [73], the puck/mallet interaction has little compliance, making the interaction time interval for a player with a human-type sensing/manipulation capability too short to execute any corrective action during the impact. In ping-pong, however, players are trained in agile execution of a rotational wrist motion, for example, in the authors’ experience, by engaging a specially designed rubber wheel with a bare hand and “feeling out” the engagement phase. This skill is then brought into the game through the choice of a racket with a soft elastic cover that provides the ball/racket engagement phase compatible in duration with the capability of an advanced player for changing within this phase the ball velocity both in magnitude and direction. The player in this setting can be viewed as generating a constraint and actively controlling its properties during the engagement phase. This player action gives rise to a new concept of active, or controlled, constraints, either naturally present or created through actuation, capable of radically changing the attainability set of the post-impact system state. The engagement phase of the system with such constraint can then be termed active singularity.

These concepts, initially proposed by the authors in [4], [78], and [49], and developed in [50], [5], [6], and [7] then naturally lead to the introduction of a new class of systems, dynamic systems with active, or controlled, singularities: a class of systems that admit control actions during the singular phases of their motion. As indicated in [5], the latter systems can be also viewed as a new class of hybrid systems characterized by impulsively-controlled discrete transitions.

In power systems, for example, such actions could be plausibly implemented during the fault clearing switching using the ultra-high-speed signal processors and fast power electronic systems [34], [70]. In smart armor applications [29] sensor-microprocessor-defeat mechanism system is expected to detect impact location, velocity, and diameter of an incoming projectile and divert projectile away from the vehicle. Similarly, complex nonsmooth cutting patterns requiring, for example, instantaneous combination of reverse and rotation at the line breakpoints could be implemented using direct drives with bandwidth much higher than that of the substrate melting.

Utilization of active singularities calls for modeling framework that can i) incorporate them in a physically meaningful manner, and ii) permit the overall motion planning and controller synthesis for both regular and singular phases to enforce this motion. Satisfying these conflicting requirements turns out, however, to be highly nontrivial. Indeed, examining the first requirement, for example, in the context of modeling of mechanical collisions, known to typically result in the velocity jumps in the system motion, one finds that the conventional techniques, such as complementarity, make use of either the so-called collision mapping or the restitution law, both of which give an expression for the velocity after the impact in terms of the velocity and position before the impact [2], [31], [17], [18], [22], [60]. These tools, however, do not provide mathematical structure rich enough to incorporate the impulsively controlled contact forces.

The second requirement could be addressed by approximating the fast almost discontinuous motion of the actual system in the singular phase by the considerably simpler completely discontinuous, or limit, motion and, on the basis of the phenomenologically accurate model, generating the corresponding tight control-oriented models of limit dynamics with enough regularity to admit controller synthesis in both phases. Furthermore, the implications of applying the results obtained on the basis of such limit description to the actual system should be clarified, as well. The techniques currently available for modeling the limit behavior of systems with singular motions include quasi-differential equations [61] and differential inclusions [67], [55], [57], and complementarity [20] and penalty methods [63], [72], [66], with modeling reported in [64], [65]. The first two representations give system evolution not in terms of an isolated trajectory, but a set of trajectories, referred to as integral funnel. As a result, system motion is vaguely defined and its precise computation and control are problematic. In [56], [57], and [37] it is indicated that the extraction of an isolated system trajectory from the funnel can be accomplished through the use of differential equations with measure; however, the approach provided is not extensible to controlled singularities. In the penalty method, the terms are devised to include the loss of energy at impact, but no guidance is provided on obtaining well-posed limit models. Although effective in approximating the velocity jumps in the complementary limit models, the method does not admit the straightforward extension to the case of controlled impacts, since systems with controlled singularities have no natural barrier or gradient functions to penalize and, therefore, are not naturally compatible with this methodology. Complementarity also runs into difficulties, since, as indicated earlier, it is incapable of encompassing controlled impact phase. Finally, existing results in impulsive control (cf. [30], [46], [45], [47], [53], [58], [77], [48], [53]) of systems with impacts (cf. [13]–[15], [12]) are limited to the open-loop commands, whereas active singularity is the intrinsic manifestation of the system dynamics through a controlled force appearing as an impulsive feedback and represents a new type of impulsive control.

Refining and completing [50] and [5]–[7], the present work resolves the issues raised above as follows.

- A novel conceptual framework, that of systems with controlled singularities, is proposed; within this framework a class of dynamic systems with controlled, or active unilateral constraints characterized by the constraint parameter, \( \mu \), corresponding in mechanical systems to the elastic deformation coefficient, is introduced.
- Admitting a constraint violation for the finite value of \( \mu \) and prohibiting it in the limit as \( \mu \) tends to infinity, a rigorous
system modeling and controller synthesis setting for this class of systems is developed, specifically, the following.

- A system of differential equations representing sequentially occurring natural and singular phases, referred to as the physically-based, or the original, system, is introduced; this system consistently incorporates controlled collisions into its singular phase, but is not well suited for motion planning and controller synthesis due to the unbounded control signals.

- A topological map—a one-to-one space-time transformation parametrized by $\mu$ is found that in the limit as $\mu \uparrow \infty$ reduces each singular phase equation of the original system to the corresponding limit representation, referred to as the controlled infinitesimal dynamics equation that has bounded control signals and therefore lends itself to the singular phase control law synthesis using regular techniques; viewed as evolving in auxiliary stretched time on an infinitesimally small natural time interval, this equation also clearly brings out the intrinsic control capability of singular phase by stripping away its nonessential dynamics.

- A new type of description of the impulsive action in the form of the controlled shift operator along the trajectories of the infinitesimal dynamics equation is introduced; this operator provides an explicit control-dependent map between the pre-impact and the post-impact system state and, therefore, can be viewed as active collision mapping, replacing the traditional one.

- Replacing in the original system the singular phase equations by the corresponding controlled shift operators, the full limit system is obtained and shown to have the desired modeling properties, namely, to uniquely describe isolated paths with discontinuities, have bounded rhs delta-function coefficients and control signals, and retain all the details of the original system dynamics, and it is demonstrated that the corresponding solutions of the original and the full limit systems can be made arbitrarily close to each other uniformly except, possibly, in the vicinities of the jump points by the appropriate setting of the value of $\mu$ in the original system.

- Implementation of the control laws designed through the use of the infinitesimal dynamics equations is shown to be accomplished by simply substituting the time-rescaled bounded control signals found into the actual physical system and slightly extending them, if necessary, resulting in the behavior of the latter close to that of the full limit system for sufficiently large values of $\mu$ and sufficiently accurate original system description.

- Using this framework, a detailed example of modeling and control of a mechanical system actuated through collisions with a manipulatable nonstationary constraint, is given, clearly bringing out the control implications of the active impact.

The traditional approach to describing systems under unilateral constraints makes, at the outset, use of Lagrange multipliers—measures activated at the instants the system engages a constraint and containing, in implicit form, either impulsive or continuous parts. The multipliers physically represent the contact force and mathematically guarantee the absence of constraint violation. This is followed by addressing an inverse-type problem of obtaining sensible restitution laws mainly through energy considerations. As shown in Example 1 in Section II, this set of analytical tools does not provide sufficiently rich mathematical structure to accommodate the problem at hand. The proposed methodology fundamentally differs from the traditional one in taking the opposite route: namely, arriving at a limit description of a system with regular and singular motions in terms of a pair of dynamic equations—the full limit system and the controlled infinitesimal dynamics equation—starting from a physically meaningful prelimit model. The latter, admitting constraint violation of elastic type, represents the entire phase of a constraint engagement as a dynamic process that in the limit, for a rigid constraint, becomes a measure. This measure, however, is now explicitly characterized by an internal dynamic structure given by the shift operator along the trajectories of the controlled infinitesimal dynamics equation. This operator can be thought of as describing imaginary infinitesimally small controlled constraint violation that can be viewed in hindsight as a penalty regularizing the limit model.

In a broader measure-theoretic context, the present work focuses on the impulsive part of a constraint-induced measure, bringing out an explicit dependence of this part on the position and the velocity preceding the velocity jump, time, and, most importantly, the external “data”—exogenous impulsive action applied to the system through the constraint during the engagement phase. This restricts phenomenology considered to only single jumps and finite jump sequences, but almost entirely obviates the need for measure-theoretic machinery. The framework proposed is also easily shown to capture the standard impact mechanics phenomena in the absence of exogenous actions. The infinite jump sequences with the finite-time accumulation points are addressed by the authors in [52] through augmenting the technique proposed by the detailed measure-theoretic setting. The latter permits obtaining the system representation with the unique extensibility of solutions beyond these points in terms of the continuous measure components.

The structure of this paper is as follows. Section II starts with examples of systems that do not lend themselves to obtaining good limit models using the existing theory and demonstrates the necessity of taking into account impact dynamics as the model complexity increases. This section, then, presents the physically-based model of a system with active constraints, the space-time transformation, and the multi-scale system description, and formulates the desirable capabilities of the controller synthesis setting to be developed. Section III introduces the infinitesimal dynamics equation, the corresponding controlled shift operator, and the full limit system with finite time sequences of controlled singularities. This section also demonstrates how the impulsive control actions that impose the desired behavior on the limit system should be implemented in the actual system to impose similar behavior on the latter. Section IV provides an example demonstrating all the elements of the framework proposed. Finally, Section V draws the conclusions. Appendix provides the proofs of several technical results.
II. PRELIM SYSTEM REPRESENTATIONS AND CONTROLLER SYNTHESIS SETTING OBJECTIVE

A. Motivating Examples

Synthesis of control laws for systems with active singularities starts with obtaining a well posed formalization of their discontinuous limit behavior. To motivate the development of a new methodology for addressing this task, two examples of impact systems with control actions are considered below. All the constraints in the examples are assumed to be characterized by the elastic-deformation-type parameter $\mu$: Elastically deformable for finite and perfectly rigid for infinite $\mu$, respectively. Systems and/or motions corresponding to the finite and the infinite values of $\mu$ are referred to later as the limit and the prelimit systems and/or motions, respectively. Motion in the domain of elastic constraint deformation is referred to below as inhibited. The elasticity in limit models is permitted to be imperfect, in general, with the imperfection characterized by the restitution coefficient $\alpha \in [0, 1]$. The constraint, whether elastically deformable or perfectly rigid, is considered perfectly elastic ($\alpha = 1$), when impact is not accompanied by the loss of energy.

Example 1 considers a system consisting of a unit mass point with phase coordinates $(x_p, x_v)$ corresponding to the position and the velocity, respectively, moving along the straight line $x_p(t) \geq 0$ until it impinges on a constraint at the point $x_p = 0$. The force, $f(t)$, acting in the negative direction along $x_p$ coordinate, depends on the state and is not necessarily bounded. Example 2 considers a ball/racket interaction, where the rate of the racket velocity change during an impact is of the same order as that of the bouncing ball velocity.

The first example brings out the richness of dynamic features of the impact phenomenon, indicates how to accommodate them in the limit model, and demonstrates the basic controlled singularity. The second example considers a system with controlled singularity that does not admit limit system derivation using the available techniques.

The examples illustrate the need to: i) constructively formalize the concept of active singularity by introducing the controlled prelimit elastic deformations characterized by their own dynamics [11], [21], [38]), and ii) regularize the latter for control applications through obtaining the corresponding well-posed dynamics of the controlled infinitesimal deformations in the respective limit models. These requirements are met by the technique, further referred to as the infinitesimal dynamics approach, developed in the present work.

1) Example 1: Matching the Impact Dynamics with the Constraint: Consider the prelimit system

\[ \dot{x}_p(t) = x_v(t) \]
\[ \dot{x}_v(t) = -f(t) \]

in the area $x_p(t) > 0$, with the restitution law

\[ x_v(t^+) = -x_v(t^-) \] (3)

at a point of impact $x_p(t) = 0$ at an instant $t$.

If in the case of bounded external force $f(t)$ the point $t$ is such that $x_p^*_{p}(t^*) = 0$ and $x_v^*(t^*) < 0$, then for sufficiently large $\mu$ there exists time of impact termination $t^*_\mu > t$ such that

\[ x_p^*(t^*_\mu) = 0, \quad x_v^*(t^*_\mu) > 0, \quad \text{and} \quad \mu \to \infty \]

\[ t^*_\mu \to \tau \quad x_v^*(t^*_\mu) \to -x_v(t^-). \] (4)

Therefore, the limit solution $\bar{x}_p(t) = \lim_{\mu \to \infty} x_p^*(t)$, $\bar{x}_v(t) = \lim_{\mu \to \infty} x_v^*(t)$ satisfies the relations

\[ x_p(t^+) = x_v(t^-), \quad x_v(t^+) = -x_v(t^-). \] (5)

The situation, however, changes if we admit an impulsive behavior of the external, or control, force within the phase of contact. For this purpose, assume the control force to be unbounded and satisfy the integral constraint

\[ f(t) \geq 0, \quad \int_{0}^{T} f(t) dt \leq M < \infty \]

and the initial conditions at point $t$ to be given by $x_p^*(t) = 0$, $x_v^*(t) = 0$. It is, then, possible to coordinate this force with the elastic deformation. Indeed, let us set

\[ f^\Delta t(t) = \begin{cases} M/\Delta t, & \text{if } t \leq \tau + \Delta t \\ 0, & \text{otherwise} \end{cases} \] (6)

which yields the impulsive action $\lim_{\Delta t \to 0} f^\Delta t(t) = M \delta(t - \tau)$. Then, (6) gives rise to the solution of (1) such that if $\mu^{3/2}/\Delta t \neq 2\pi k$, then there exists the time, $t^*_\mu$, of the first exit from the constraint defined by

\[ x_p^*(t^*_\mu) = 0, \quad x_v^*(t^*_\mu) = \frac{M}{\mu^{3/2}} \left( 2(1 - \cos(\mu^{3/2} \Delta t)) \right) > 0 \]

and $t^*_\mu$ lies in

\[ \tau + \frac{\pi}{2\mu^{3/2}} + \Delta t \leq t^*_\mu \leq \tau + \frac{\pi}{\mu^{3/2}} + \Delta t. \] (7)
Consider now a sequence $\Delta t_n \to 0, \mu_n \to \infty$ that for the state of the first exit of the ball yields

$$\bar{F}_p(\tau^+) = \lim_{\Delta t_n, \mu_n \to \infty} F_p(\tau^+),$$

$$\bar{F}_v(\tau^+) = \lim_{\Delta t_n, \mu_n \to \infty} F_v(\tau^+).$$

This gives rise to the following two distinct cases.

i) If the impulsive control corresponds to the external impact, like that exerted by a hammer or a bullet, with the initial state shown in Fig. 1(a), one can assume that

$$\lim_{\Delta t_n, \mu_n \to \infty} \mu_n^{1/2} \Delta t_n = \Delta P < \infty,$$

and as follows from (7), one can obtain various motions with nonzero positive velocity $\bar{F}_v(t)$ such that

$$\bar{F}_v(\tau^+) = \frac{M}{\Delta T} \sqrt{2(1 - \cos(\Delta P))} \in (0, M]$$

as shown in Fig. 1(b).

ii) If, however, $\lim_{\Delta t_n, \mu_n \to \infty} \mu_n^{1/2} \Delta t_n = \infty$, one obtains in the limit the null solution with $\bar{F}_v(t) = 0$ as shown in Fig. 1(c).

The complementarity equation corresponding to both cases with the same external impulsive force $-M\delta(t - \tau)$ has, however, the form

$$\bar{F}_p(t) = \bar{F}_v(t)$$

along with the restitution law (3) and admits only null solution, since according to (3) $\bar{F}_v(\tau^+) = \bar{F}_v(\tau-) = 0$, and the corresponding measure $d\bar{x}(t) = M\delta(t - \tau)dt \geq 0$. It is, therefore, seen that complementarity approach can take into account only the geometry of constraints and the kinematics of the impact in the form of a restitution law (3), but not the dynamic features of the impact phenomenon, such as, for example, assigning the appropriate spatiotemporal scaling factors that match motion of the system with its infinitesimal elastic deformation in the limit system.

The change of modes of system behavior under impulsive action depending on coordination of the external impulsive force with the scale of elasticity indicates that the impact in this setting is, in fact, a controlled singularity, and points at new control capabilities during the impact phase.

2) Example 2: Rotational Impact Controllability of a Ball Colliding with a Moving Racket: As indicated in the Introduction, in [73], Spong considers an impact controllability of an air hockey puck in a puck/mallet interaction, with the mallet position being a continuous function at the point of impact. In the present example, however, the change in the racket position during the phase of contact is admitted, making the angle of the racket a discontinuous function at the impact time moment. This discontinuity represents the limit description of an agile wrist rotation frequently demonstrated by the highly skilled ping-pong players during the contact phase of a ball with an elastic racket. The additional force component arising in this case due to the change of the racket surface orientation is shown in Fig. 2, where $\alpha$ and $\alpha'$ denote, respectively, the angle of incidence and the bounce-off angle of the ball with respect to the normals to the racket surface.

In this setting, the change of the surface position coordinated with the scale of elasticity creates a controlled collision, or active singularity, and gives a player a considerable advantage in the game through the radically enhanced impact controllability properties of the ball/racket interaction. Attempts to derive for this system a collision map through the use of mechanical conservation laws quickly run into difficulty — abrupt change change in both the orientation and the value of the reaction force during the impact renders these laws inadequate for describing the impulsive reaction of the rotating elastic surface. This example presents a challenge in deriving a limit model due, also, to the abrupt change of several position coordinates during contact. This challenge is met in Section IV using the approach developed in Sections II and III.

B. The Physically-Based System Description

The development of the formal setting for addressing the problems raised by the examples starts with noting that, in reality, a collision with a constraint in most physical systems is accompanied by a constraint violation due to some degree of constraint elasticity and, therefore, is not instantaneous. Indeed, [32] indicates that in a broad range of relative impact velocities above 0.1 m/s collisions can be represented by completely or nearly purely elastic behavior; the accuracy of this hypothesis can be gauged by tests [28] that showed the generation of an optically observable dimple produced by the drop of a small steel sphere on a hardened steel surface from a height of only 2.5 mm. In such cases the energy loss is but a small portion of the total mechanical energy involved and can be neglected as
long as plastic wave propagation or fracture is avoided. This regime may also encompass dissipation mechanisms such as viscosity and internal friction.” Thus, in the range of speeds corresponding to a sustainable system operation, collision represents, in fact, a very fast, but continuous motion phase that in the natural time scale of system velocities can be viewed only as approximately discontinuous. Furthermore, for systems with the sufficient constraint elasticity this phase becomes compatible with fast sensing and actuation. As indicated in [74], p.222, local compliance can be represented by discrete elements such as springs and dashpots, with coefficients for these elements selected, for example, via extended Hertz theory ([74], Section 4). The coefficients “depend on the geometry of the contact region, the material properties, and the contact pressure.” The latter reference further indicates that “incorporating these explicit force-displacement relations in a rigid-body impact analysis implicitly involves two different displacement scales. Very small relative displacements must be considered in order to obtain the time-dependent interaction forces at points of constraint - the displacements, however, must be so small that they have negligible effect on the inertia properties of the system.” Thus, a simplified limit motion description of a system with elastic impact would require considering impact on a spatiotemporal scale different from that of the natural system motion, but related to the latter via a specially constructed matching space-time transformation.

In [43], [44], and [54], the latter approach is used to derive the equations of collision mapping for robotic manipulator; however, the description of the discontinuous system behavior is not rigorously obtained. Further on, unlike the “passive constraints” case considered in [43], [44], and [54], in the case of the active constraints the control action, impulsive in the limit, should be admitted during the contact and properly represented in both the limit and the prelimit system models.

Based on these considerations, a physically motivated approach to describing systems with the contact forces is proposed with the following features.

- The contact force is considered to be the result of a small violation of a constraint (“dimple generation”) that starts taking place as soon as the system impinges on the boundary of a constraint.
- This force resists the penetration of the system into the domain, inhibited by the constraint, and causes a change of the sign of normal velocity component to the opposite one.
- The controlled motion in the inhibited domain is admitted and described by the nonstandard controlled singularly perturbed differential equation with infinitely growing right-hand-side, so that the solution of this equation behaves in the limit as a stepwise function with respect to the components of the generalized velocity.

This representation is then utilized for the derivation of the limit models describing full systems motion and development, through the inverse theorems, of a limit-model-based rigorous control synthesis setting.

1) Natural and Singular Motion Phases: Let the controlled dynamic system with an elastic constraint be described by the set of variables \(\mathbf{x}_p(t) \in \mathbb{R}^n, \mathbf{x}_v(t) \in \mathbb{R}^n, t \in [0, T]\), where vectors \(\mathbf{x}_p\) and \(\mathbf{x}_v\) are referred to as the sets of generalized positions and generalized velocities, respectively. Assume that the domain occupied by the constraint is given as

\[\{(\mathbf{x}_p(t), \mathbf{x}_v(t)) : G(\mathbf{x}_p(t), \mathbf{x}_v(t)) \leq 0\} \quad G : \mathbb{R}^n \times [0, T] \to \mathbb{R} \quad (11)\]

where \(G(\mathbf{x}, t)\) is a continuous function: \(G(\mathbf{x}, t) \in C^2(\mathbb{R}^n \times [0, T])\). Assume also that constraint elasticity is parametrized by some coefficient \(\mu\), so that for finite \(\mu\) constraint admits a system motion, although inhibited, in the domain occupied by it, and in the limit as \(\mu \to \infty\), it becomes rigid. Further on, let the entire system motion consist of the two modes of motion, an unconstrained one in the constraint-free domain \(\{(\mathbf{x}_p(t), \mathbf{x}_v(t)) : G(\mathbf{x}_p(t), \mathbf{x}_v(t)) > 0\}\), and an “inhibited” one in the domain \(\{(\mathbf{x}_p(t), \mathbf{x}_v(t)) : G(\mathbf{x}_p(t), \mathbf{x}_v(t)) \leq 0\}\), further referred to as the natural and the singular motion phases, respectively. Let also the superscripts \(\nu\) and \(s\) designate the relation of a particular quantity to the regular and the singular phases, respectively. As seen from (11), \(G(\mathbf{x}, t)\) is assumed to be scalar-valued: This assumption holds throughout the paper. Generalizing theoretical development to a vector-valued constraint is nontrivial and is not pursued in the present work.

a) Motion in the Natural Phase: In the domain \(\{(\mathbf{x}_p(t), \mathbf{x}_v(t)) : G(\mathbf{x}_p(t), \mathbf{x}_v(t)) > 0\}\) the system of differential equations for \((\mathbf{x}_p, \mathbf{x}_v)\) has the form

\[
\begin{align*}
\dot{\mathbf{x}}_p(t) &= F^p_\nu(\mathbf{x}_p(t), \mathbf{x}_v(t), \mathbf{u}(t), t) \\
\dot{\mathbf{x}}_v(t) &= F^v_\nu(\mathbf{x}_p(t), \mathbf{x}_v(t), \mathbf{u}(t), t)
\end{align*}
\quad (12)
\]

where \(\mathbf{u}(t) \in \mathcal{U} \subset \mathbb{R}^n\) is a control signal, \(\mathcal{U}\) is a compact set, and \(F^p_\nu(\mathbf{x}_p, \mathbf{x}_v, t)\) and \(F^v_\nu(\mathbf{x}_p, \mathbf{x}_v, t)\) have the standard properties of continuity and smoothness sufficient for existence and uniqueness of the solution of system (12) for a given measurable control \(u(\cdot)\) and arbitrary initial conditions \(\mathbf{x}_p(0), \mathbf{x}_v(0)\).

For example, they could be assumed to be continuous with respect to all variables and smooth with respect to \((\mathbf{x}_p, \mathbf{x}_v)\).

b) Motion in the Singular Phase: In the domain \(\{(\mathbf{x}_p(t), \mathbf{x}_v(t)) : G(\mathbf{x}_p(t), \mathbf{x}_v(t)) \leq 0\}\) the system of differential equations for \((\mathbf{x}_p, \mathbf{x}_v)\) has the form

\[
\begin{align*}
\dot{\mathbf{x}}_p(t) &= F^p_s(\mathbf{x}_p(t), \mathbf{x}_v(t), t) \\
\dot{\mathbf{x}}_v(t) &= \mu F^v_s(\mathbf{x}_p(t), \mathbf{x}_v(t), \mathbf{u}(t), t, \mu) + F^v_\nu(\mathbf{x}_p(t), \mathbf{x}_v(t), \mathbf{u}(t), t)
\end{align*}
\quad (13)
\]

where \(\mu F^v_s(\mathbf{x}_p, \mathbf{x}_v, \mathbf{u}(t), t, \mu)\) describes an additional controlled contact force, with \(\mathbf{u}(t) \in W \subset \mathbb{R}^n\) being the external contact force control signal to be implemented in the singular phase, and \(W\)—a compact set including zero element. As before, this contact force is considered to arise due to the constraints violation. Function \(F^v_\nu(\mathbf{x}_p, \mathbf{x}_v, \mathbf{u}(t), t, \mu)\) is assumed
to be continuous and smooth in \( \{ (x_p, t) : G(x_p, t) \leq 0 \} \), and satisfy the constraints

\[
F^u_v(x_p, x_v, u^t, t, \mu) = 0, \quad \text{if } G(x_p, t) = 0 \quad \text{and} \quad \frac{d}{dt} \bigg|_{F^u_v} G(x_p, t) = G'_{x_p}(x_p, t) F^u_v(x_p, x_v, t) + G'_t(x_p, t) = 0
\]

where \( G'_{x_p} \) and \( G'_t \) denote partial derivatives with respect to \( x_p \) and \( t \), respectively, as well as the following

**General Assumption:** Let \( F^u_v \) satisfy the Lipschitz condition in the following form: There exist \( L > 0 \) and \( \mu_0 > 0 \) such that for any \( (x_p, x'_p, x_v, x'_v), t \in [0, T], T < \infty, u^t \in W, \) and \( \mu \geq \mu_0 \)

\[
\| F^u_v(x_p, x_v, u^t, t, \mu) - F^u_v(x'_p, x'_v, u^t, t, \mu) \| \leq L \{ |x_p - x'_p| + \mu^{-1/2} |x_v - x'_v| \}\tag{14}
\]

where \( \| \cdot \| \) is any norm in \( \mathbb{R}^n \). Condition \( (14) \) is satisfied for the well known models of mechanical systems with viscoelastic constraints (Kelvin-Voigt, Maxwell, etc.) and is typically related to Rayleigh’s dissipative potentials in \( \{ (x_p, t) : G(x_p, t) \leq 0 \} \) [38]. Equations \((12)\) and \((13)\) together with the constraint \((11)\) and \( \mu_0 \leq \mu < \infty \) will be further referred to as the physically based, or the original, system. This system is assumed to have the unique solution for any given measurable control signals \( u(\cdot), u^t(\cdot) \). In general, system can have a finite sequence of singular phases, say \( N \), of the form \((13)\) with control signals \( u^t_i(t), \ i = 1, \ldots, N \).

To obtain a simplified system description, consider the original system motion in the limit as \( \mu \to \infty \). If this limit exists, one can treat it as the generalized solution of a dynamic system with unilateral constraints, which would then be described by the limiting form of the original system, further referred to as the limit system. The latter could possibly assume the form of a generalized differential equation with \( \delta \)-functions in the r.h.s. or, more generally, with measure.

**Remark 1:** The principal feature of the systems considered is that their generalized positions are continuous while their velocities admit jumps. At the same time, the generalized positions are exactly the ones responsible for the appearance of the contact forces. Therefore, in order to clearly bring out the impulsive nature of the impacts, one needs to introduce the multi-scale motion representation obtained through “opening up” of a singularity, i.e., through modeling of its fine structure. The latter can be accomplished through the use of the space-time transformation in the vicinity of the singular point, as shown in the next subsection.

### C. Space-Time Transformation at the Singularity Point and the Multiscale System Description

Let the system motion start from the initial condition \( x_p(0), x_v(0) \) such that \( G(x_p(0), 0) > 0 \), and \( \tau \) be the first point where the system engages the constraint, so that

\[
G(x_p(\tau), \tau) = 0, \quad \frac{d}{dt} \bigg|_{F^{\mu}_v} G(x_p(\tau), \tau) < 0, \tag{15}
\]

Let us now coordinate control \( u^t(\tau) \) with the time scale of impulsive interaction. For this purpose, introduce control signal \( u_\tau(\cdot) \) such that

\[
uu^t(\tau) = \begin{cases} 
\sqrt{\mu}(t - \tau), & \text{if } t \geq \tau, \\
0, & \text{otherwise}.
\end{cases}	ag{16}
\]

This coordinate will be shown to permit obtaining the singular phase control signals \( u^t_\tau(\cdot) \) for the original system through synthesizing \( u_\tau(\cdot) \) in the well-posed limit systems and using the converse theorems.

A typical case in mechanics is that of viscoelastic constraints where the spatial and the temporal intervals of the constraint violation are of the order \( \sim \mu^{-1/2} \) [38], [62], as depicted in Figs. 3 and 4. Therefore, for finite value of \( \mu \) there exists a nonzero time interval of the constraint violation, and the following space–time transformation in the vicinity of the jump point can be used for \( s > 0 \):

\[
y^u_p(s) = x_p(\tau) + \mu^{1/2} [x_p(\tau - 1/2 s) - x_p(\tau)], \\
y^u_v(s) = x_v(\tau + \mu^{-1/2} s), \\
t = \tau + \mu^{-1/2} s. \tag{17}
\]

Then, the new variables \((y^u_p(s), y^u_v(s))\) satisfy the equations

\[
y^u_p(s) = F^u_v \left[ (y^u_p(s) - x_p(\tau)) / \mu^{1/2} 
+ x_p(\tau), y^u_v(s), \tau + \mu^{-1/2} s \right], \\
y^u_v(s) = \mu^{1/2} F^u_v \left[ (y^u_v(s) - x_v(\tau)) / \mu^{1/2} 
+ x_p(\tau), y^u_v(s), \tau + \mu^{-1/2} s, \mu \right] 
+ \mu^{-1/2} F^u_v \left[ (y^u_v(s) - x_v(\tau)) / \mu^{1/2} 
+ x_p(\tau), y^u_v(s), u(\tau + \mu^{-1/2} s), \tau + \mu^{-1/2} s \right] \\
y^u_p(0) = x_p(\tau), \quad y^u_v(0) = x_v(\tau - \tau). \tag{18}
\]

The system of \((12)\) for the nonsingular phase, the coordinate map \((17)\), and the system of \((18)\) for the singular phase will be jointly referred to as the multi-scale motion representation of the original system. This term arises due to the decomposition of the original system \((12)\) and \((13)\) that contain mixed scales into subsystems separately describing the slow regular
and the fast singular phases. Also, from the previous subsection it follows that the multiscale system can have a finite sequence of singular phases of the form (18) with control signals $w_{\tau_i}(\cdot), i = 1, \ldots, N$.

D. Controller Synthesis Setting Objectives

Having developed the physically based and the multiscale representations, the problems stated in the Introduction can now be recast into the following specific tasks:

Controller synthesis setting objectives: provide an analytical setting that permits: i) consistent approximation of motion of (11)-(13) by discontinuous motion, and ii) reduction of an ill-posed problem of synthesis of the singular phase control signals $w^u(t)$ or $w^u_i(t), i = 1, \ldots, N$, in (13) to a well-posed two-step approximation procedure: a) synthesis of bounded signals $w^u(\cdot)$ or $w^u_i(\cdot), i = 1, \ldots, N$ in the corresponding well-posed limit representations, and b) calculation of $w^u(t)$ or $w^u_i(t), i = 1, \ldots, N$, implementable in the original system using signals $w^u(\cdot)$ or $w^u_i(\cdot), i = 1, \ldots, N$, synthesized in i). These objectives, that can be viewed as the direct and the converse ones, respectively, are addressed in the next section by Theorems 1–6.

III. LIMIT SYSTEM CONTROLLER SYNTHESIS SETTING

This section presents the key technical results that form a framework for synthesizing control laws for the original system. Using auxiliary Lemma 1, Theorem 1 introduces the controlled infinitesimal dynamics equation that describes the limit behavior of the multi-scale subsystem (18) as $\mu \uparrow \infty$. This equation incorporates control into the singular motion phase and lends itself to controller synthesis, thereby creating a bridge between the impact mechanics and the impulsive control theory. Theorem 1 and its corollary also demonstrate that the velocity jumps in the limit behavior of (18) can be represented by means of the controlled shift operator along the paths of the infinitesimal dynamics equation. A sufficient condition for the existence of the shift operator is given by Proposition 1. Using the explicit realization of this operator for representing a constraint-induced controlled single velocity jump or a sequence of jumps, the limit systems corresponding to the entire original system motion, referred to as the full limit systems, are given by “direct” Theorems 2 and 3, respectively. Finally, converse Theorems 4 and 6 explicitly provide impulsive control law realizations for single jumps and finite jump sequences, completing all the steps in addressing the objectives stated above. Controller synthesis techniques that employ the proposed framework can be found in [51] where the new types of necessary optimality conditions for systems with active singularities are derived and in [71] where a nontrivial output-based optimal singularity control problem is solved.

A. Controlled Infinitesimal Dynamics Equation and Calculation of a Velocity Jump

Lemma 1: For any continuous and smooth function $f(\cdot)$ define for some given $\alpha > 0$ an exit time

$$\tau^e(f) = \begin{cases} \inf\{s > \alpha : f(s) \geq \delta\} \\ \infty, & \text{if the set is empty.} \end{cases}$$

Let for some function $f^0(\cdot) \in C^1$ such that $f^0(\alpha) < 0$ the exit time $\tau^e(f^0) < \infty$ and $f^0$ have the positive derivative at $t = \tau^e(f^0)$. Then, $\tau^e(f)$ is continuous at $f^0(\cdot)$ in the topology of uniform convergence. In other words, if $f^0(s) \rightarrow f^0(s)$ uniformly on some interval $[\alpha, \tau^e(f^0) + \varepsilon], \varepsilon > 0$, then

$$\tau^e(f^0) \rightarrow \tau^e(f^0).$$

Proof: The proof is given in the Appendix.

In order to formulate the next theorem, introduce equation

$$\begin{align*}
\dot{y}_p(s) &= \tilde{F}^p(x_p(\tau), y_k(s), \tau) \\
\dot{y}_k(s) &= \tilde{F}^p(y_k(s), y_k(s), s, w_{\tau}(\tau), x_p(\tau), \tau) \\
y_p(0) &= x_p(\tau) \\
y_k(0) &= x_k(\tau)
\end{align*}$$

(19)

further referred to as the controlled infinitesimal dynamics equation, where

$$\tilde{F}^p(y_p, y_k, s, w_{\tau}(s), x_p, \tau) \triangleq \lim_{\mu \to \infty} \mu^{1/2} F^p_{\mu}\left((y_p-x_p)/\mu^{1/2}+x_p, y_k, s, w_{\tau}(s), \tau+\mu^{-1/2}s, \mu\right).$$

(20)

As it will be seen further in the paper, (19) and its solution constitute the key analytical/computational tools for the controller synthesis in the singular phase and full limit system representation, respectively. Introduce also two time instants in the fast time $s$ defined as

$$s^*(\tau) \triangleq \inf \begin{cases} 
G^r_{x_p(\tau), \tau}(s) \\
G^l_{x_p(\tau), \tau}(y_p(s)) \\
G^l_{x_p(\tau), \tau}(\dot{y}_p(s)) \\
G^l_{x_p(\tau), \tau}(y_k(s), \tau) > 0
\end{cases}$$

s > 0 : \begin{align*}
G^r_{x_p(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s}(s) \\
G^l_{x_p(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s}(y_p(s)) \\
G^l_{x_p(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s}(\dot{y}_p(s)) \\
G^l_{x_p(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s}(y_k(s), \tau)
\end{align*}$$

and

$$s^*_\mu(\tau) \triangleq \inf \begin{cases} 
G^r_{x_p(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s}(s) \\
G^l_{x_p(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s}(y_p(s)) \\
G^l_{x_p(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s}(\dot{y}_p(s)) \\
G^l_{x_p(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s}(y_k(s), \tau+\mu^{-1/2}s, \tau+\mu^{-1/2}s) > 0
\end{cases}$$

(21)
The quantity \( s^a(\tau) \equiv s^a(\tau, x_p(\tau), x_v(\tau-)) \) in (21) will be shown below to be the time moment, in the fast time \( s \), of the limit system (19) exit from the constraint. This quantity depends on the original system state \( x_p(\tau), x_v(\tau-) \) at the time moment \( \tau \) of the collision onset through setting initial conditions for (19) as \( y_p(0) = x_p(\tau), \ y_v(0) = x_v(\tau-) \) and it is found by solving (19) until (21) is met. Analogously, \( s^*_h(\tau) \equiv s^*_{h}(\tau, x_p(\tau), x_v(\tau-)) \) is the corresponding time moment of the exit of the prelimit system (18) from the constraint found by solving system (18) with initial conditions \( y^*_p(0) = x_p(\tau) \), \( y^*_v(0) = x_v(\tau-) \) and a given fixed \( \mu \) until (22) is met.

**Theorem 1:** Assume that

1. for any admissible control \( w_p(\cdot) \) and any \( (x_p, \tau) \) such that \( G(x_p, \tau) = 0 \) and \((d/dt)\left|_{\tau} \right. G(x_p, \tau) < 0 \) there exists (20) where convergence is uniform in any bounded vicinity of \((y_p, y_v, s)\), and
2. equation (19) has the unique solution on some interval \([0, s^a(\tau) + \varepsilon]\) where \( \varepsilon > 0 \) and \( s^a(\tau) \) is given by (21).

Then, the limit function \( F_p^\varepsilon \) is continuous with respect to all variables and Lipschitz in \( y_p, y_v \),

\[
(y^*_p(s), y^*_v(s)) \to (y_p(s), y_v(s)) \quad \text{uniformly on} \quad [0, s^a(\tau) + \varepsilon]
\]

as \( \mu \to \infty \), and for all sufficiently large \( \mu \) there exists \( s^*_h(\tau) \) given by (22) such that as \( \mu \to \infty \),

\[
s^*_h(\tau) \to s^a(\tau).
\]

**Proof:** Due to the continuity of \( F_p^\varepsilon \) it is obvious that

\[
\lim_{\mu \to \infty} F_p^\varepsilon [y_p - x_p, y_v, x_v - x_0, \tau + \mu^{-1/2}s] = F_p^\varepsilon [x_p, y_v, x_v, \tau].
\]

As follows from (14), (20) \( F_p^\varepsilon [y_p, y_v, s, w_p(s), x_p, \tau] \) satisfies the Lipschitz condition with respect to \( (y_p, y_v) \). The continuity of the ordinary differential equation solution with respect to the parameters (cf. [26, p. 71] or [67, Th. 7, Sec. 1]) implies that \( (y^*_p(s), y^*_v(s)) \) converges to \( (y_p(s), y_v(s)) \) uniformly on \([0, s^a(\tau) + \varepsilon]\).

Now, define

\[
f^0(s) = G^d(x_p(\tau), \tau)(y_p(s) - x_p(\tau)) + G^l(x_p(\tau), \tau)s
\]

and

\[
f^\mu(s) = G^d(x_p(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s)(y^*_p(s) - x_p(\tau)) + G^l(x_p(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s)s,
\]

Then, by assumption

\[
f^\mu(s) = G(x_p(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s) - G(x_p(\tau), \tau)
\]

is positive at the point \( s^a(\tau) \) and \( f^\mu(s) \to f^0(s) \) uniformly on \([0, s^a(\tau) + \varepsilon]\). The proof now follows from the uniform convergence of \( (y^*_p(s), y^*_v(s)) \) to \( (y_p(s), y_v(s)) \) and Lemma 1. Indeed, due to the conditions of theorem, there exists \( a \) such that \( f^0(s) < 0 \) for \( s \in [0, a] \). According to Lemma 1 there exists \( \tau^\varepsilon(f^0) \rightarrow \tau^\varepsilon(f^0) = s^a(\tau) \). Then

\[
G(x_p(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s) - G(x_p(\tau), \tau) = G(x_p(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s)
\]

\[
= \mu^{-1/2}[G^d(x_v(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s)(y^*_p(s) - x_p(\tau)) + G^l(x_v(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s)s]
\]

\[
O(\mu^{-1})
\]

where \( O(\mu^{-1})/\mu^{-1/2} \to 0 \) if \( \mu \to \infty \) uniformly in \( s \in [0, s^a(\tau) + \varepsilon] \). Since \( G(x_p(\tau), \tau) = 0 \), we have

\[
f^\mu(\tau^\varepsilon(f^0)) \geq \frac{\delta}{2} \quad \text{for} \quad \tau^\varepsilon(f^0) \leq s^a(\tau) + \varepsilon.
\]

Then, the uniform convergence of \( f^\mu \) to \( f^0 \) for chosen \( \delta \) and sufficiently large \( \mu > \mu_2(\delta) \) yields the inequalities

\[
f^\mu(\tau^\varepsilon(f^0)) \geq \frac{\delta}{2} \quad \text{for} \quad \tau^\varepsilon(f^0) \leq s^a(\tau) + \varepsilon.
\]

At the same time one can choose \( \mu_2(\delta) \) such that for \( \mu > \mu_2(\delta) \) one obtains \( |O(\mu^{-1})|/|\mu^{-1/2}| \leq \delta/4 \). Therefore, from (23) for sufficiently large \( \mu \) one has

\[
\mu^{-1/2}G(x_p(\tau + \mu^{-1/2}\tau^\varepsilon(f^0)), \tau + \mu^{-1/2}\tau^\varepsilon(f^0)) > \frac{\delta}{4}
\]

\[
\mu^{-1/2}G(x_p(\tau + \mu^{-1/2}\tau^\varepsilon(f^0)), \tau + \mu^{-1/2}\tau^\varepsilon(f^0)) < \frac{\delta}{4}
\]

This implies that \( G(x_p(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s) \) changes the sign within the interval \( (\tau^\varepsilon(f^0), \tau^\varepsilon(f^0)) \). Due to the uniform convergence and the representation (23) there are no other sign inversions at \( s < \tau^\varepsilon(f^0) \). Therefore, there exists

\[
s^*_h(\tau) \in (\tau^\varepsilon(f^0), \tau^\varepsilon(f^0))
\]
satisfying the equality in (22). The inequality in (22) follows from the representation:

\[
\int \mu(s) = G_{xp}(x_p(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s)|\mu(s) + O(\mu^{-1/2})
\]

and from the uniform convergence of \((y_p^\mu(s), y_p^{\mu}(s))\) to \((y_p(s), y_p(s))\). Finally, convergence of \(s_p^\mu(\tau)\) to \(s^\mu(\tau)\) follows from (24) and the arbitrariness of \(\delta\).

**Remark 2:** The signification of Theorem 1 is in i) bringing out the controlled infinitesimal dynamics (19) that represents singular phase dynamics, yet has no unbounded terms, i.e., is well-posed, admitting regular methods for synthesis of \(w_p(\cdot)\), ii) demonstrating that for sufficiently large \(\mu\) (19) displays almost the same behavior as the multi-scale subsystem (18), traceable further to the original system through (17), and iii) pointing at the possibility of representing the velocity jumps in the prospective limit system corresponding to the original system (11)-(13) in terms of the shift operator map along the solutions of (19). The latter implication is formalized by the corollary below. The functions \(\overline{f}_p(t), \overline{x}_p(t)\) introduced in this corollary will be used further in Theorems 2 and 3 to represent the state of the full limit system corresponding to (11)-(13).

**Corollary 1:** For sufficiently small \(\varepsilon > 0\) on the interval \([0, \tau + \varepsilon]\) the solution of the original system (11)-(13) converges to some discontinuous functions \(\overline{f}_p(t), \overline{x}_p(t)\), such that

\[
\overline{f}_p(t) = x_p(t) \quad \overline{x}_p(t) = x_v(t) \quad t < \tau
\]

and

\[
\overline{f}_p(\tau) = \overline{x}_p(\tau +) = \lim_{\mu \to \infty} x_p(\tau + \mu^{-1/2}s_p^\mu(\tau)) = x_p(\tau)
\]

\[
\overline{x}_p(\tau) = \overline{x}_v(\tau +) = \lim_{\mu \to \infty} x_v(\tau + \mu^{-1/2}s_p^\mu(\tau)) = y_v(s^*(\tau)).
\]

**Remark 3:** If in Assumption 2 of Theorem 1 \(\varepsilon = \infty\), one can establish only the existence of sequence \(s_p^\mu(\tau) \to s^*(\tau)\) such that

\[
\lim_{\mu \to \infty} G[x_p(\tau + \mu^{-1/2}s_p^\mu(\tau)), \tau + \mu^{-1/2}s_p^\mu(\tau)] = 0.
\]

However, Corollary 1 is still valid.

**Remark 4:** In the case of a single impact, the result obtained for unilaterally constrained mechanical systems in [62], [63], [72] by penalization methods coincides with that presented previously. The latter, however, is not restricted to some special equation structure, as in the case of purely and partially elastic obstacles in impact mechanics, and is valid for general unilaterally constrained dynamic systems, encompassing mechanical systems as a particular case.

**Remark 5:** Construction of the full limit system, given in the next two theorems, utilizes the explicit jump representation in terms of the shift-operator along the solutions of (19). The latter representation is admissible for any unconstrained discrete-continuous system that has trajectories robust with respect to variation of impulsive input (cf. [53, p. 23], [45], [47], [77]). For systems with constraints, however, one cannot guarantee at the outset that such representation exists for any initial conditions \(x_p(\tau), x_v(\tau)\) and the impulsive control \(w_p(\cdot)\) in (19), since the existence of \(s^*(\tau) < \infty\) in (21) can not be guaranteed. From Assumption 2 of Theorem 1 it follows that to assure this existence, the constraints need to be “repulsive.” This condition is formalized in the next subsection and is assumed henceforth to hold.

**B. The Sufficient Condition for the Constraint Repulsiveness**

Assumption 2 of Theorem 1 means that “force” \(\overline{f}_p\) has the property to repulse the system from the inhibited domain. Let us express this property in terms of the so-called restitution force. Consider the motion in the area \(\{(y_p(s), s) : \overline{f}_p(x_p(s), \tau)[y_p(s) - x_p(s)] + \overline{f}_v(x_v(s), \tau)s < 0\}\) along the paths of the limit system (19). The term

\[
Z(y_p(s), s) = \overline{f}_p(x_p(s), \tau)[y_p(s) - x_p(s)] + \overline{f}_v(x_v(s), \tau)s
\]

characterizes the principal part of constraint violation, whereas the restitution force is typically expressed in terms of \(Z\) as \(F(s) = d^2Z(s)/ds^2\). Suppose that one can ascertain that this restitution force is of the viscoelastic type and such that

\[
d^2Z(s)/ds^2 \geq -k_1 Z(s) - k_2 \dot{Z}(s).
\]

One can then obtain the following criterion, the proof of which is given in the Appendix.

**Proposition 1:**

1. Let the restitution force satisfy (25) in the area \(Z < 0\). Let also \(k_1, k_2 \geq 0\) and satisfy the inequality \((k_2/2)^2 < k_1\). Suppose that \(Z(0) = 0\) and \(\dot{Z}(0) < 0\). Then there exists

\[
s^*(\tau) \equiv \inf_{s > 0} \{Z(s) = 0, \dot{Z}(s) > 0\} < \infty
\]

with \(s^* \leq \pi/\omega\), where \(\omega^2 = k_1 - (k_2/2)^2\).

2. If in the area \(Z < 0\) the restitution force satisfies the inequality

\[
d^2Z(s)/ds^2 \leq -k_1 Z(s) - k_2 \dot{Z}(s)
\]

then the exit time \(s^*\), if it exists, satisfies the opposite inequality \(s^* \geq \pi/\omega\).

**Remark 6:** This proposition means that the restitution force (25) guarantees the repulsion in the finite time. Thus, if the limit system (19) satisfies Assumption 2 of Theorem 1, then the constraint is repulsive. This condition is easily shown to hold for typical models of the impact mechanical systems [17], [38].

**C. Full Limit System Representations**

1. **Single Jump Representation:** Let the original system start from the initial condition \(x_p(0), x_v(0)\) such that \(G(x_p(0), 0) >
0, and \( \tau \) be the time instant corresponding to the first system engagement with the constraint under some singular phase control \( u^p(t) \). In order to describe the discontinuity at a point \( \tau \), assume that conditions (15) and (16) hold, consider (19) with control signal

\[
w_\tau(s) \in W \subset R^k
\]  

(27)

on \( s \in [0, \infty) \), and introduce a shift operator along the paths of (19) denoted by

\[
\Psi(\cdot, w_\tau(\cdot), \tau) = \begin{bmatrix}
\Psi_{1}(\cdot, w_\tau(\cdot), \tau) \\
\Psi_{2}(\cdot, w_\tau(\cdot), \tau)
\end{bmatrix}.
\]  

(28)

Then, the discontinuity of the path \( x_\nu(\cdot) \) at instant \( t = \tau \) can be viewed as the result of the action of this shift operator and described by

\[
x_\nu(\tau) = x_\nu(\tau^-) + \Psi_{1}(x_\nu(\tau), x_\nu(\tau^-), w_\tau(\cdot), \tau).
\]  

(29)

Relation (29) describes the jump at \( t = \tau \) in terms of the controlled infinitesimal dynamics equation (19), so that if

\[
\varphi(\cdot, s, w(\cdot), \tau) = \begin{bmatrix}
\varphi_{1}(\cdot, s, w_{\tau}(\cdot), \tau) \\
\varphi_{2}(\cdot, s, w_{\tau}(\cdot), \tau)
\end{bmatrix}
\]  

(30)

is the general solution of (19), then

\[
y_\nu(s^*(\tau)) = y_\nu(\tau^-) = \varphi_{1}(x_\nu(\tau), x_\nu(\tau^-), s^*(\tau), w_{\tau}(\cdot), \tau)
\]  

(31)

with \( s^*(\tau) \) defined in (21),

\[
G_{x}(x_\nu(\tau^-), \tau) s + G_{x}(x_\nu(\tau^-), \tau) (\varphi_{1}(x_\nu(\tau), x_\nu(\tau^-), s^*(\tau), w_{\tau}(\cdot), \tau) - x_\nu(\tau^-)) < 0
\]  

(32)

on the interval \((0, s^*(\tau))\), and

Let \((x_{\mu}(t), x_{\mu}(t))\) denote the state of (11)–(13) with any fixed \( \mu \in [\mu_0, \infty) \). Then, functions \( \bar{x}(t) \) and \( \bar{\nu}(t) \) introduced in Corollary 1 define the pointwise limit \((\bar{x}(t), \bar{\nu}(t))\) of the ordinary solution \((x_{\mu}(t), x_{\mu}(t))\) of system (11)–(13) as \( \mu \to \infty \). This limit, in general discontinuous in \( t \), can be thought of as the generalized, or limit, solution of (11)–(13). With this notation, Corollary 1 and the shift-operator jump representation introduced previously yield the following result.

**Theorem 2:** The generalized solution \((\bar{x}(t), \bar{\nu}(t))\) of (11)–(13) under (16) satisfies on the interval \([0, \tau + \varepsilon]\) the system of nonlinear generalized differential equations

\[
\dot{\bar{x}}(t) = F_{\nu}((\bar{x}(t), \bar{\nu}(t), t), t)
\]

\[
\dot{\bar{\nu}}(t) = F_{\nu}((\bar{x}(t), \bar{\nu}(t), t), u(t), t)
\]

\[
+ \Psi_{1}(x_\nu(\tau), x_\nu(\tau^-), w_{\tau}(\cdot), \tau)(t - \tau)
\]

\[
\bar{x}(0) = x_\nu(0), \bar{\nu}(0) = \dot{x}_{\nu}(0)
\]

\[
\bar{x}(\tau) = x_\nu(\tau), \bar{\nu}(\tau^-) = x_{\nu}(\tau^-),
\]  

(33)

Equation (33) constitutes the full limit system representation of the original system (11)–(13) with a single active singularity.

**2) Jump Sequence Representation:** In many applications, such as cutting described in the Introduction, an open-loop optimization-based path planning might call for a sequence of generalized velocity jumps. The latter also takes place in the closed-loop, such as in the case of a ping-pong player bouncing a ball against a racket. To extend the previous result to these cases, suppose that originally \( G(\bar{x}(0), 0) > 0 \). Fix an arbitrary admissible control \( u(t), t \in [0, T] \), and define recursively a sequence of the intersection times

\[
0 < \tau_1 < \ldots < \tau_i < \ldots < T, i \leq N < \infty
\]  

(34)

as follows. First, set \( \tau_1 \) to satisfy

\[
G(\bar{x}(\tau_1), \tau_1) = 0, \quad G_{\nu}(\bar{x}(\tau_1), \tau_1) + G_{x}(\bar{x}(\tau_1), \tau_1) F_{\nu}(\bar{x}(\tau_1), \tau_1) < 0
\]  

(35)

for \( i = 1 \), so that

\[
\tau_1 = \begin{cases}
\inf_{0 \leq \theta \leq T} \{ t : G(\bar{x}(t), t) = 0, G_{\nu}(\bar{x}(t), t) + G_{x}(\bar{x}(t), t) F_{\nu}(\bar{x}(t), t) < 0 \} \\
T, \text{ if the set is empty.}
\end{cases}
\]  

(36)

Here, \( \bar{x}(t) \) can be replaced by \( x_\nu(t) \), since they are identical on \( t \in [0, \tau_1] \), and (12) can be solved with a given \( u(t) \) until \( \tau_1 \) to yield \( x_\nu(\tau_1) \) and \( \bar{x}(\tau_1) \) to yield \( x(\tau_1) \). Now, assuming that conditions (15) and (16) hold for the first singular phase, find a solution of the limit system (19). This gives \( y_\nu(s^*(\tau_1)) \) and, hence, a shift operator

\[
\Psi_{1}(y_\nu(0), y_\nu(0), \tau_1) = y_\nu(s^*(\tau_1)) - y_\nu(0).
\]  

Next, taking the values \( \bar{x}(\tau_1) = x_\nu(\tau_1) \) and \( \bar{\nu}(\tau_1) = y_\nu(s^*(\tau_1)) \) as the initial ones, define a moment \( \tau_2 \) analogously to \( \tau_1 \) by
solving (12) with a given $u(t)$ and $(x_p, x_v)$ replaced by $(\bar{x}_p, \bar{x}_v)$ until
\begin{align}
\tau_2 = \begin{cases} 
\inf_{t < T} \{ t \in G(\bar{x}_p(t), t) = 0, \ G'_e(\bar{x}_p(t), t) + G'_e(\bar{x}_p(t), t) F_{\bar{x}_p}(\bar{x}_p(t), \bar{x}_v(t), t) < 0 \} \\
T, \text{ if the set is empty}
\end{cases}
\end{align}
(37)
to yield $\bar{x}_p(\tau_2)$ and $\bar{x}_v(\tau_2)$, and so on, so that
\begin{align}
\bar{x}_v(\tau_i) = \bar{x}_v(\tau_{i-1}) + \Psi_v(\bar{x}_p(\tau_i), \bar{x}_v(\tau_{i-1}), w_{\tau_i}(\cdot), \tau_i), \quad i = 1, \ldots, N.
\end{align}
As a result, the following generalization of Theorem 2 is obtained.

**Theorem 3:** The solution $(\bar{x}_p(t), \bar{x}_v(t))$ generated by the procedure described previously satisfies on $[0, T]$ the system of generalized differential equations
\begin{align}
\dot{\bar{x}}_p(t) &= F_{\bar{x}_p}(\bar{x}_p(t), \bar{x}_v(t), t) \\
\dot{\bar{x}}_v(t) &= F_{\bar{x}_v}(\bar{x}_p(t), \bar{x}_v(t), t, u(t), t) + \sum_{\tau_i \leq t} \Psi_v(\bar{x}_p(\tau_i), \bar{x}_v(\tau_{i-1}), w_{\tau_i}(\cdot), \tau_i) \delta(t - \tau_i), \\
i &= 1, \ldots, N
\end{align}
with $\bar{x}_p(0) = x_p(0)$, $\bar{x}_v(0) = x_v(0)$.

**Proof:** The proof of this theorem is a direct consequence of Corollary 1 and the construction preceding formulation of the theorem.

**Remark 7:** Since the ordinary solution $(x_p^0(t), x_v^0(t))$ of system (11)–(13) converges to $(\bar{x}_p(t), \bar{x}_v(t))$ elsewhere on $[0, T]$, except, possibly, at the points $\{\tau_i\}$, as $\mu \to \infty$, (39) constitutes the full limit system representation of the original system (11)–(13) with a finite sequence of controlled singular phases. In this representation, the state of the limit system (39) changes continuously on half-intervals $[0, \tau_1], \ldots, [\tau_{i-1}, \tau_i)$ and undergoes a discontinuous change at every instant $\tau_i$. Due to (29), the values of these changes depend on the state immediately preceding the jump and the impulsive control signal $w_{\tau_i}(\cdot)$ applied during the singularity phase corresponding to the instant $\tau_i$. The shift operator representation of jumps implies the Lipschitzian character of function $\Psi_v(\cdot)$, thereby guaranteeing the existence and uniqueness of the solution of (39), [53].

**Remark 8:** The physical sense of Theorem 3 is that given the original system (11)–(13) with some fixed $\mu$ and a sequence of controlled jumps, the corresponding well-posed system (39) with the same number of jumps, but controlled impulsively, is established in the limit as $\mu \to \infty$. Further on, this limit system could be demonstrated to have discontinuous solutions close to the continuous solutions of the original system in a weak-* topology of the space of functions of bounded variation, and the increase in the value of $\mu$ in the original system makes these solutions closer.

**Remark 9:** The problem of bouncing ball stopping by an active constraint involving a finite jump sequence is solved in [71] in the framework proposed using the output-feedback-based optimal control. In this case, in addition to regular controllability/observability requirements, the individual control action needs to be more subtle than that considered here and involves the concept of multi-impulse.

### D. Realization of the Impulsive Control Laws

1) **Single Jump Realization:**

**Theorem 4:** Let the generalized solution $(\bar{x}_p(t), \bar{x}_v(t))$ of the limit system (33) satisfy the desired performance objective under control signal $\bar{u}(t) \in U$ and the corresponding control function $\bar{u}_\tau(s) \in W_s, s \in [0, s^*(\tau)]$, where $s^*(\tau)$ is defined by (21). Then, if $\mu$ is sufficiently large, the solution of the original system (11)–(13) is guaranteed to be in the close vicinity of the solution of the limit system (33) on $[0, T]$ if the control signals introduced into the original system are of the form
\begin{align}
u^u(t) &= \begin{cases} \bar{u}(t), \quad t \in [0, \tau] \\
\begin{cases} \bar{u}_\tau(\sqrt{\mu}(t - \tau)), \quad t \in [\tau, \tau + \tau^*_e], \\
\begin{cases} \text{any admissible}, \quad t > \tau
\end{cases}
\end{cases}
\end{cases}
\end{align}
(40)
where $\tau^*_e = \min\{s^*(\tau)/\sqrt{\mu}, \ s^*(\tau)/\sqrt{\mu}, \ t^*_u, \ \mu \to \infty\}$ and $s^*(\tau)$ is defined by (22).

**Proof:** This theorem is a direct consequence of the following result that is the converse of Theorem 2.

**Theorem 5:** Let $(\bar{x}_p(t), \bar{x}_v(t))$ be a solution of the system (33) generated by some admissible control signals $\bar{u}(t)$ and $\bar{u}_\tau(\cdot)$. Assume that for any fixed $\mu \in [\mu_0, \infty)$ the control signals $u(t)$ and $u^u(t)$ in the system (11)–(13) are given in terms of these $\bar{u}(t)$ and $\bar{u}_\tau(\cdot)$ according to (40), and system (11)–(13) has the corresponding solution $(x_p^u(t), x_v^u(t))$. Then, as $\mu \to \infty$, a sequence of solutions $(x_p^u(t), x_v^u(t))$ of (11)–(13) converges to $(\bar{x}_p(t), \bar{x}_v(t))$ on an interval $[0, \tau + \tau^*_e]$ pointwise.

**Proof:** To obtain the solution $(\bar{x}_p(t), \bar{x}_v(t))$ under $\bar{u}(t)$ and $\bar{u}_\tau(\cdot)$, integrate the original system (11)–(13) with control $\bar{u}(t)$, $t \in [0, \tau]$, and then calculate the shift operator $\Psi_v(\bar{x}_p(\tau), \bar{x}_v(\tau), \bar{u}_\tau(\cdot), \tau)$. To do this, integrate the system (19), (21) under a given control signal $\bar{u}_\tau(\cdot)$, $s \in [0, s^*(\tau)]$. Now, let us consider the system (13) with some fixed $\mu > \mu_0$ under the control signals (40), i.e., with $\bar{u}_\tau(\cdot)$ extended, if necessary, beyond the point $s^*(\tau)$ in an arbitrary admissible manner. Denote a solution of that system by $(x_p^u(t), x_v^u(t))$. Then, from (40) $(x_p^u(t), x_v^u(t)) = (\bar{x}_p(t), \bar{x}_v(t))$ for $t \in [0, \tau]$, and according to Theorem 1 there exists an instant $t^*_{u_\tau} = \tau + \mu^{1/2} s^*(\tau)$ such that $t^*_{u_\tau} \to \tau$ and $(x_p^u(t^*_{u_\tau}), x_v^u(t^*_{u_\tau})) \to (\bar{x}_p(t), \bar{x}_v(t))$, as $\mu \to \infty$.

**Remark 10:** The proof of Theorem 4 simply follows by noting that the uniformity of convergence of $(x_p^u(t), x_v^u(t))$ to $(\bar{x}_p(t), \bar{x}_v(t))$ with respect to $u$ on $[0, T]$, except, possibly, at the points $\{\tau_i\}$ implies that if $u$ in (11)–(13) is finite, but sufficiently large, the distance between $(x_p^u(t), x_v^u(t))$ and $(\bar{x}_p(t), \bar{x}_v(t))$ on $[0, T]$ is negligible, except, possibly, in the small open intervals around points $\{\tau_i\}$.

2) **Jump Sequence Realization:**

**Theorem 6:** Let the generalized solution $(\bar{x}_p(t), \bar{x}_v(t))$ of the limit system (33) satisfy the desired performance objective under control signal $\bar{u}(t) \in U$, the finite set of instants $\{\tau_i\} \subset
[0, T), i = 1, ..., N, and the set of the corresponding control functions \( \{ \bar{w}_i(s) \in W \} \) defined for each \( \tau_i \). Then, if \( \mu \) is sufficiently large, the solution of the original system (11)–(13) is guaranteed to be in the close vicinity of the solution of the limit system (3) on \([0, T]\) if the control signals introduced into the original system are of the form

\[
\begin{align*}
\mu^i(t) &= \bar{n}(t), \quad t \in [0, T], \\
\mu^i(t) &= \begin{cases} \\
\bar{w}_i(\mu^{1/2}(t - \tau_i^*)^+), & \text{if } t \in \tau_i^*, \tau_i^{**} - \tau_i^* \leq s^*(\tau_i)/\sqrt{\mu} \\
\text{any admissible}, t \in [\tau_i^*, \tau_i^{**}], & \tau_i^{**} - \tau_i^* \geq s^*(\tau_i)/\sqrt{\mu} \\
\end{cases}
\end{align*}
\]

where a finite sequence of instants

\[
0 < \tau_1^* < \tau_1^{**} < ... < \tau_i^* < \tau_i^{**} < ... < T, \quad i \leq N < \infty
\]
of intersections of system (11)–(13) trajectory \((x^p_i(t), x^v_i(t))\) with the constraint boundary is defined recursively as follows.

Set \( \tau_1^* = \tau_1 = \tau, \) i.e.,

\[
\tau_1^* = \inf_{0 < t < T} \{ t : G(x^p_1(t), t) = 0, \ G^*_i(x^v_i(t), t) \}
\]

1) Uncontrolled Regular Phase: Let the dynamics of the ball be represented by a pair of vectors \((x^p, x^v) \in \mathbb{R}^8\), corresponding to the position and the velocity, respectively, and the nonsingular phase equation be given by

\[
\dot{x}^p_p(t) = x^v(t) \quad \dot{x}^v(t) = 0.
\]

The position of the racket surface plane is given by the relation \(\langle \eta, n(t) \rangle = 0\), where \(\langle \cdot, \cdot \rangle\) is the scalar product, \(\mu\) is the elasticity coefficient, and \(n(t)\) is a unit vector normal to the surface. For simplicity, consider the case when the center of rotation coincides with the point of contact that is the coordinate system origin, and the rotation axis is orthogonal to the plane formed by vectors \(x^p_i\) and \(n\).

2) Controlled Singular Phase: In the singular phase, that takes place whenever the ball hits the surface, the contact force is proportional to the value of the surface deformation and directed perpendicularly to the surface, so that if

\[
G(x^p(t), t) = \langle x^p(t), n(t) \rangle < 0
\]

the equations have the form

\[
\dot{x}^p(t) = x^v(t) \quad \dot{x}^v(t) = -\mu(n(t)x^p(t), n(t)).
\]

The controlled singularity is introduced into this system through the differential equation for \(n(t)\)

\[
\dot{n}(t) = \omega(t, \mu) \times n(t)
\]

where \(\times\) denotes the vector product, and \(\omega(t, \mu)\) is the angular velocity of the surface rotation. This velocity could be interpreted as an impulsive control that abruptly changes the angle of the surface during the contact phase. Therefore, this velocity admits the representation

\[
\omega(t, \mu) \equiv w^\theta(t) = \omega_0 \mu^{1/2} w_\tau(\mu^{1/2}(t - \tau))
\]

where \(\omega_0\) is the unit vector directed along the rotation axis, \(w_\tau(\cdot)\) is an impulsive control, and \(\tau\) is the impact time.
B. Coordinate Transformation and the Controlled Infinitesimal Dynamics Equation

The use of the transformation (17)

\[ \begin{align*}
    y_p^\mu(s) &= \mu^{1/2} x_p(\tau + \mu^{1/2}s) \\
    y_v^\mu(s) &= x_v(\tau + \mu^{1/2}s) \\
    n^\mu(s) &= n(\tau + \mu^{1/2}s) \\
    t &= \tau + \mu^{1/2}s
\end{align*} \tag{46} \]

yields the following controlled infinitesimal dynamics equation for limit variables \((y_p(s), y_v(s), n(s))\) defined for \(s > 0\):

\[ \begin{bmatrix}
    \dot{y}_p(s) \\
    \dot{y}_v(s) \\
    \dot{n}(s)
\end{bmatrix} = \begin{bmatrix}
    y_v(s) \\
    -\bar{n}(s)(y_p(s), n(s)) \\
    (\omega_0 \times \bar{n}(s)) y_v(s)
\end{bmatrix}. \tag{47} \]

This system has to be solved with the initial conditions

\[ y_p(0) = 0 \quad y_v(0) = x_v(\tau-), \quad \bar{n}(0) = n(\tau-), \]

such that \(\langle x_v(\tau-), n(\tau-), \rangle < 0\), until the time

\[ s^* = \min \left\{ s > 0 : \frac{d}{ds} \langle y_p(s), \bar{n}(s) \rangle = 0 \right\}. \tag{48} \]

Define a coordinate system such that

\[ \bar{n}(0) = n(\tau-) = \begin{bmatrix} n_1(\tau-) \\ n_2(\tau-) \\ 0 \end{bmatrix}. \]

Then the solution of the third equation of (47) is given by

\[ \bar{n}(s) = \begin{bmatrix}
    n_1(\tau-) \cos \phi(s) + n_2(\tau-) \sin \phi(s) \\
    -n_1(\tau-) \sin \phi(s) + n_2(\tau-) \cos \phi(s) \\
    0
\end{bmatrix} \]

where

\[ \phi(s) = \int_0^s w_\tau(u) \, du \tag{49} \]

and \(w_\tau(\cdot)\) are, respectively, the rotation angle of the contacting surface and the racket rotation velocity in the spatiotemporal scale of the infinitesimal dynamics. Therefore, for an arbitrary control law \(w_\tau(\cdot)\), one obtains a linear system of differential equations

\[ \begin{bmatrix}
    \dot{y}_p(s) \\
    \dot{y}_v(s)
\end{bmatrix} = \begin{bmatrix}
    y_v(s) \\
    n(\tau-) y_v(s)
\end{bmatrix} = A(\phi(s), \bar{n}(\tau-)) y_p(s) \tag{50} \]

for \((y_p, y_v)\), where \(A(\phi, n)\) is some matrix-valued function depending on the current value of the rotation angle, \(\phi(s)\), and the initial orientation, \(n(\tau-)\), of the plane. Thus, (50) represents the equation of controlled infinitesimal dynamics with explicit dependence on the initial racket orientation.

C. Controlled Shift Operator and Full Limit System Representation

The general solution of system (50) has the form

\[ \begin{bmatrix}
    y_p(s) \\
    y_v(s)
\end{bmatrix} = \Phi(s, 0, \phi(\cdot)) \begin{bmatrix}
    y_p(0) \\
    y_v(0)
\end{bmatrix}. \]

A block-matrix \(\Phi(s, 0, \phi(\cdot))\) can be represented as

\[ \Phi(s, 0, \phi(\cdot)) = \begin{bmatrix}
    \Phi_{pp}(s, 0, \phi(\cdot)) & \Phi_{pv}(s, 0, \phi(\cdot)) \\
    \Phi_{vp}(s, 0, \phi(\cdot)) & \Phi_{vv}(s, 0, \phi(\cdot))
\end{bmatrix}. \]

If there exists an exit time \(s^*\) defined by the relation (48), the collision mapping in this model is then calculated as

\[ \Delta x_v(\tau) = y_v(s^*) - y_v(0) = \{\Phi_{vv}(s^*, 0, \phi(\cdot)) - I\} x_v(\tau-) \]

where \(I\) is a \(3 \times 3\) unity matrix, and the full limit system is given by

\[ \begin{align*}
    \dot{y}_p(t) &= \tau_v(t) \\
    \dot{y}_v(t) &= \{\Phi_{vv}(s^*, 0, \phi(\cdot)) - I\} x_v(\tau-) \delta(t - \tau) \\
    \delta x_v(0) &= x_v(0) \tau_v(t) = x_v(0) \tau_v(\tau-) = x_v(\tau-)
    \end{align*} \tag{52} \]

D. Limit Control Law Design

First, we note that without racket rotation during impact the modulus of the initial ball velocity is equal to that of the velocity after the impact and the angle of incidence of the ball is equal to its bounce-off angle. The racket rotation permits changing of both the bounce-off angle and the velocity modulus of the ball. Therefore, given the desired bounce-off angle increment and assuming constant racket rotation velocity in (49), one can determine the required racket rotation velocity value. This is illustrated as follows. Let

\[ \begin{align*}
    x_v1(\tau-) &= -\sqrt{3}/2, x_v2(\tau-) &= 1/2 \\
    x_v3(\tau-) &= 0, n_1(\tau) &= 1, n_2(\tau) = n_3(\tau) = 0.
    \end{align*} \]

This means that the modulus of initial velocity is equal to one and the angle of incidence is equal to \(-\pi/6\). Consider two cases, one corresponding to clockwise and the other to counter-clockwise rotation of the racket with constant angular velocities \(w_\tau(s) = \bar{\omega} = 0.1\) and \(w_\tau(s) = \bar{\omega} = -0.1\), respectively.

Figs. 5 and 6 show the results of the solution calculation of the infinitesimal dynamics system (50) subject to the above control...
signal and initial conditions for clockwise and counterclockwise rotation, respectively.

Using the curves obtained, $s^*$ is seen to be the first instant where $Z(s) = (y_{h1}(s), y_{h2}(s)) = 0$ and $dZ(s)/ds > 0$. The angle of the racket rotation can then be taken as $\Delta \phi(s^*) = \overline{ms^*}$, and the ball velocity components after the impact—as $(y_{h1}(s^*), y_{h2}(s^*))$. By using the above calculation, the change of the modulus of the ball velocity $\Delta V$ and the increment $\Delta \psi$ of the bounce-off angle with respect to the nominal value of $\pi/6$ are obtained as follows:

\[
\Delta V = \sqrt{(y_{h1}(s^*))^2 + (y_{h2}(s^*))^2} - 1
\]

\[
\Delta \psi = \arctan(y_{h2}(s^*)/y_{h1}(s^*)) - \pi/6.
\]

The first case corresponding to the clockwise rotation yields

\[
s^* = 3.32 \quad \Delta \phi(s^*) = -0.332 \text{ rad}
\]

\[
\Delta V = 0.1 > 0 \quad \Delta \psi = -0.343 \text{ rad}
\]

and the second case corresponding to the counter-clockwise rotation—

\[
s^* = 2.87 \quad \Delta \phi(s^*) = 0.287 \text{ rad}
\]

\[
\Delta V = -0.12 < 0 \quad \Delta \psi = 0.404 \text{ rad}.
\]

E. Limit Control Law Implementation

The implementation of this control law depends on the racket stiffness. For example, taking the contact phase duration equal to $\Delta t_{\mu}$ yields the necessary value of angular velocity of the racket rotation (modulus of control

\[
|\omega(t, \mu)| = \mu^{1/2}w_{\tau}((\mu^{1/2}(t - \tau)))
\]

during the contact phase equal to

\[
|\omega(t, \mu)| = \Delta \phi(s^*)/\Delta t_{\mu} \quad \text{for} \quad t \in [\tau, \tau + \Delta t_{\mu}].
\]

Consequently, for example, the rotation angle increment $\Delta \phi(s^*) = -0.332$ rad could be attained during $\Delta t_{\mu} = 0.1$ s with the angular velocity $|\omega(t, \mu)| \approx 3$ rad/s by the hard racket characterized by $\mu = (s^* / \Delta t_{\mu})^2 \approx 10^5$, and during $\Delta t_{\mu} = 0.3$ s with the angular velocity $|\omega(t, \mu)| \approx 1$ rad/s by the soft racket characterized by $\mu \approx 10^2$, respectively. Thus, in accordance with physical intuition, the soft racket imposes noticeably weaker demands on the player in attaining the specified bounce-off angle increment in a rotationally controllable impact than the hard racket by requiring significantly lower racket rotation velocity and longer rotation phase.

V. CONCLUSION

A new class of systems: dynamic systems with controlled singularities is proposed. A subset of this class - systems with controlled elastic impacts—is considered. For the latter subset, a physically based mathematical framework is developed that provides consistent description of controlled impact dynamics and yields control-oriented models suitable for system redesign, motion planning, and control law synthesis and implementation. To demonstrate the use of the technique developed, an example detailing the introduction of control actions into the singular phase, limit model derivation, and control signal implementation in the physically based model is presented, and physical significance of the results is clearly brought out.

The approach proposed also offers a general, mathematically consistent, and convenient framework for representing various types of mechanical collisions even in the absence of the deliberately introduced impulsive actions. The main contribution of the approach, however, is the introduction of the equation of controlled infinitesimal dynamics that opens up the singular motion phase for optimal control and observation problems in systems with controlled singularities. These problems are nonstandard due to the possible dimension change as system evolves through normal and singular phases. This class of problems calls for methods based on nonsmooth analysis [24] and multiprocess optimization [25]. The first approach has already been utilized
in [51] to derive the new types of necessary optimality conditions for systems with active singularities. The output-based optimal singularity control problem has been considered in [71] and shown to give rise to new types of control signals.

**APPENDIX**

Proof: [Proof of Lemma 1]: Due to the positivity of the derivative at $\tau^0(f^0)$ there exists $\delta_0 > 0$ such that $\tau^\delta(f^0) < \tau^0(f^0) < \tau^\delta(f^0)$ for any $0 < \delta < \delta_0$. Moreover, $\tau^0(f^0) \downarrow \tau^0(f^0)$ and $\tau^\delta(f^0) \uparrow \tau^0(f^0)$ if $\delta \to 0$. Due to the uniform convergence of \( f^\nu(\cdot) \) to $f^0(\cdot)$ for any $\delta < \delta_0$ one can choose $n(\delta)$ such that for $n \geq n(\delta)$

$$f^\nu(\tau^\delta(f^0)) > f^0(\tau^\delta(f^0)) - \frac{\delta}{2} > 0$$

and

$$f^\nu(s) < f^0(\tau^{-\delta}(f^0)) + \frac{\delta}{2} = -\frac{\delta}{2} < 0, \text{ for } s \leq \tau^{-\delta}(f^0).$$

Hence, by continuity of $f^\nu$ for $n \geq n(\delta)$, we have

$$\tau^0(f^\nu) \in (\tau^{-\delta}(f^0), \tau^\delta(f^0))$$

which establishes the convergence due to the arbitrariness of $\delta$.

Proof: [Proof of Proposition 1]: (First assertion) Function $Z(s)$ satisfies the equation

$$d^2 Z(s)/ds^2 = f(s) - k_1 Z(s) - k_2 \dot{Z}(s)$$

with some function $f(s) \geq 0$. This equation has an explicit solution

$$Z(s) = e^{-\lambda s} \left[ Z(0) \frac{\sin \omega s}{\omega} + \int_0^s e^{\lambda \tau} \frac{\sin \omega (s - \tau)}{\omega} f(\tau) d\tau \right]$$

(53)

where $\omega^2 = k_1 - (k_2/2)^2$ and $\lambda = k_2/2$. It is easily seen that $Z(\pi/\omega) \geq 0$. Consider first the case when the Lebesgue measure

$$\text{mes} \{ s : f(s) > 0 \} > 0.$$

Then

$$\int_0^{\pi/\omega} e^{\lambda \tau} \frac{\sin \omega (\pi/\omega - \tau)}{\omega} f(\tau) d\tau > 0$$

due to positivity of $\sin \omega (\pi/\omega - \tau)$ and nonnegativity of $f$. Since $Z(0) < 0$, there exist an interval, $[0, \varepsilon]$, where $Z(s) < 0$. Therefore, there exists $s^* \in (0, \pi/\omega)$ such that $Z(s^*) = 0$ and $Z(s) < 0$ for $s \in [0, s^*]$. At the same time

$$Z(s^*) = e^{-\lambda s^*} \left[ Z(0) \frac{\cos \omega s^* - \lambda}{\omega \sin \omega s^*} \right]$$

(54)

Thus, taking into account (53) and (54) we obtain

$$\dot{Z}(s^*) \sin \omega s^* = e^{-\lambda s^*} \int_0^s e^{\lambda \tau} \sin \omega f(\tau) d\tau > 0.$$

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