

# Maximum principle in nonlinear optimal stochastic singular control problems

Francois Dufour and Boris Miller

**Abstract**—In this paper, necessary conditions of optimality, in the form of a maximum principle, are obtained for singular stochastic control problems. This maximum principle is derived for a state process satisfying a general stochastic differential equation where the coefficient associated to the control process can be dependent on the state. We consider the class of so called robust nonlinear impulsive systems, those discontinuous solutions can be considered also as point-wise limits of ordinary solutions. The special conditions of robustness permit to derive the backward equations for adjoint variables in concise form of differential equation with measure and thereby to derive the optimality condition in the form of strong (point-wise) maximum principle.

## I. INTRODUCTION AND THE PROBLEM STATEMENT

Stochastic systems with impulsive or singular controls constitute a very important class of dynamic systems, where the application of the control actions causes very fast (almost abrupt) changes in the system state. They arise in numerous applications including: flight dynamics, medicine, information processes, queuing systems, power production control, stocks management, and financing. The general approach to the optimal control of such systems was suggested in the early 70's and is based on so-called quasivariational inequalities which generalize the dynamic programming method [8]. However only a few problems in stochastic settings have been solved till now and only for systems with linear dependence on impulse controls [23]. The reason for this is the non-robustness of general nonlinear impulse control systems, which becomes apparent in the instability of paths with respect to variation of impulse controls, the failure of approximation of the impulse controls by ordinary ones, absence of the optimal controls, and difficulty in applying of direct variational methods.

The importance of optimal control problems with singular controls and generalized solutions is well recognized and motivated by numerous applications in various areas, including but not exhausting: space flights [7], [9], [24], [25], irreversible investment policy [6], [12]; rational harvesting planning [1], [2], [26]; inflation control [11]; portfolio management [14]; advertising [15]; observation control [31], telecommunications [27]. Meanwhile the theory of stochastic

singular controls exists only for a restricted class of simple linear models, while the applications demand a deeper understanding. The prospective approach is based on the idea of *robustness* extending this conception to stochastic systems with impulse control.

In this paper, we consider the following stochastic optimal control problem:

minimize the cost function

$$E \left[ G \left( \int_0^T |u(s)| dv(s), x(T) \right) \right], \quad (1)$$

subject to

$$\begin{aligned} x(t) &\doteq \zeta + \int_0^t A(s, x(s)) ds + \int_{[0,t]} B(s, x(s)) u(s) dv^c(s) \\ &+ \int_0^t D(s, x(s)) dW(s) \\ &+ \sum_{n \in \mathbb{N}^*} \Psi(\tau_n, x(\tau_n -), u(\tau_n) \Delta v(\tau_n)) I_{\{\tau_n \leq t\}}. \end{aligned} \quad (2)$$

In the previous equation,  $\{(u_t, v_t)\}$  are the control variables defined on a filtered probability space  $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ . The set of admissible controls is defined as the class of right continuous with left hand limits (*corlol*) processes  $\{(u_t, v_t)\}$ ,  $\{\mathcal{F}_t\}$ -progressively measurable satisfying the following conditions:  $\{v_t\}$  is increasing,  $v_T \leq M$  *P*-a.s., and  $u_t \in K$ ,  $|u_t| \leq 1$  where  $K$  is a subset of  $\mathbb{R}^r$ ,  $T > 0$  is the finite horizon.  $\{v^c(t)\}$  is the continuous part of the control  $\{v(t)\}$ .  $\{\tau_n\}_{n \in \mathbb{N}^*}$  denotes the sequence of  $\{\mathcal{F}_t\}$ -stopping times exhausting the jumps of  $\{v(t)\}$ , and  $\Delta v(\tau_n)$  is size of the jump of  $\{v(t)\}$  at time  $\tau_n$ . The functions  $A$ ,  $B$ ,  $D$ ,  $\Psi$ , and  $G$  are deterministic.

This paper is a continuation of Dufour and Miller [18], [19] that applies a time change method to study the maximum principle for singular control problems (see [17], [18] for a detailed discussion of this technique of time change applied to singular control problem). The stochastic maximum principle for singular controls was considered by A. Cadenillas and U. Haussmann [10] and more recently by S. Bahlali and A. Chala in [4], [5] and by F. Dufour and B. Miller in [18]. In [19], the authors derive a stochastic maximum principle for the problem (1),(2), where the singular part was described by linear term

$$\int_{[0,t]} B(s) u(s) dv(s).$$

A natural generalization of this work would have been to consider a state process satisfying (2) with singular part in

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F. Dufour: Université Bordeaux 1, MAB, Talence Cedex, France, E-mail: francois.dufour@math.u-bordeaux1.fr

B. Miller: School of Mathematical Sciences, Monash University, Clayton, 3800, Victoria, Australia and Institute for Information Transmission Problems, RAS, Moscow, Russia, E-mail: boris.miller@sci.monash.edu.au

the form

$$\int_{[0,t]} B(s, x(s))u(s)dv(s),$$

where the *gain* of the control (namely the function  $B$ ) depends on the state process  $\{x(t)\}$ .

However, in the present paper, the control acts upon the state through the term

$$\int_{[0,t]} B(s, x(s))u(s)dv^c(s) + \sum_{n \in \mathbb{N}^*} \Psi(\tau_n, x(\tau_n^-), u(\tau_n) \Delta v(\tau_n)) I_{\{\tau_n \leq t\}},$$

where the coefficients  $B$  and  $\Psi$  are not arbitrary but satisfy the following relations and are connected by equation

$$\begin{aligned} & \Psi(t, x, u_1 + u_2) \\ &= \Psi(t, x, u_1) + \Psi(t, x + \Psi(t, x, u_1), u_2) \\ & B(t, x) = \frac{\partial \Psi}{\partial u}(t, x, 0) \end{aligned} \quad (3)$$

which may appear artificial. Nevertheless, our approach to defining state trajectories finds its justification in the deterministic context. Indeed, in the deterministic case (taking  $D=0$  in the previous equation) it has been shown that the optimal control may not exist for a state process defined by the previous equation (see for example [30] end reference therein). Roughly speaking, one of the main reason explaining this non-existence property is that the state trajectories may not be *stable* or *robust* with respect to perturbations of the control processes. This point has been widely studied in the deterministic context. It would be useful to underline that examples of non-robustness are already found in financial stochastic models [32], [33]

In particular it has been shown that if the state process satisfies the deterministic equation (2) with  $D \equiv 0$  and  $B, \Psi$  satisfying (3) then an optimal control may exist under some technical hypotheses. Under these assumptions the deterministic maximum principle has been studied in [3], [13], [16], [28], [29], [36], [39], [40].

Consequently, it is very natural to consider the same framework but in the stochastic context leading to a state process defined by the equation (2). The hypotheses that the functions  $\Psi$ , and  $B$  need to satisfy are given in the next section (see assumptions A2-A4).

Moreover, a very interesting property is that this singular control has a connection with respect to a classical control problem with integral constraint on the control studied by the authors in [19]. Indeed, the approach we use to solve singular control problem can be described in three steps. The first step is to convert the original singular control problem into a classical control problem by using a time change driven by the control. The second step is to solve this classical control problem, the last step consisting to recover the original control problem by using an inverse time change. It appears that the singular control problem studied here and the classical control problem analyzed by the authors in [19] have the same auxiliary control structure.

The paper is organized as follows. In section 3, we formulate the singular control problem. The time change and the auxiliary control problem is briefly described in section 4. Section 5 deals with the auxiliary maximum principle and its properties. In section 6, the main results are presented and in particular the stochastic maximum principle for singular controls (see Theorem 4).

## II. ASSUMPTIONS AND PRELIMINARY RESULTS

### A. Notations

The Lebesgue measure on  $\mathbb{R}$  is denoted by  $\lambda$ .

$\mathbb{N}_N$  is the set of the first  $N$  integers, that is  $\mathbb{N}_N = \{1, \dots, i, \dots, N\}$ .

$\mathbb{N}^* \doteq \{k \in \mathbb{N} : k > 0\}$  and  $\mathbb{R}_+ \doteq \{x \in \mathbb{R} : x \geq 0\}$ .

If  $V$  is a vector,  $V_i$  denotes the  $i$ th component of  $V$ .

If  $M$  is a matrix,  $M_i$  denotes a vector given by the  $i$ th column of the matrix  $M$ , and  $M_{ij}$  is the element corresponding to  $i$ th row and the  $j$ th column.

$(\top)$  denotes the transpose operation.

For  $x \in \mathbb{R}^k$ ,  $|x|$  denotes its Euclidean norm and for a matrix

$A \in \mathbb{R}^{k \times d}$  the norm of  $A$  is defined by  $|A| = \sqrt{\text{tr}[AA^\top]}$ .

For  $K \subset \mathbb{R}^k$ ,  $B_1(K) \doteq \{x \in K : |x| \leq 1\}$ , and  $S_1(K) \doteq \{x \in K : |x| = 1\}$ .

$\mathcal{S}_n$  denotes the set of all  $(n \times n)$  real symmetric matrices.

$0_n \in \mathbb{R}^n$  is the zero vector.

The indicator function of a set  $A$  is defined as  $I_A(x)$ .

The function  $\delta$  defined on  $\mathbb{N} \times \mathbb{N}$  is such that  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  otherwise.

For  $x \in \mathbb{R}$ ,  $x^+$  is defined by  $x^+ = \frac{1}{x}$  if  $x \neq 0$ ; and  $x^+ = 0$  if  $x = 0$ .

If  $V$  is a metric space then  $\mathcal{B}(V)$  denotes its associated borel  $\sigma$ -field.

A filtered probability space  $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$  is said to satisfy the usual hypotheses, if the probability space  $(\Omega, \mathcal{F}, P)$  is complete and if the filtration  $\{\mathcal{F}_t\}$  is right-continuous and if every  $\mathcal{F}_t$  contains all  $P$ -null sets of  $\mathcal{F}$ .

Let  $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$  be a filtered probability space and  $\{\theta_t\}$  be a  $[0, 1]$ -valued,  $\{\mathcal{F}_t\}$ -progressively measurable process.

Suppose that  $V$  denotes any of the spaces  $\mathbb{R}^k$ ,  $\mathbb{R}^{k \times d}$  or  $\mathcal{S}_k$ . Then,  $L^2(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}; [0, T]; V)$  (respectively,

$L^2_\theta(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}; [0, T]; V)$ ) denotes the set of  $V$ -valued processes  $\{x(t)\}$  which are  $\{\mathcal{F}_t\}$ -progressively measurable

and satisfy  $E_P \left[ \int_0^T |x(s)|^2 ds \right] < +\infty$  (respectively,

$E_P \left[ \int_0^T |x(s)|^2 [1 - \theta(s)] ds \right] < +\infty$ ).

If  $\mathbb{F}$  denotes the filtered probability space  $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ , then  $L^2(\mathbb{F}; [0, T]; V)$ , (respectively,  $L^2_\theta(\mathbb{F}; [0, T]; V)$ ) is used to write in a more compact form

$L^2(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}; [0, T]; V)$  (respectively,  $L^2_\theta(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}; [0, T]; V)$ ).

Moreover,  $L^2(\Omega, \mathcal{F}, P; V)$  denotes the set of  $V$ -valued random variables  $X$  defined on the probability space  $(\Omega, \mathcal{F}, P)$  such that  $E_P[|X|^2] < +\infty$ .

Let  $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$  be a filtered probability space, for a *corlol*, adapted processes  $\{A(t)\}$  of finite variation on each interval  $[0, t]$ ,  $\{A(t)\}$  is the distribution function of a measure denoted by  $dA$ . For  $\{H(t)\}$  a progressively measurable process, the integral process  $\int_0^\cdot H(s)dA(s)$  is denoted by  $H \cdot A$ . If  $\{A(t)\}$  is an increasing *corlol*, adapted processes, the measure defined on  $(\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F})$  by  $E_P \left[ \int_0^{+\infty} I_C(s)dA(s) \right]$  for  $C \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$  is denoted by  $\mathcal{M}_A$ .

Let  $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$  be a filtered probability space satisfying the usual hypotheses and supporting a standard  $m$ -dimensional Brownian motion  $\{W_t\}$ . Then  $\{\mathcal{F}_t^W\}$  denotes the augmentation of the natural filtration generated by  $\{W_t\}$ .

In order to define the state processes, let us introduce the following data:

- $T$  and  $M$  are fixed real numbers.
- $K$  is a subset of  $\mathbb{R}^r$ .
- $\zeta$  is a fixed vector in  $\mathbb{R}^n$ .
- $A : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ .
- $D : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ .
- $\Psi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^n$ .
- $G : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ .
- $N : \mathbb{R} \rightarrow \mathbb{R}^2$  such that  $N(t) = \begin{pmatrix} t - T \\ t^2 - T^2 \end{pmatrix}$ .

The following assumptions will be used in the paper:

*Assumptions 1:* The maps  $A$ ,  $D$ , and  $G$  are  $C^2$ .

*Assumptions 2:* The maps  $\Psi$  is  $C^1$  satisfying  $(\forall(t, x, u_1, u_2) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^r)$

$$\Psi(t, x, 0) = 0,$$

$$\Psi(t, x, u_1 + u_2) = \Psi(t, x, u_1) + \Psi\left(t, x + \Psi(t, x, u_1), u_2\right).$$

The second partial derivatives of  $\Psi$  with respect to its second variable exists and is continuous. The first and the second partial derivatives of  $\Psi$  with respect to its second variable is bounded.

*Assumptions 3:* Let us introduce the map  $B : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times r}$  defined by

$$B(t, x) = \frac{\partial \Psi}{\partial u}(t, x, 0), \quad (4)$$

where  $\frac{\partial \Psi}{\partial u}$  denotes the partial derivative of the function  $\Psi$  with respect to its third variable. The map  $B$  is  $C^2$ .

*Assumptions 4:* The first and second derivatives of  $A$ ,  $B$ ,  $D$  and the second derivative of  $G$  are bounded. The maps  $A(t, x)$ ,  $B(t, x)$ ,  $D(t, x)$  are bounded by  $C(1 + |t| + |x|)$ . The first derivative of  $G(w, x)$  is bounded by  $C(1 + |w| + |x|)$ .

*Assumptions 5:*  $(\forall x \in \mathbb{R}^n)$ ,  $(\forall(w_1, w_2) \in \mathbb{R} \times \mathbb{R})$  if  $w_1 \leq w_2$  then  $G(w_1, x) \leq G(w_2, x)$ .

Let us introduce the following notation:

$\mathcal{A} : \mathbb{R} \times \mathbb{R}^n \times B_1(K) \times [0, 1] \rightarrow \mathbb{R}^{n+2}$   
is defined by  $\mathcal{A}(t, x, u, z) \doteq$

$$\begin{pmatrix} 1 - z \\ A(t, x)(1 - z) + zB(t, x)u \\ z|u| \end{pmatrix},$$

and

$\mathcal{D} : \mathbb{R} \times \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^{n+2}$  is defined by

$$\mathcal{D}(t, x, z) \doteq \begin{pmatrix} 0 \\ D(t, x)\sqrt{1 - z} \\ 0 \end{pmatrix};$$

$(\forall(t, x, u, z, p, q, P) \in \mathbb{R} \times \mathbb{R}^n \times B_1(K) \times [0, 1] \times \mathbb{R}^{2+n} \times \mathbb{R}^{(2+n) \times m} \times \mathbb{R}^{n \times n})$ ,

$$\mathcal{H}(t, x, u, z, p, q) \doteq \mathcal{A}(t, x, u, z)^\top p + \text{tr}[\mathcal{D}(t, x, z)^\top q]; \quad (5)$$

$(\forall(t, x, u, z, p^1, p^2, p^3, q, P, \psi) \in \mathbb{R} \times \mathbb{R}^n \times B_1(K) \times [0, 1] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times n} \times \mathbb{R}^3)$ ,

$$J(t, x, p^2, q) \doteq A(t, x)^\top p^2 + \text{tr}[D(t, x)^\top q] \quad (6)$$

$$\begin{aligned} H(t, x, u, z, p^1, p^2, p^3, q, P, \psi) \doteq & (\psi_2 p^1 - \psi_1 - 2T\psi_3)(1 - z) \\ & + \psi_2 [(1 - z)A(t, x) + zB(t, x)u]^\top p^2 \\ & + \psi_2 z|u|p^3 + \psi_2 \sqrt{1 - z} \text{tr}[D(t, x)^\top q] \\ & + \frac{\psi_2}{2} \text{tr}[D(t, x)^\top P D(t, x)](1 - z). \end{aligned} \quad (7)$$

In the rest of the paper, the partial derivatives of the function  $A$  (respectively  $B$ ,  $\mathcal{H}$ ,  $J$ ,  $\Psi$ , and  $D_j$  for  $j \in \mathbb{N}_m$ ) with respect to the first variable will be denoted by  $A_t$  (respectively  $B_t$ ,  $\mathcal{H}_t$ ,  $J_t$ ,  $\Psi_t$ , and  $D_{jt}$ ) and with respect to the second variable it will be denoted by  $A_x$  (respectively  $B_x$ ,  $\mathcal{H}_x$ ,  $J_x$ ,  $\Psi_x$ , and  $D_{jx}$ ). The partial derivative of  $G$  with respect to the first variable will be denoted by  $G_w$  and with respect to the second variable it will be denoted by  $G_x$ . The second partial derivatives of the function  $G$  (respectively  $\mathcal{H}$ ,  $J$ , and  $\Psi_j$  for  $j \in \mathbb{N}_n$ ) with respect to the second variable will be denoted by  $G_{xx}$  (respectively  $\mathcal{H}_{xx}$ ,  $J_{xx}$ , and  $\Psi_{jxx}$ ).

## B. Preliminary results

In this section, we give some important technical results to derive a the equation for adjoint variables in maximum principle.

*Theorem 1:* Assume that  $\Psi$  satisfies the hypotheses A2-A4. Let  $(t_0, t_1, \tau) \in \mathbb{R}^3$  with  $t_0 < t_1$ , and  $y_0 \in \mathbb{R}^n$ ,  $\bar{y}_1 \in \mathbb{R}^{n \times n}$ , and  $Y_1 \in \mathcal{S}_n$ . Let  $u(t)$  be an  $\mathbb{R}^r$ -mapping defined on  $[t_0, t_1]$  satisfying  $\int_{t_0}^{t_1} |u(s)|ds < \infty$ . Then, the solution of the following system of differential equations

$$\begin{aligned} y(t) &= y_0 + \int_{t_0}^t B(\tau, y(s))u(s)ds, \\ \bar{y}(t) &= \bar{y}_1 + \sum_{j=1}^r \int_t^{t_1} B_{jx}(\tau, y(s))^\top u_j(s)\bar{y}(s)ds, \\ \tilde{y}(t) &= \int_t^{t_1} \bar{y}(s)^\top B_t(\tau, y(s))u(s)ds, \\ Y(t) &= Y_1 + \\ & \sum_{j=1}^r \int_t^{t_1} \left[ B_{jx}(\tau, y(s))^\top Y(s) + Y(s)B_{jx}(\tau, y(s)) \right] u_j(s)ds \\ & + \sum_{j=1}^n \sum_{i=1}^r \int_t^{t_1} \bar{y}_j(s)B_{jixx}(\tau, y(s))u_i(s)ds, \end{aligned} \quad (8)$$

for  $t \in [t_0, t_1]$  is given by

$$\begin{aligned} y(t) &= y_0 + \Psi\left(\tau, y(t_0), \int_{t_0}^t u(s)ds\right), \\ \bar{y}(t) &= \left[ I_n + \Psi_x\left(\tau, y(t), \int_{t_0}^t u(s)ds\right) \right]^\top \bar{y}(t_1), \\ \tilde{y}(t) &= \bar{y}(t)^\top \Psi_t\left(\tau, y(t_0), \int_{t_0}^t u(s)ds\right), \\ Y(t) &= \left[ I_n + \Psi_x\left(\tau, y(t), \int_{t_0}^t u(s)ds\right) \right]^\top Y(t_1) \times \\ &\quad \left[ I_n + \Psi_x\left(\tau, y(t), \int_{t_0}^t u(s)ds\right) \right] \\ &\quad + \sum_{j=1}^n \Psi_{jxx}\left(\tau, y(t), \int_{t_0}^t u(s)ds\right) \bar{y}_j(t_1). \end{aligned} \quad (9)$$

*Lemma 1:* There exists a constant  $k$  such that  $(\forall(\tau, x, y, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times B_M(\mathbb{R}^r))$ ,

$$\begin{aligned} \left| \Psi(\tau, x, u) \right| &\leq k(1 + |x|)|u| \\ \left| \Psi(\tau, x, u) - \Psi(\tau, y, u) \right| &\leq k|x - y||u| \\ \left| \left( I_n + \Psi_x(\tau, x, u) \right)^{-1} - I_n \right| &\leq k|u|. \end{aligned} \quad (10)$$

*Proposition 1:* Let  $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$  be a filtered probability space satisfying the usual hypotheses. Assume that  $\{W(t)\}$  is a standard  $m$ -dimensional  $\{\mathcal{F}_t\}$ -Brownian motion and  $\{u(t), v(t)\}$  is a  $B_1(K) \times \mathbb{R}_+$ -valued, *corlol*,  $\{\mathcal{F}_t\}$ -progressively measurable process such that  $\{v(t)\}$  is increasing and satisfies  $v(T) \leq M$ . Let  $\{\tau_n\}_{n \in \mathbb{N}^*}$  be the sequence of  $\{\mathcal{F}_t\}$ -stopping times which exhausts the jumps of  $\{v(t)\}$ . Then the following stochastic differential equation (2) admits a unique solution in the set of *corlol*  $\{\mathcal{F}_t\}$ -adapted processes.

*Proposition 2:* Assume that  $\{Y(t), Z(t)\} \in L^2\left(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^W\}; [0, T]; \mathbb{R}^n \times \mathbb{R}^{n \times m}\right)$  with  $\{Y(t)\}$  *corlol* is solution of the following backward stochastic differential equation

$$\begin{aligned} dY(t) &= A(t)Y(t)dt + U(t)dt + B(t)Y(t)dv^c(t) \\ &\quad + C(t)Y(t-)dv^d(t) + \sum_{j=1}^m D_j(t)Z_j(t)dt + Z(t)dW(t) \end{aligned} \quad (11)$$

with  $Y(T) = \zeta \in \mathbb{R}^n$ ,  $\{v(t)\}$  an increasing *corlol*,  $\mathcal{F}_t^W$ -progressively measurable process ( $v(T) \leq M$ ), and  $\{A(t), B(t), C(t)\}$  (respectively  $\{D_j(t)\}$  for  $j \in \mathbb{N}_m$ ) an  $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$  (respectively  $\mathbb{R}^{n \times n}$ )  $\mathcal{F}_t^W$ -progressively measurable process such that  $|A(t)| + |B(t)| + |C(t)| + \sum_{j=1}^m |D_j(t)| \leq K$ , for a constant  $K$ , and  $\{U(t)\} \in L^2\left(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^W\}; [0, T]; \mathbb{R}^n\right)$ .

Then the solution is unique in this class of processes.

### III. SINGULAR CONTROL PROBLEM STATEMENT

In this section, we formulate the singular stochastic control problem presented in the introduction using the formulation described in [22].

*Definition 1:* A singular control is defined by the following term:

$$C \doteq (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, \{u(t), v(t)\}, \{W(t)\}, \{x(t)\})$$

where:

- $(\Omega, \mathcal{F}, P)$  is a complete probability space with a right continuous complete filtration  $\{\mathcal{F}_t\}$ .

- $\{W(t)\}$  is a standard  $m$ -dimensional  $\{\mathcal{F}_t\}$ -Brownian motion.
- $\{u(t), v(t)\}$  is a  $B_1(K) \times \mathbb{R}_+$ -valued, *corlol*,  $\{\mathcal{F}_t\}$ -progressively measurable process such that  $\{v(t)\}$  is increasing and satisfies

$$v(T) \leq M. \quad (12)$$

- $\{x(t)\}$  is an  $\mathbb{R}^n$ -valued, *corlol*  $\{\mathcal{F}_t\}$ -progressively measurable process satisfying equation (2), where  $\{\tau_n\}_{n \in \mathbb{N}^*}$  denotes the sequence of  $\{\mathcal{F}_t\}$ -stopping times which exhausts the jumps of  $\{v(t)\}$ , and  $x(0-) = \zeta$ .

We write  $\mathcal{C}$  for the set of controls satisfying the previous conditions.

The cost is given by

$$J[C] \doteq E_P \left[ G \left( \int_0^T |u(s)| dv(s), x(T) \right) \right]. \quad (13)$$

The set  $\mathcal{C}^a$  of admissible controls is defined by

$$\mathcal{C}^a \doteq \{C \in \mathcal{C} : J[C] < \infty\}. \quad (14)$$

The singular control problem is defined by the minimization of  $J[C]$  on  $\mathcal{C}^a$ . Assuming the existence of an optimal singular control  $\tilde{C}$ , the aim of the paper is to derive necessary conditions for  $\tilde{C}$  to be optimal in terms of variational inequalities (see the maximum principle presented in Theorem 4).

### IV. MAXIMUM PRINCIPLE FOR AUXILIARY CONTROL VARIABLES

Next result shows that by using a time change technique the original control problem can be converted into an auxiliary classical control problem.

*Proposition 3:* Denote the process  $\{t + \hat{v}(t)\}$  by  $\{\hat{\Gamma}(t)\}$ . Let  $\{\eta^*(t)\}$  be the right inverse of  $\{\hat{\Gamma}(t)\}$ . Then,  $\{\eta^*(t)\}$  is a continuous time change such that the probability space  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_{\eta^*(t)}^W\})$  satisfies the usual hypotheses. Moreover, there exists a  $[0, 1]$ -valued,  $\{\hat{\mathcal{F}}_t^W\}$ -progressively measurable process  $\{\hat{z}(t)\}$  such that

$$\hat{v}(t) = \int_{[0, t]} \hat{z}(s) d\hat{\Gamma}(s). \quad (15)$$

Define the  $B_1(K) \times [0, 1]$ -valued,  $\{\hat{\mathcal{F}}_{\eta^*(t)}^W\}$ -progressively measurable process  $\{(\alpha^*(t), \theta^*(t))\}$  by

$$\alpha^*(t) = \hat{u}(\eta^*(t)) \quad \text{and} \quad \theta^*(t) = \hat{z}(\eta^*(t)). \quad (16)$$

Then

$$\eta^*(t) = \int_0^t (1 - \theta^*(s)) ds, \quad (17)$$

$$J[\hat{C}] = E_{\hat{P}} \left[ G(\mu^*(T+M), \xi^*(T+M)) \right], \quad (18)$$

$$E_{\hat{P}} \left[ N(\eta^*(T+M)) \right] = 0, \quad (19)$$

where the processes  $\{\xi^*(t)\}$  and  $\{\mu^*(t)\}$  are solution of the following equations

$$\begin{aligned} \xi^*(t) &\doteq \zeta + \int_0^t A(\eta^*(s), \xi^*(s))(1 - \theta^*(s))ds \\ &+ \int_0^t \theta^*(s)B(\eta^*(s), \xi^*(s))\alpha^*(s)ds \\ &+ \int_0^t D(\eta^*(s), \xi^*(s))d\widehat{W}(\eta^*(s)), \end{aligned} \quad (20)$$

$$\mu^*(t) \doteq \int_0^t |\alpha^*(s)|\theta^*(s)ds. \quad (21)$$

Moreover,

$$\widehat{x}(t) = \xi^*(\widehat{\Gamma}(t)) \quad \text{and} \quad \mu^*(T+M) = \int_{[0,T]} |\widehat{u}(s)|d\widehat{v}(s) \quad (22)$$

*Definition 2:* The processes  $\{\eta^*(t)\}$ ,  $\{\xi^*(t)\}$ ,  $\{\mu^*(t)\}$  are called the auxiliary state variables and  $\{\alpha^*(t)\}$ ,  $\{\theta^*(t)\}$  are called the auxiliary control variables.

For the rest of the paper, we shall use the following notation:

$$\widehat{\mathbb{F}}_{\eta^*}^{\widehat{W}} \doteq \left( \widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}, \{\widehat{\mathcal{F}}_{\eta^*}^{\widehat{W}}(t)\} \right), \quad \mathbb{F}^W \doteq (\Omega, \mathcal{F}, P, \{\mathcal{F}_t^W\}).$$

*Theorem 2:* There exist  $\psi \in S_1(\mathbb{R}^3)$  such that  $(\forall(\alpha, \theta) \in B_1(K) \times [0, 1])$

$$\begin{aligned} &H(\eta^*(t), \xi^*(t), \alpha, \theta, p^*(t), r^*(t), P^*(t), \psi) \leq \\ &H(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^*(t), r^*(t), P^*(t), \psi) \end{aligned}$$

$\lambda \otimes \widehat{P}$  - a.s. on  $[0, T+M] \times \widehat{\Omega}$ , with  $\{p^*(t)\}$ , and  $\{q^*(t)\}$  where  $\{p^*(t)\} \doteq \left\{ \begin{pmatrix} p^{*1}(t) \\ p^{*2}(t) \\ p^{*3}(t) \end{pmatrix} \right\}$ ,  $\{q^*(t)\} \doteq \left\{ \begin{pmatrix} q^{*1}(t) \\ q^{*2}(t) \\ q^{*3}(t) \end{pmatrix} \right\}$ ,  $\{P^*(t)\}$ ,  $(\{Q^{*j}\})_{j \in \mathbb{N}_m}$  are the unique solutions, in the following class of processes

$$\begin{aligned} \{p^*(t), P^*(t)\} &\in L^2\left(\widehat{\mathbb{F}}_{\eta^*}^{\widehat{W}}; [0, T+M]; \mathbb{R}^{2+n} \times \mathcal{S}^n\right) \\ &\{q^*(t), (Q^{*j}(t))_{j \in \mathbb{N}_m}\} \in \\ &L_{\theta^*}^2\left(\widehat{\mathbb{F}}_{\eta^*}^{\widehat{W}}; [0, T+M]; \mathbb{R}^{(2+n) \times m} \times [\mathcal{S}^n]^m\right) \end{aligned}$$

with  $\{p^*(t), P^*(t)\}$  corlol, of the system of backward stochastic differential equations defined by

$$\begin{aligned} dp^*(t) &= q^*(t)d\widehat{W}(\eta^*(t)) - \\ &\left( \begin{array}{c} \mathcal{H}_t(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^*(t), \sqrt{1-\theta^*(t)}q^*(t)) \\ \mathcal{H}_x(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^*(t), \sqrt{1-\theta^*(t)}q^*(t)) \\ 0 \end{array} \right) dt. \end{aligned} \quad (23)$$

with

$$p^*(T+M) = - \begin{pmatrix} 0 \\ G_x(\mu^*(T+M), \xi^*(T+M)) \\ G_w(\mu^*(T+M), \xi^*(T+M)) \end{pmatrix} \quad (24)$$

and

$$\begin{aligned} dP^*(t) &= -A_x^*(t)^\top P^*(t)(1 - \theta^*(t))dt \\ &- P^*(t)A_x(t)^*(1 - \theta^*(t))dt \\ &- \sum_{j=1}^r \left( [B_{jx}^*(t)]^\top P^*(t) + P^*(t)B_{jx}^*(t) \right) \alpha_j^*(t)\theta^*(t)dt \\ &- \sum_{j=1}^m [D_{jx}^*(t)]^\top P^*(t)D_{jx}^*(t)(1 - \theta^*(t))dt \\ &- \sum_{j=1}^m \left( [D_{jx}^*(t)]^\top Q^{*j}(t) + Q^{*j}(t)D_{jx}^*(t) \right) (1 - \theta^*(t))dt \\ &- \mathcal{H}_{xx}(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^*(t), \sqrt{1-\theta^*(t)}q^*(t))dt \\ &+ \sum_{j=1}^m Q^{*j}(t)d\widehat{W}(\eta^*(t)) \end{aligned} \quad (25)$$

with

$$P^*(T+M) = -G_{xx}(\mu^*(T+M), \xi^*(T+M)) \quad (26)$$

and  $\{r^*(t)\}$  is defined by

$$r^*(t) \doteq \left[ q^{*2}(t) - P^*(t)D(\eta^*(t), \xi^*(t)) \right] \sqrt{1 - \theta^*(t)}, \quad (27)$$

where functions  $A_x^*$ ,  $B_{jx}^*$ ,  $D_{jx}^*$ ,  $\mathcal{H}_{xx}^*$  are calculated along the optimal path and control.

## V. THE ADJOINT VARIABLES IN SINGULAR CONTROL PROBLEM

In the following definition, we introduce the backward stochastic differential equations that will be satisfied by the adjoint variables for the original control problem. Note the special form of the right hand side of these equation that gives a singular part for these adjoint variables.

*Definition 3:* Let  $C \in \mathfrak{C}^a$  be a singular control

$$C \doteq (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, \{u(t), v(t)\}, \{W(t)\}, \{x(t)\})$$

such that  $\{u(t), v(t)\}$  is  $\{\mathcal{F}_t^W\}$ -progressively measurable. Let us use the notations:  $A_t(s), D_{jt}(s), B_t(s), \Psi_t(\tau), A_x(s), B_{jx}(s), \Psi_x, D_{jx}(s), A_{jxx}(s), B_{jix}(s), D_{jix}(s), \Psi_{jix}(\tau)$  for values of functions calculated along the control  $\{x(t)\}, \{u(t), v(t)\}$ . If the system of backward stochastic differential equations

$$\begin{aligned} p^1(t) &= \int_t^T A_t(s)^\top p^2(s)ds + \sum_{j=1}^m \int_t^T D_{jt}(s)^\top q_j^2(s)ds \\ &- \int_t^T q^1(s)dW(s) + \int_t^T p^2(s)^\top B_t(s)u(s)dv^c(s) \\ &+ \sum_{n=1}^{\infty} p^2(\tau_n)^\top \Psi_t(\tau_n)I_{\{t < \tau_n\}}, \end{aligned} \quad (28)$$

$$\begin{aligned}
p^2(t) &= -G_x \left( \int_{[0,T]} |u(s)| dv(s), x(T) \right) \\
&+ \int_t^T A_x(s)^\top p^2(s) ds \\
&+ \sum_{j=1}^{\infty} \int_t^T B_{jx}(s)^\top p^2(s) u_j(s) dv^c(s) \\
&+ \sum_{n=1}^{\infty} \Psi_x(\tau_n)^\top p^2(\tau_n) I_{\{t < \tau_n\}} \\
&- \int_t^T q^2(s) dW(s) + \sum_{j=1}^m \int_t^T D_{jx}(s)^\top q_j^2(s) ds,
\end{aligned} \tag{29}$$

$$\begin{aligned}
p^3(t) &= -G_w \left( \int_0^T |u(s)| dv(s), x(T) \right) \\
&- \int_t^T q^3(s) dW(s),
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
P(t) &= -G_{xx} \left( \int_{[0,T]} |u(s)| dv(s), x(T) \right) \\
&+ \int_t^T \left[ A_x(s)^\top P(s) + P(s) A_x(s) \right] ds \\
&+ \sum_{j=1}^m \int_t^T \left[ B_{jx}(s)^\top P(s) + P(s) B_{jx}(s) \right] u_j(s) dv^c(s) \\
&+ \sum_{j=1}^m \int_t^T \left[ D_{jx}(s)^\top P(s) D_{jx}(s) \right. \\
&+ \left. D_{jx}(s)^\top Q^j(s) + Q^j(s) D_{jx}(s) \right] ds \\
&+ \sum_{j=1}^n \int_t^T \left[ A_{jxx}(s) p_j^2(s) + \sum_{i=1}^m D_{jix}(s) q_{ji}^2(s) \right] ds \\
&+ \sum_{j=1}^n \sum_{i=1}^r \int_t^T p_j^2(s) B_{jix}(s) u_i(s) dv^c(s) \\
&- \int_t^T \sum_{j=1}^m Q^j(s) dW(s) \\
&+ \sum_{n=1}^{\infty} \left[ I_n + \Psi_x(\tau_n) \right] P(\tau_n) \left[ I_n + \Psi_x(\tau_n) \right] I_{\{t < \tau_n\}} \\
&+ \sum_{n=1}^{\infty} \sum_{j=1}^n \left[ p_j^2(\tau_n) \Psi_{jxx}(\tau_n) - P(\tau_n) \right] I_{\{t < \tau_n\}}
\end{aligned} \tag{31}$$

admits a solution in the following class of processes

$$\begin{aligned}
\{p^1(t), p^2(t), p^3(t)\} &\in L^2(\mathbb{F}^W; [0, T]; \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}), \\
\{q^1(t), q^2(t), q^3(t)\} &\in \\
&L^2(\mathbb{F}^W; [0, T]; \mathbb{R}^{1 \times m} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{1 \times m}), \\
&\left( \{P(t)\}, \{Q^j(t)\}_{j \in \mathbb{N}_m} \right) \\
&\in L^2(\mathbb{F}^W; [0, T]; \mathcal{S}^n) \times \left[ L^2(\mathbb{F}^W; [0, T]; \mathcal{S}^n) \right]^m,
\end{aligned}$$

with  $\{p^1(t), p^2(t), p^3(t), P(t)\}$  corlorol.

Then  $\left( (\{p^i(t)\}, \{q^i(t)\})_{i \in \mathbb{N}_3}, \{P(t)\}, (\{Q^j(t)\})_{j \in \mathbb{N}_m} \right)$  are called the adjoint variables associated to the control  $\hat{C}$ . The

solution is said unique if the solution of the previous system is unique in this class of processes.  $\square$

By using a time transformation, we show that from

$$\left( \{p^*(t)\}, \{q^*(t)\}, \{P^*(t)\}, (\{Q^{*j}(t)\})_{j \in \mathbb{N}_m} \right)$$

we can obtain the adjoint variables for the original optimal control.

*Theorem 3:* Define the following processes:

$$\begin{aligned}
\hat{p}(t) &\doteq p^*(\hat{\Gamma}(t)), \quad \hat{q}(t) \doteq q^*(\hat{\Gamma}(t)), \quad \hat{P}(t) \doteq P^*(\hat{\Gamma}(t)), \\
&\text{and for } j \in \mathbb{N}_m \quad \hat{Q}^j(t) \doteq Q^{*j}(\hat{\Gamma}(t)),
\end{aligned} \tag{32}$$

where  $\left( \{p^*(t)\}, \{q^*(t)\}, \{P^*(t)\}, (\{Q^{*j}(t)\})_{j \in \mathbb{N}_m} \right)$  are solutions of equations (23)-(26). Write  $\hat{p}(t)$  in the form

$$\hat{p}(t) = \begin{pmatrix} \hat{p}^1(t) \\ \hat{p}^2(t) \\ \hat{p}^3(t) \end{pmatrix},$$

where  $\hat{p}^1(t) \in \mathbb{R}$ ,  $\hat{p}^2(t) \in \mathbb{R}^n$  and  $\hat{p}^3(t) \in \mathbb{R}$  and similarly

$$\hat{q}(t) = \begin{pmatrix} \hat{q}^1(t) \\ \hat{q}^2(t) \\ \hat{q}^3(t) \end{pmatrix}$$

where  $\hat{q}^1(t) \in \mathbb{R}^{1 \times n}$ ,  $\hat{q}^2(t) \in \mathbb{R}^{n \times n}$  and  $\hat{q}^3(t) \in \mathbb{R}^{1 \times n}$ . Then  $\left( (\{\hat{p}^i(t)\}, \{\hat{q}^i(t)\})_{i \in \mathbb{N}_3}, \{\hat{P}(t)\}, (\{\hat{Q}^j(t)\})_{j \in \mathbb{N}_m} \right)$  are the unique adjoint variables associated to the control  $\hat{C}$ .  $\square$

*Lemma 2:* For  $\psi \in \mathbb{R}^3$ , and  $(\alpha, \theta) \in B_1(K) \times [0, 1]$ , write

$$\begin{aligned}
L(t, \alpha, \theta) &= H(\eta^*(t), \xi^*(t), \alpha, \theta, p^*, r^*(t), P^*(t), \psi) \\
&- H(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^*(t), r^*(t), P^*(t), \psi).
\end{aligned}$$

Then, for all  $t \in [\hat{\Gamma}(\tau_i-), \hat{\Gamma}(\tau_i)]$

$$\begin{aligned}
L(t, \alpha, \theta) &= \psi_2 [B(\tau_i, \xi^*(t))(\theta\alpha - \hat{u}(\tau_i))]^\top p^{*2}(t) + \\
&\psi_2(\theta|\alpha| - |\hat{u}(\tau_i)|) \hat{p}^3(\tau_i) + (1 - \theta) \left\{ -\psi_1 - 2T\psi_3 \right. \\
&+ \psi_2 \left[ p^{*1}(t) + A(\tau_i, \xi^*(t))^\top p^{*2}(t) + \right. \\
&\left. \left. \frac{1}{2} \text{tr} \left[ D(\tau_i, \xi^*(t))^\top P^*(t) D(\tau_i, \xi^*(t)) \right] \right] \right\},
\end{aligned} \tag{33}$$

with

$$\xi^*(t) = \hat{x}(\tau_i-) + \Psi(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-))), \tag{34}$$

$$\begin{aligned}
p^{*1}(t) &= \hat{p}^1(\tau_i-) - \Psi_t(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top \\
&\times \left[ I_n + \Psi_x(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top \right]^{-1} \hat{p}^2(\tau_i-),
\end{aligned} \tag{35}$$

$$\begin{aligned}
p^{*2}(t) &= \\
&\left[ I_n + \Psi_x(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top \right]^{-1} \hat{p}^2(\tau_i-),
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
P^*(t) &= \left[ I_n + \Psi_x(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top \right]^{-1} \\
&\times \left[ \hat{P}(\tau_i-) - \sum_{i=1}^n \Psi_{ixx}(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top \right] \\
&\times \left[ I_n + \Psi_x(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top \right]^{-1}, \\
\end{aligned} \tag{37}$$

$\hat{P}$  – a.s. on  $\{\tau_i \leq T\}$ .

Moreover, for  $(\alpha, \theta) \in B_1(K) \times [0, 1]$ ,

$$\begin{aligned}
&(\forall t \in [\hat{\Gamma}(\tau_i-), \hat{\Gamma}(\tau_i)]), \\
&L(t, \alpha, \theta) \leq 0 \quad \hat{P} - \text{a.s. on } \{\tau_i \leq T\}. \\
&L(\hat{\Gamma}(t), \alpha, \theta) \leq 0, \quad \mathcal{M}_{\hat{\Gamma}} - \text{a.s. on } [0, T] \times \hat{\Omega}.
\end{aligned} \tag{38}$$

## VI. THE MAXIMUM PRINCIPLE FOR SINGULAR CONTROL PROBLEM

We can now establish a necessary condition for any control  $\tilde{C} \in \mathfrak{C}^a$  to be optimal. In fact similarly to the maximum principle for non-singular control problem, see for example [35], [41] and the references therein, we show that if a control  $\tilde{C} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_t\}, \{\tilde{u}(t), \tilde{v}(t)\}, \{\hat{W}(t)\}, \{\tilde{x}(t)\})$  is optimal it maximizes a certain Hamiltonian almost surely with respect to the measure of Doleans-Dade generated by the optimal control  $\{\tilde{v}(t)\}$ .

Let  $\{v(t)\}$  be an  $\mathbb{R}_+$ -valued, *corlol*, progressively measurable increasing process defined on a filtered probability space  $(\Omega, \mathcal{F}, P)$ . The process  $\{v'(t)\}$  defined by

$$v'(t) = \begin{cases} \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} \frac{v(t+\epsilon) - v(t)}{\epsilon} & \text{if the limit exists in } \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$

is an  $\mathbb{R}_+$ -valued,  $\{\mathcal{F}_t\}$ -progressively measurable process (see Lemma 5.6 in [19]).

*Theorem 4:* Assume the existence of an optimal singular control denoted by

$$\tilde{C} \doteq (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_t\}, \{\tilde{u}(t), \tilde{v}(t)\}, \{\hat{W}(t)\}, \{\tilde{x}(t)\}).$$

such that  $\{\tilde{u}(t), \tilde{v}(t)\}$  is  $\{\hat{\mathcal{F}}_t^{\hat{W}}\}$ -progressively measurable.

Denote by  $((\{\tilde{p}^i(t)\}, \{\tilde{q}^i(t)\})_{i \in \mathbb{N}_3}, \{\tilde{P}(t)\}, (\{\tilde{Q}^j(t)\})_{j \in \mathbb{N}_m})$  the adjoint variables associated  $\tilde{C}$  and let  $\{\tilde{\tau}_i\}_{i \in \mathbb{N}^*}$  the sequence of  $\{\hat{\mathcal{F}}_t^{\hat{W}}\}$ -stopping times which exhausts the jumps of  $\{\tilde{v}(t)\}$ . Then, there exist  $\psi \in S_1(\mathbb{R}^3)$  such that for all  $(u, z) \in B_1(K) \times [0, 1]$

$$\begin{aligned}
&[\tilde{z}(t) - z] \left\{ -\psi_1 - 2T\psi_3 + \right. \\
&\psi_2 \left[ \tilde{p}^1(t) + \frac{1}{2} \text{tr} [D(t, \tilde{x}(t))^\top \tilde{P}(t) D(t, \tilde{x}(t))] \right. \\
&\left. \left. + A(t, \tilde{x}(t))^\top \tilde{p}^2(t) \right] \right\} \\
&+ \psi_2 \left[ [B(t)[zu - \tilde{z}(t)\tilde{u}(t)]]^\top \tilde{p}^2(t) + [z|u| - \tilde{z}(t)|\tilde{u}(t)|] \tilde{p}^3(t) \right. \\
&\left. + \text{tr} [D(t, \tilde{x}(t))^\top \tilde{r}(t)] (\sqrt{1-z} - \sqrt{1-\tilde{z}(t)}) \right] \leq 0
\end{aligned} \tag{39}$$

$\mathcal{M}_{\tilde{v}^{ac}}$  – a.s. on  $[0, T] \times \hat{\Omega}$ , where

$$\begin{aligned}
\tilde{r}(t) &\doteq \left[ \tilde{q}^2(t) - \tilde{P}(t) D(t, \tilde{x}(t)) \right] \sqrt{1-\tilde{z}(t)}, \\
\tilde{z}(t) &\doteq \frac{\tilde{v}'(t)}{1+\tilde{v}'(t)},
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
&\psi_2 \left[ [zu - \tilde{u}(t)]^\top B(t)^\top \tilde{p}^2(t) + [z|u| - |\tilde{u}(t)|] \tilde{p}^3(t) \right] \\
&+ (1-z) \left\{ -\psi_1 - 2T\psi_3 \right. \\
&+ \psi_2 \left[ \tilde{p}^1(t) + \frac{1}{2} \text{tr} [D(t, \tilde{x}(t))^\top \tilde{P}(t) D(t, \tilde{x}(t))] \right. \\
&\left. \left. + A(t, \tilde{x}(t))^\top \tilde{p}^2(t) \right] \right\} \leq 0
\end{aligned} \tag{41}$$

$\mathcal{M}_{\tilde{v}^s}$  – a.s. on  $[0, T] \times \hat{\Omega}$ ,

and for all  $i \in \mathbb{N}$  and  $\gamma \in [0, 1]$

$$\begin{aligned}
&\psi_2 \left[ [B(\tilde{\tau}_i, \tilde{x}^\gamma(\tilde{\tau}_i))(zu - \tilde{u}(\tilde{\tau}_i))]^\top \tilde{p}^{\gamma 2}(\tilde{\tau}_i) \right. \\
&+ (z|u| - |\tilde{u}(\tilde{\tau}_i)|) \tilde{p}^3(\tilde{\tau}_i) \left. \right] \\
&+ (1-z) \left\{ -\psi_1 - 2T\psi_3 \right. \\
&+ \psi_2 \left[ \tilde{p}^{\gamma 1}(\tilde{\tau}_i-) + A(\tilde{\tau}_i, \tilde{x}^\gamma(\tilde{\tau}_i))^\top \tilde{\gamma} \tilde{p}^2(\tilde{\tau}_i) \right. \\
&\left. \left. + \frac{1}{2} \text{tr} \left[ D(\tilde{\tau}_i, \tilde{x}^\gamma(\tilde{\tau}_i))^\top \tilde{P}^\gamma(\tilde{\tau}_i) D(\tilde{\tau}_i, \tilde{x}^\gamma(\tilde{\tau}_i)) \right] \right] \right\} \leq 0,
\end{aligned} \tag{42}$$

$\hat{P}$  – a.s. on  $\{\tilde{\tau}_i \leq T\}$ , with

$$\begin{aligned}
&\tilde{x}^\gamma(\tau_i) = \tilde{x}(\tau_i-) + \Psi(\tau_i, \tilde{x}(\tau_i-), \gamma \tilde{u}(\tau_i) \Delta \tilde{v}(\tau_i)), \\
&\tilde{p}^{\gamma 1}(\tau_i) = \tilde{p}^1(\tau_i-) - \Psi_t(\tau_i, \tilde{x}(\tau_i-), \gamma \tilde{u}(\tau_i) \Delta \tilde{v}(\tau_i))^\top \\
&\times \left[ I_n + \Psi_x(\tau_i, \tilde{x}(\tau_i-), \gamma \tilde{u}(\tau_i) \Delta \tilde{v}(\tau_i))^\top \right]^{-1} \tilde{p}^2(\tau_i-), \\
&\tilde{p}^{\gamma 2}(\tau_i) \\
&= \left[ I_n + \Psi_x(\tau_i, \tilde{x}(\tau_i-), \gamma \tilde{u}(\tau_i) \Delta \tilde{v}(\tau_i))^\top \right]^{-1} \tilde{p}^2(\tau_i-),
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
&\tilde{P}^\gamma(\tau_i) = \left[ I_n + \Psi_x(\tau_i, \tilde{x}(\tau_i-), \gamma \tilde{u}(\tau_i) \Delta \tilde{v}(\tau_i))^\top \right]^{-1} \\
&\times \left[ \tilde{P}(\tau_i-) - \sum_{i=1}^n \Psi_{ixx}(\tau_i, \tilde{x}(\tau_i-), \gamma \tilde{u}(\tau_i) \Delta \tilde{v}(\tau_i))^\top \right. \\
&\left. \times \left[ I_n + \Psi_x(\tau_i, \tilde{x}(\tau_i-), \gamma \tilde{u}(\tau_i) \Delta \tilde{v}(\tau_i))^\top \right]^{-1} \right].
\end{aligned} \tag{46}$$

## VII. CONCLUSIONS

To compare the results obtained in the present paper with the works presented in [4], [5], [10], [19], it turns out that the maximum principle we obtained is completely novel. In particular, the crucial generalization and the main difference with the literature [4], [5], [10] and our earlier work [19] is that the coefficient associated with control process and the size of the jump of the state process  $\{x(t)\}$  (see the expression of  $B$ , and  $\Psi$  in the previous equation) are allowed to be dependent on the state variable. An important consequence of this generalization is that all the adjoint variables have now

a singular part contrary to the case studied in [4], [5], [10]. Moreover, in the above works the maximum principle has a form of integral inequality, i.e. gives an optimality conditions in weak form. Our results gives the maximum principle in the form of point-wise inequalities (see Theorem 4), i.e. in strong form like in well known work of S. Peng [35].

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