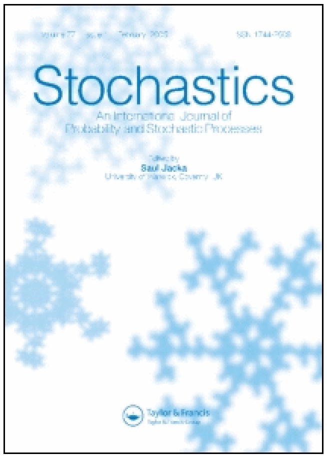


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# Necessary conditions for optimal singular stochastic control problems

F. Dufour [a]; B. Miller [b]
[a] Université Bordeaux 1, MAB, Talence Cedex, France
[b] School of Mathematical Sciences, Monash University, Clayton, Vic., Australia

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# Necessary conditions for optimal singular stochastic control problems

F. DUFOUR†*‖ and B. MILLER‡¶§

†Université Bordeaux 1, MAB, 351 Cours de la Libération, 33405 Talence Cedex, France
‡School of Mathematical Sciences, Monash University, Clayton 3800, Vic., Australia
¶Institute for Information Transmission Problems, RAS, Moscow, Russia

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In this paper, necessary conditions of optimality, in the form of a maximum principle, are obtained for singular stochastic control problems. This maximum principle is derived for a state process satisfying a general stochastic differential equation where the coefficient associated to the control process can be dependent on the state, extending earlier results of the literature.

*Keywords*: Stochastic control; Singular control; Maximum principle; Stochastic differential equation

*AMS Subject Classification*: 49J30; 49N25; 93E20

## 1. Introduction

In this paper, we consider the following stochastic optimal control problem: minimize the cost function

$$E\left[G\left(\int_0^T |u(s)| \mathrm{d}v(s), x(T)\right)\right],$$

subject to

$$x(t) \doteq \zeta + \int_0^t A(s, x(s))\mathrm{d}s + \int_{[0,t]} B(s, x(s))u(s)\mathrm{d}v^c(s) + \int_0^t D(s, x(s))\mathrm{d}W(s)$$
$$+ \sum_{n \in \mathbb{N}^*} \Psi(\tau_n, x(\tau_n -), u(\tau_n)\Delta v(\tau_n)) I_{\{\tau_n \leq t\}}. \quad (1)$$

*Corresponding author. Email: dufour@math.u-bordeaux1.fr

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In the previous equation, $\{(u_t, v_t)\}$ are the control variables defined on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$. The set of admissible controls is defined as the class of right continuous with left hand limits (*corlol*) processes $\{(u_t, v_t)\}$, $\{\mathcal{F}_t\}$-progressively measurable satisfying the following conditions: $\{v_t\}$ is increasing, $v_T \leq M$ $P$-a.s., and $u_t \in K$, $|u_t| \leq 1$ where $K$ is a subset of $\mathbb{R}^r$, $T > 0$ is the finite horizon. $\{v^c(t)\}$ is the continuous part of the control $\{v(t)\}$. $\{\tau_n\}_{n\in\mathbb{N}^*}$ denotes the sequence of $\{\mathcal{F}_t\}$-stopping times exhausting the jumps of $\{v(t)\}$, and $\Delta v(\tau_n)$ is size of the jump of $\{v(t)\}$ at time $\tau_n$. The functions $A$, $B$ $D$, $\Psi$ and $G$ are deterministic.

This paper is a continuation of Dufour and Miller [1] that applies a time change method to study the maximum principle for singular control problems (see [1,2] for a detailed discussion of this technique of time change applied to singular control problem). There are many works concerning singular control problems. In particular, the connection between singular control problem and optimal stopping problem has been studied by many authors including Alvarez [3,4], Boetius [5,6], Boetius and Kohlmann [7], Chow, Menaldi and Robin [8], Dufour and Miller [2], El Karoui and Karatzas [9,10], Karatzas [11,12], Karatzas and Shreve [13–15]. Results on the dynamic programming principle can be found in Boetius [6], Haussmann and Suo [16], Fleming and Soner [17], Zhu [18]. Sufficient conditions for the existence optimal singular control for general non-linear models have been obtained in Dufour and Miller [2] and Haussmann and Suo [19]. Recent applications of singular stochastic control have been studied in diverse areas such as mathematical finance by Framstad, Oksendal and Sulem [20], the control of a satellite by Jacka [21,22], optimal harvesting by Lungu and Oksendal [23] and by Alvarez [24]. This list of works does not pretend to be an exhaustive panorama of the literature relative to singular control problems (see the book by Oksendal and Sulem [25] and the thesis report by Boetius [6] for more complete expositions on this topic).

To the best knowledge of the authors the stochastic maximum principle for singular controls was only considered by Cadenillas and Haussmann [26] and more recently by Bahlali and Chala in Refs. [27,28] and by Dufour and Miller in Ref. [1] using different approaches. These differences may be briefly described as follows: in Ref. [26], the state process satisfies a linear stochastic differential equation. In Refs. [27,28], Bahlali and Chala generalizes the work of Cadenillas and Haussmann to the case where the state process satisfies a non-linear stochastic differential equation. In Refs. [26–28], the control process is of finite variation. In Ref. [1], the authors considers a state process defined by a non-linear stochastic differential equation, and the admissible control is of bounded variation ($v(T)$ is bounded). In many aspects the results obtained in Refs. [1,26–28] are different and complementary. To compare the results obtained in the present paper with the works presented in Refs. [1,26–28], it turns out that the maximum principle we obtained is completely novel. In particular, the crucial generalization and the main difference with the literature [26–28] and our earlier work [1] is that the coefficient associated with control process and the size of the jump of the state process $\{x(t)\}$ (see the expression of $B$, and $\Psi$ in the previous equation) are allowed to be dependent on the state variable. An important consequence of this generalization is that all the adjoint variables have now a singular part contrary to the case studied in Refs. [26–28].

In [1], the authors derive a stochastic maximum principle for the following state process:

$$x(t) \doteq \zeta + \int_0^t A(s, x(s))\mathrm{d}s + \int_{[0,t]} B(s)u(s)\mathrm{d}v(s) + \int_0^t D(s, x(s))\mathrm{d}W(s).$$

where the cost to minimize is defined by

$$E\left[G\left(\int_0^T |u(s)|\mathrm{d}v(s), x(T)\right)\right],$$

the functions $A$, $B$ $D$ and $G$ are deterministic, and $\{(u_t, v_t)\}$ is the control process satisfying the following conditions: $\{v_t\}$ is increasing, $v_T \leq M$, and $u_t \in K \subset \mathbb{R}^r$, $|u_t| \leq 1$.

A natural generalization of this work would have been to consider a state process satisfying the following stochastic differential equation

$$x(t) \doteq \zeta + \int_0^t A(s, x(s))\mathrm{d}s + \int_{[0,t]} B(s, x(s))u(s)\mathrm{d}v(s) + \int_0^t D(s, x(s))\mathrm{d}W(s),$$

where the *gain* of the control (namely the function $B$) depends on the state process $\{x(t)\}$.

However, in the present paper, the control acts upon the state through the terms $\int_{[0,t]} B(s, x(s))u(s)\mathrm{d}v^c(s)$, and $\sum_{n\in\mathbb{N}^*}\Psi(\tau_n, x(\tau_n-), u(\tau_n)\Delta v(\tau_n))I_{\{\tau_n\leq t\}}$ where the coefficients $B$ and $\Psi$ are not arbitrary but are connected by equation (2) which may appear artificial. Nevertheless, our approach to defining state trajectories finds its justification in the deterministic context. Indeed, in the deterministic case (taking $D = 0$ in the previous equation) it has been shown that the optimal control may not exist for a state process defined by the previous equation (see for example [29]). Roughly speaking, one of the main reason explaining this non-existence property is that the state trajectories may not be *stable* with respect to perturbations of the control processes. This point has been widely studied in the deterministic context [30–36]. In particular, it has been shown that if the state process satisfies the following equation

$$\begin{aligned} x(t) \doteq \zeta + \int_0^t A(s, x(s))\mathrm{d}s + \int_{[0,t]} B(s, x(s))u(s)\mathrm{d}v^c(s) \\ + \sum_{n\in\mathbb{N}^*}\Psi(\tau_n, x(\tau_n-), u(\tau_n)\Delta v(\tau_n))I_{\{\tau_n\leq t\}} \end{aligned}$$

where $\Psi$ satisfies a semi-group property $\Psi(t, x, u_1 + u_2) = \Psi(t, x, u_1) + \Psi(t, x + \Psi(t, x, u_1), u_2)$ and the function $B$ is defined through $\Psi$ by the following equation $B(t, x) = (\partial\Psi/\partial u)(t, x, 0)$ then an optimal control may exist under some technical hypotheses [36–40]. Under these assumptions the deterministic maximum principle has been studied in Refs. [32,41–47].

Consequently, it is very natural to consider the same framework but in the stochastic context leading to a state process defined by the equation (1). The hypotheses that the functions $\Psi$, and $B$ need to satisfy are given in the next section (see assumptions A2, A3 and A4).

Moreover, a very interesting property is that this singular control has a connection with respect to a classical control problem with integral constraint on the control studied by the authors in Ref. [48]. Indeed, the approach we use to solve singular control problem can be described in three steps. The first step is to convert the original singular control problem into a classical control problem by using a time change driven by the control. The second step is to solve this classical control problem, the last step consisting to recover the original control problem by using an inverse time change. It appears that the singular control problem studied here and the classical control problem analysed by the authors in Ref. [48] have the same auxiliary control structure.

The paper is organized as follows. In Section 2, we formulate the singular control problem. The time change and the auxiliary control problem is briefly described in Section 3. Section 4 deals with the auxiliary maximum principle and its properties. In Section 5, the main results are obtained and in particular the stochastic maximum principle for singular controls (see Theorem 5.5).

### *1.2 Notation*

The Lebesgue measure on $\mathbb{R}$ is denoted by $\lambda$.

$\mathbb{N}_N$ is the set of the first $N$ integers, that is $\mathbb{N}_N = \{1, \ldots, i, \ldots, N\}$.

$\mathbb{N}^* \doteq \{k \in \mathbb{N} \ : \ k > 0\}$ and $\mathbb{R}_+ \doteq \{x \in \mathbb{R} \ : \ x \geq 0\}$.

If $V$ is a vector, $V_i$ denotes the $i$th component of $V$.

If $M$ is a matrix, $M_i$ denotes a vector given by the $i$th column of the matrix $M$, and $M_{ij}$ is the element corresponding to $i$th row and the $j$th column. $(^\top)$ denotes the transpose operation.

For $x \in \mathbb{R}^k$, $|x|$ denotes its Euclidean norm and for a matrix $A \in \mathbb{R}^{k\times d}$ the norm of $A$ is defined by $|A| = \sqrt{\mathrm{tr}[AA^\top]}$.

For $K \subset \mathbb{R}^k$, $B_1(K) \doteq \{x \in K : |x| \leq 1\}$, and $S_1(K) \doteq \{x \in K : |x| = 1\}$.

$\mathcal{S}_n$ denotes the set of all $(n \times n)$ real symmetric matrices.

$0_n \in \mathbb{R}^n$ is the zero vector.

The indicator function of a set $A$ is defined as $I_A(x)$.

The function $\delta$ defined on $\mathbb{N} \times \mathbb{N}$ is such that $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

For $x \in \mathbb{R}$, $x^+$ is defined by $x^+ = (1/x)$ if $x \neq 0$; and $x^+ = 0$ if $x = 0$.

If $V$ is a metric space then $\mathcal{B}(V)$ denotes its associated borel $\sigma$-field.

A filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ is said to satisfy the usual hypotheses, if the probability space $(\Omega, \mathcal{F}, P)$ is complete and if the filtration $\{\mathcal{F}_t\}$ is right-continuous and if every $\mathcal{F}_t$ contains all P-null sets of $\mathcal{F}$.

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a filtered probability space and $\{\theta_t\}$ be a [0,1]-valued, $\{\mathcal{F}_t\}$-progressively measurable process. Suppose that $V$ denotes any of the spaces $\mathbb{R}^k$, $\mathbb{R}^{k\times d}$ or $\mathcal{S}_k$. Then, $L^2(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}; [0, T]; V)$ (respectively, $L^2_\theta(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}; [0, T]; V)$) denotes the set of $V$-valued processes $\{x(t)\}$ which are $\{\mathcal{F}_t\}$-progressively measurable and satisfy $E_P[\int_0^T |x(s)|^2 \mathrm{d}s] < +\infty$ (respectively, $E_P[\int_0^T |x(s)|^2[1 - \theta(s)]\mathrm{d}s] < +\infty$).

If $\mathbb{F}$ denotes the filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$, then $L^2(\mathbb{F}; [0, T]; V)$ (respectively, $L^2_\theta(\mathbb{F}; [0, T]; V)$) is used to write in a more compact form $L^2(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}; [0, T]; V)$ (respectively, $L^2_\theta(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}; [0, T]; V)$).

Moreover, $L^2(\Omega, \mathcal{F}, P; V)$ denotes the set of $V$-valued random variables $X$ defined on the probability space $(\Omega, \mathcal{F}, P)$ such that $E_P[|X|^2] < +\infty$.

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a filtered probability space, for a *corlol*, adapted processes $\{A(t)\}$ of finite variation on each interval $[0,t]$, $\{A(t)\}$ is the distribution function of a measure denoted by $\mathrm{d}A$. For $\{H(t)\}$ a progressively measurable process, the integral process $\int_0^{\cdot} H(s)\mathrm{d}A(s)$ is denoted by $H{\cdot}A$. If $\{A(t)\}$ is an increasing *corlol*, adapted processes, the measure defined on $(\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F})$ by $E_P[\int_0^{+\infty} I_C(s)\mathrm{d}A(s)]$ for $C \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ is denoted by $\mathcal{M}_A$.

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a filtered probability space satisfying the usual hypotheses and supporting a standard $m$-dimensional Brownian motion $\{W_t\}$. Then $\{\mathcal{F}_t^W\}$ denotes the augmentation of the natural filtration generated by $\{W_t\}$.

In order to define the state processes, let us introduce the following data:

- $T$ and $M$ are fixed real numbers.
- $K$ is a subset of $\mathbb{R}^r$.
- $\zeta$ is a fixed vector in $\mathbb{R}^n$.
- $A \ : \ \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.
- $D \ : \ \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$.
- $\Psi \ : \ \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^n$.
- $G \ : \ \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$.
- $N \ : \ \mathbb{R} \rightarrow \mathbb{R}^2$ such that $N(t) = \begin{pmatrix} t - T \\ t^2 - T^2 \end{pmatrix}$.

The following assumptions will be used in the paper:

*Assumption A*1. The maps $A$, $D$ and $G$ are $C^2$.

*Assumption A*2. The maps $\Psi$ is $C^1$ satisfying $(\forall (t, x, u_1, u_2) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^r)$

$$\Psi(t, x, 0) = 0, \quad \Psi(t, x, u_1 + u_2) = \Psi(t, x, u_1) + \Psi(t, x + \Psi(t, x, u_1), u_2).$$

The second partial derivatives of $\Psi$ with respect to its second variable exists and is continuous. The first and the second partial derivatives of $\Psi$ with respect to its second variable is bounded.

Let us introduce the map $B \ : \ \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times r}$ defined by

$$B(t, x) = \frac{\partial \Psi}{\partial u}(t, x, 0), \tag{2}$$

where $(\partial \Psi / \partial u)$ denotes the partial derivative of the function $\Psi$ with respect to its third variable.

*Assumption A*3. The map $B$ is $C^2$.

*Assumption A*4. The first and second derivatives of $A$, $B$, $D$ and the second derivative of $G$ are bounded. The maps $A(t, x)$, $B(t, x)$, $D(t, x)$ are bounded by $C(1 + |t| + |x|)$. The first derivative of $G(w, x)$ is bounded by $C(1 + |w| + |x|)$.

*Assumption A*5. $(\forall x \in \mathbb{R}^n)$, $(\forall (w_1, w_2) \in \mathbb{R} \times \mathbb{R})$ if $w_1 \leq w_2$ then $G(w_1, x) \leq G(w_2, x)$.

*Remark* 1. Examples of functions $\Psi$ satisfying hypotheses A2–A3 can be found in chapter 2, Section 2.2, examples 2.3–2.6 in the book [29].

Let us introduce the following notation: $\mathcal{A} : \mathbb{R} \times \mathbb{R}^n \times B_1(K) \times [0, 1] \rightarrow \mathbb{R}^{n+2}$ is defined by

$$\mathcal{A}(t, x, u, z) \doteq \begin{pmatrix} 1 - z \\ A(t, x)(1 - z) + zB(t, x)u \\ z|u| \end{pmatrix},$$

and

$$\mathcal{D} \;:\; \mathbb{R}\times\mathbb{R}^n\times[0,1]\to\mathbb{R}^{n+2} \text{ is defined by } \mathcal{D}(t,x,z)\doteq\begin{pmatrix}0\\ D(t,x)\sqrt{1-z}\\ 0\end{pmatrix}$$

$$(\forall(t,x,u,z,p,q,P)\in\mathbb{R}\times\mathbb{R}^n\times B_1(K)\times[0,1]\times\mathbb{R}^{2+n}\times\mathbb{R}^{(2+n)\times m}\times\mathbb{R}^{n\times n}),$$

$$\mathcal{H}(t,x,u,z,p,q)\doteq\mathcal{A}(t,x,u,z)^\top p+\mathrm{tr}[\mathcal{D}(t,x,z)^\top q] \tag{3}$$

$$(\forall(t,x,u,z,p^1,p^2,p^3,q,P,\psi)\in\mathbb{R}\times\mathbb{R}^n\times B_1(K)\times[0,1]\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{n\times n}\times\mathbb{R}^3),$$

$$J(t,x,p^2,q)\doteq A(t,x)^\top p^2+\mathrm{tr}[D(t,x)^\top q] \tag{4}$$

$$\begin{aligned} H(t,x,u,z,p^1,p^2,p^3,q,P,\psi)\doteq{}&(\psi_2p^1-\psi_1-2T\psi_3)(1-z)+\psi_2[(1-z)A(t,x)\\ &+zB(t,x)u]^\top p^2+\psi_2 z|u|p^3+\psi_2\sqrt{1-z}\,\mathrm{tr}[D(t,x)^\top q]\\ &+\frac{\psi_2}{2}\mathrm{tr}[D(t,x)^\top PD(t,x)](1-z). \end{aligned} \tag{5}$$

In the rest of the paper, the partial derivatives of the function $A$ (respectively $B$, $\mathcal{H}$, $J$, $\Psi$ and $D_j$ for $j\in\mathbb{N}_m$) with respect to the first variable will be denoted by $A_t$ (respectively $B_t$, $\mathcal{H}_t$, $J_t$, $\Psi_t$ and $D_{jt}$) and with respect to the second variable it will be denoted by $A_x$ (respectively $B_x$, $\mathcal{H}_x$, $J_x$, $\Psi_x$ and $D_{jx}$). The partial derivative of $G$ with respect to the first variable will be denote by $G_w$ and with respect to the second variable it will be denoted by $G_x$. The second partial derivatives of the function $G$ (respectively $\mathcal{H}$, $J$ and $\Psi_j$ for $j\in\mathbb{N}_n$) with respect to the second variable will be denoted by $G_{xx}$ (respectively $\mathcal{H}_{xx}$, $J_{xx}$ and $\Psi_{jxx}$).

## 2. Preliminaries

In this section, we prove some important technical results to derive a maximum principle.

THEOREM 2.1. Assume that $\Psi$ satisfies the hypotheses A2, A3 and A4. Let $(t_0,t_1,\tau)\in\mathbb{R}^3$ with $t_0<t_1$, and $y_0\in\mathbb{R}^n$, $\bar{y}_1\in\mathbb{R}^{n\times n}$, and $Y_1\in\mathcal{S}_n$. Let $u(t)$ be an $\mathbb{R}^r$-mapping defined on $[t_0,t_1]$ satisfying $\int_{t_0}^{t_1}|u(s)|\mathrm{d}s<\infty$. Then, the solution of the following system of differential equations

$$y(t)=y_0+\int_{t_0}^{t}B(\tau,y(s))u(s)\mathrm{d}s, \tag{6}$$

$$\bar{y}(t)=\bar{y}_1+\sum_{j=1}^{r}\int_t^{t_1}B_{jx}(\tau,y(s))^\top u_j(s)\bar{y}(s)\mathrm{d}s, \tag{7}$$

$$\tilde{y}(t)=\int_t^{t_1}\bar{y}(s)^\top B_t(\tau,y(s))u(s)\mathrm{d}s, \tag{8}$$

$$\begin{aligned} Y(t)={}&Y_1+\sum_{j=1}^{r}\int_t^{t_1}[B_{jx}(\tau,y(s))^\top Y(s)+Y(s)B_{jx}(\tau,y(s))]u_j(s)\mathrm{d}s\\ &+\sum_{j=1}^{n}\sum_{i=1}^{r}\int_t^{t_1}\bar{y}_j(s)B_{jixx}(\tau,y(s))u_i(s)\mathrm{d}s, \end{aligned} \tag{9}$$

for $t \in [t_0, t_1]$ is given by

$$y(t) = y_0 + \Psi\left(\tau, y(t_0), \int_{t_0}^{t} u(s)\mathrm{d}s\right), \tag{10}$$

$$\bar{y}(t) = \left[I_n + \Psi_x\left(\tau, y(t), \int_{t}^{t_1} u(s)\mathrm{d}s\right)\right]^\top \bar{y}(t_1), \tag{11}$$

$$\tilde{y}(t) = \bar{y}(t)^\top \Psi_t\left(\tau, y(t_0), \int_{t_0}^{t} u(s)\mathrm{d}s\right), \tag{12}$$

$$\begin{aligned} Y(t) = {} & \left[I_n + \Psi_x\left(\tau, y(t), \int_{t}^{t_1} u(s)\mathrm{d}s\right)\right]^\top Y(t_1)\left[I_n + \Psi_x\left(\tau, y(t), \int_{t}^{t_1} u(s)\mathrm{d}s\right)\right] \\ & + \sum_{j=1}^{n} \Psi_{jxx}\left(\tau, y(t), \int_{t}^{t_1} u(s)\mathrm{d}s\right)\bar{y}_j(t_1). \end{aligned} \tag{13}$$

*Proof*. In the context of deterministic optimal control problems, it is known that the solution of equation (6) (respectively equations (7), (8)) is given by equation (10) (respectively equations (11), (12)) (see for example the reference [29]). We have only to show that $Y(t)$ defined in (13) is solution of equation (9). As it will appear in the Proposition 4.2 and its proof, this additional equation comes from the fact that in stochastic optimal control problems, the maximum principle is given in terms of two adjoint variables (first and second-order adjoint variables) by opposition to deterministic control problems where only one adjoint variable is necessary.

Denote by $\Phi(\tau, y_0, t_0, t) = y_0 + \Psi(\tau, y_0, \int_{t_0}^{t} u(s)\mathrm{d}s)$. The partial derivative of $\Phi$ with respect to the second variable will be denoted by $\Phi_x$ and the second partial derivative of $\Phi_i$ (for $i \in \mathbb{N}_n$) with respect to the second variable will be denoted by $\Phi_{ixx}$. From equations (6) and (10), it satisfies the following equation

$$\Phi(\tau, y_0, t_0, t) = y_0 + \int_{t_0}^{t} B(\tau, \Phi(\tau, y_0, t_0, s))u(s)\mathrm{d}s. \tag{14}$$

Combining equations (7), and (14), we obtain by direct calculation that

$$\begin{aligned} \sum_{i=1}^{n} \Phi_{ixx}(\tau, y_0, t_0, t)\bar{y}_i(t) = {} & \int_{t_0}^{t} \Phi_x(\tau, y_0, t_0, s)^\top \\ & \times \left[\sum_{i=1}^{n}\sum_{j=1}^{r} \bar{y}_i(s)B_{ijxx}(\tau, \Phi(\tau, y_0, t_0, s))u_j(s)\right]\Phi_x(\tau, y_0, t_0, s)\,\mathrm{d}s. \end{aligned} \tag{15}$$

Moreover, from equation (14) it follows that

$$\Phi_x^{-1}(\tau, y_0, t_0, t) = I_n - \int_{t_0}^{t} \Phi_x^{-1}(\tau, y_0, t_0, s)\left[\sum_{j=1}^{r} B_{jx}(\tau, \Phi(\tau, y_0, t_0, s))u_j(s)\right]. \tag{16}$$

Define for $Z_0 \in \mathcal{S}_n$

$$Z(t) = \Phi_x^{-1}(\tau, y_0, t_0, t)^\top \left[ Z_0 - \sum_{i=1}^{n} \Phi_{ixx}(\tau, y_0, t_0, t)\bar{y}_i(t) \right] \Phi_x^{-1}(\tau, y_0, t_0, t). \tag{17}$$

Remark that $Z(t_0) = Z_0$ since $\Phi_{ixx}(\tau, y_0, t_0, t_0) = 0_{n,n}$ and $\Phi_x(\tau, y_0, t_0, t_0) = I_n$. Now, by using equations (15), and (16) we obtain that $Z(t)$ satisfies the differential equation defined by (9) with the initial condition $Z(t_0) = Z_0$. However, equation (17) implies that

$$Z(t_0) = \Phi_x(\tau, y_0, t_0, t)^\top Z(t) \Phi_x^{-1}(\tau, y_0, t_0, t) + \sum_{i=1}^{n} \Phi_{ixx}(\tau, y_0, t_0, t)\bar{y}_i(t),$$

showing that

$$Z(t) = \Phi_x(\tau, y(t), t, t_1)^\top Y_1 \Phi_x^{-1}(\tau, y(t), t, t_1) + \sum_{i=1}^{n} \Phi_{ixx}(\tau, y(t), t, t_1)\bar{y}_i(t_1), \tag{18}$$

is the solution of the differential equation (9) with the terminal condition $Y_1$. However by definition of $\Phi$, we obtain that $\Phi_x(\tau, y(t), t, t_1) = I_n + \Psi_x(\tau, y(t), \int_t^{t_1} u(s)\mathrm{d}s)$, and for $i \in \mathbb{N}_n$, $\Phi_{ixx}(\tau, y(t), t, t_1) = \Psi_{ixx}(\tau, y(t), \int_t^{t_1} u(s)\mathrm{d}s)$ giving the result. □

LEMMA 2.2. There exists a constant $k$ such that $(\forall(\tau, x, y, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times B_M(\mathbb{R}^r))$,

$$|\Psi(\tau, x, u)| \leq k \ (1 + |x|) \ |u| \tag{19}$$

$$|\Psi(\tau, x, u) - \Psi(\tau, y, u)| \leq k \ |x - y| \ |u| \tag{20}$$

$$|(I_n + \Psi_x(\tau, x, u)^\top)^{-1} - I_n| \leq k \ |u| \tag{21}$$

*Proof*. From Theorem 2.1, it follows that $\Psi(\tau, x, tu) = \int_0^t B(\tau, x + \Psi(\tau, x, su))u\,\mathrm{d}s$. Therefore, by using hypothesis A3, we have

$$\begin{aligned} |\Psi(\tau, x, tu)| &\leq \int_0^t |B(\tau, x + \Psi(\tau, x, su)) - B(\tau, x)||u|\mathrm{d}s + \int_0^t |B(\tau, x)||u|\mathrm{d}s \\ &\leq k\int_0^t \Psi(\tau, x, su)|\mathrm{d}s + k(1 + |x|)|u|, \end{aligned}$$

for a constant $k$ (which depends on $T$ and $M$). An application of Gronwall's Lemma shows equation (19).

Moreover, there exists a constant $k_1$ such that

$$\begin{aligned} |\Psi(\tau, x, tu) - \Psi(\tau, y, tu)| &\leq \int_0^T |B(\tau, x + \Psi(\tau, x, su)) - B(\tau, y + \Psi(\tau, y, su))||u|\mathrm{d}s \\ &\leq k_1|x - y| \ |u| + k_1|u|\int_0^t |\Psi(\tau, x, su) - \Psi(\tau, y, su)|\mathrm{d}s, \end{aligned}$$

by using the fact that the first derivative of $B$ is bounded. An application of Gronwall's Lemma shows equation (20).

From equation (16), we have that

$$\left(I_n + \Psi_x(\tau, x, tu)^\top\right)^{-1} = I_n - \int_0^t \left(I_n + \Psi_x(\tau, x, su)^\top\right)^{-1}\left[\sum_{j=1}^r B_{jx}(\tau, x + \Psi(\tau, x, su))u_j\right]\mathrm{d}s,$$

implying that

$$(I_n + \Psi_x(\tau,x,tu)^\top)^{-1} - I_n = -\int_0^t \left[\sum_{j=1}^r B_{jx}(\tau,x+\Psi(\tau,x,su))u_j\right]\mathrm{d}s$$
$$-\int_0^t \left[(I_n + \Psi_x(\tau,x,su)^\top)^{-1} - I_n\right]\left[\sum_{j=1}^r B_{jx}(\tau,x+\Psi(\tau,x,su))u_j\right]\mathrm{d}s.$$

Again combining the fact that the first derivative of $B$ is bounded and Gronwall's Lemma, we obtain easily equation (21). □

Proposition 2.3. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a filtered probability space satisfying the usual hypotheses. Assume that $\{W(t)\}$ is a standard $m$-dimensional $\{\mathcal{F}_t\}$-Brownian motion and $\{u(t), v(t)\}$ is a $B_1(K) \times \mathbb{R}_+$-valued, *corlol*, $\{\mathcal{F}_t\}$-progressively measurable process such that $\{v(t)\}$ is increasing and satisfies $v(T) \leq M$. Let $\{\tau_n\}_{n\in\mathbb{N}^*}$ be the sequence of $\{\mathcal{F}_t\}$-stopping times which exhausts the jumps of $\{v(t)\}$. Then the following stochastic differential equation

$$x(t) \doteq \zeta + \int_0^t A(s, x(s))\mathrm{d}s + \int_{[0,t]} B(s, x(s))u(s)\mathrm{d}v^c(s) + \int_0^t D(s, x(s))\mathrm{d}W(s)$$
$$+ \sum_{n\in\mathbb{N}^*} \Psi(\tau_n, x(\tau_n -), u(\tau_n)\Delta v(\tau_n))I_{\{\tau_n \leq t\}}, \tag{22}$$

admits a unique solution in the set of corlol $\{\mathcal{F}_t\}$-adapted processes.

*Proof*. Clearly, we have that $\{x(t)\}$ is a solution of equation (22) if and only if $\{x(t)\}$ is a solution of the following eqution

$$x(t) \doteq \zeta + \int_0^t A(s, x(s))\mathrm{d}s + \int_{[0,t]} B(s, x(s))u(s)\mathrm{d}v^c(s) + \int_0^t D(s, x(s))\mathrm{d}W(s)$$
$$+ \int_0^t \bar{\Psi}(s, x(s-), y)\mu(\mathrm{d}s, \mathrm{d}y),$$

where $\bar{\Psi}(s, x, y) \doteq \Psi(s, x, y)(I_{\{|y|>0\}}/|y|)$, and $\mu$ is the optional random measure on $\mathbb{R}_+ \times \mathbb{R}^r$ defined by $\mu(\mathrm{d}t, \mathrm{d}y) = \sum_n |u_{\tau_n}\Delta v_{\tau_n}|\delta_{(\tau_n, u_{\tau_n}\Delta v_{\tau_n})}(\mathrm{d}t, \mathrm{d}y)$. From hypotheses A1, A3, A4 and by using equations (19)–(20), it is easy to check that the hypotheses of Theorem 3 in Ref. [49] are satisfied showing the existence and the uniqueness of a solution of (22) in the set of corlol $\{\mathcal{F}_t\}$-adapted processes. □

PROPOSITION 2.4. Assume that $\{Y(t), Z(t)\} \in L^2(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^W\}; [0,T]; \mathbb{R}^n \times \mathbb{R}^{n\times m})$ with $\{Y(t)\}$ *corlol* is solution of the following backward stochastic differential equation

$$\mathrm{d}Y(t) = A(t)Y(t)\mathrm{d}t + U(t)\mathrm{d}t + B(t)Y(t)\mathrm{d}v^c(t) + C(t)Y(t-)\mathrm{d}v^d(t)$$

$$+ \sum_{j=1}^{m} D_j(t)Z_j(t)\mathrm{d}t + Z(t)\mathrm{d}W(t) \tag{23}$$

with $Y(T) = \zeta \in \mathbb{R}^n$, $\{v(t)\}$ an increasing corlol, $\mathcal{F}_t^W$-progressively measurable process ($v(T) \le M$), and $\{(\alpha^*(t), \theta^*(t))\}$ (respectively $\{D_j(t)\}$ for $j \in \mathbb{N}_m$) an $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ (respectively $\mathbb{R}^{n\times n}$) $\mathcal{F}_t^W$-progressively measurable process such that $|A(t)| + |B(t)| + |C(t)| + \sum_{j=1}^m |D_j(t)| \le K$, for a constant $K$, and $\{U(t)\} \in L^2(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^W\}; [0,T]; \mathbb{R}^n)$.

Then the solution is unique in this class of processes.

*Proof*. Let $\{X(t)\}$ be the unique solution of the following stochastic differential equation

$$\mathrm{d}X(t) = -X(t-)A(t)\mathrm{d}t - X(t-)B(t)\mathrm{d}v^c(t) - X(t-)C(t)[1 + \Delta v(t)]^{-1}\mathrm{d}v^d(t)$$

$$- \sum_{j=1}^{m} X(t-)D_j(t)\mathrm{d}W^j(t)$$

with $X(0) = I_n$. It is easy to show that $E_P[\sup_{t\in[0,T]}|X(t)|^2] < +\infty$, and the inverse matrix of $X(t)$ exists. Now, it follows that

$$\mathrm{d}X(t)Y(t) = X(t)U(t)\mathrm{d}t + X(t)Z(t)\mathrm{d}W(t) - \sum_{j=1}^{m} X(t)D_j(t)Y(t)\mathrm{d}W^j(t).$$

By using the fact that

$$E_P\left[\sqrt{\int_0^T |X(t)Y(t)|^2 + |X(t)Z(t)|^2}\right] < +\infty,$$

it gives that

$$X(t)Y(t) = E_P\left[X(T)\zeta - \int_0^T X(s)U(s)\mathrm{d}s|\mathcal{F}_t^W\right] + \int_0^t X(s)U(s)\mathrm{d}s.$$

Consequently, if $\{\hat{Y}(t), \hat{Z}(t)\} \in L^2(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^W\}; [0,T]; \mathbb{R}^n \times \mathbb{R}^{n\times m})$ with $\{\hat{Y}(t)\}$ *corlol* is an other solution of (23), then $Y(t) = \hat{Y}(t)$ since $X(t)$ is invertible and it implies that $Z(t) = \hat{Z}(t)$, showing the result. □

## 3. Problem statement

In this section, we formulate the original singular stochastic control problem presented in the introduction using the formulation described in Refs. [50,51].

Definition 3.1. A singular control is defined by the following term:

$$C \doteq (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, \{u(t), v(t)\}, \{W(t)\}, \{x(t)\})$$

where

(i) $(\Omega, \mathcal{F}, P)$ is a complete probability space with a right continuous complete filtration $\{\mathcal{F}_t\}$.
(ii) $\{W(t)\}$ is a standard $m$-dimensional $\{\mathcal{F}_t\}$-Brownian motion.
(iii) $\{u(t),\ v(t)\}$ is a $B_1(K) \times \mathbb{R}_+$-valued, *corlol*, $\{\mathcal{F}_t\}$-progressively measurable process such that $\{v(t)\}$ is increasing and satisfies

$$v(T) \leq M. \tag{24}$$

(iv) $\{x(t)\}$ is an $\mathbb{R}^n$-valued, corlol $\{\mathcal{F}_t\}$-progressively measurable process such that ($\forall t \in [0, T]$)

$$x(t) \doteq \zeta + \int_0^t A(s, x(s))\mathrm{d}s + \int_{[0,t]} B(s, x(s))u(s)\mathrm{d}v^c(s) + \int_0^t D(s, x(s))\mathrm{d}W(s) + \sum_{n \in \mathbb{N}^*} \Psi(\tau_n, x(\tau_n-), u(\tau_n)\Delta v(\tau_n))I_{\{\tau_n \leq t\}}, \tag{25}$$

where $\{\tau_n\}_{n \in \mathbb{N}^*}$ denotes the sequence of $\{\mathcal{F}_t\}$-stopping times which exhausts the jumps of $\{v(t)\}$, and $x(0-) = \zeta$.

We write $\mathfrak{C}$ for the set of controls satisfying the previous conditions.

The cost is given by

$$J[C] \doteq E_P\left[G\left(\int_0^T |u(s)|\mathrm{d}v(s), x(T)\right)\right]. \tag{26}$$

The set $\mathfrak{C}^a$ of admissible controls is defined by

$$\mathfrak{C}^a \doteq \{C \in \mathfrak{C} : J[C] < \infty\}. \tag{27}$$

The singular control problem is defined by the minimization of $J[C]$ on $\mathfrak{C}^a$. Assuming the existence of an optimal singular control $\tilde{C}$, the aim of the paper is to derive necessary conditions for $\tilde{C}$ to be optimal in terms of variational inequalities (see the maximum principle presented in Theorem 5.5).

## 4. The maximum principle for the auxiliary control variables

In this section, we present a maximum principle in terms of the auxiliary state and control variables. These results are presented here with minimal details in order to be concise. The interested reader may consult the reference [1,2] to have a complete description of this approach.

In general terms, the approach we used to obtain the maximum principle for singular control problems (completely described in Refs. [1,2]) can be divided in three steps. The first step is to convert with a time transformation the original singular control problem into an auxiliary classical control problem. The second step is to derive the maximum principle in terms of these new auxiliary state and control variables. The last step consisting to recover from the auxiliary maximum principle the original state and control variables by using a time change, and thus giving a maximum principle for the singular control problem (see Theorem 5.5) will be presented in more details in the next section.

In Proposition 4.2, it is shown that by using a time change technique the original control problem can be converted into an auxiliary classical control problem. This result is of central importance and the fundamental difference with respect to the approach used previously by the authors in Ref. [1] is that now the gain of the control depends on the state variable (the maps $B$ and $\Psi$ depend on $x$) implying some technical difficulties. In particular, it appears in the proof of Proposition 4.2 that the results of Theorem 2.1 are extremely important for applying our time change approach.

Assume the existence of an optimal singular control denoted by

$$\tilde{C} \doteq (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_t\}, \{\tilde{u}(t), \tilde{v}(t)\}, \{\hat{W}(t)\}, \{\tilde{x}(t)\}).$$

such that $\{\tilde{u}(t), \tilde{v}(t)\}$ is $\{\hat{\mathcal{F}}_t^{\hat{W}}\}$-progressively measurable.

With the next Proposition, we show how it is possible to construct an optimal singular control $\hat{C}$ satisfying $\hat{v}(T) = M$ from the optimal singular control $\tilde{C}$.

PROPOSITION 4.1. The control $\hat{C}$ defined by

$$\hat{C} \doteq (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_t\}, \{\hat{u}(t), \hat{v}(t)\}, \{\hat{W}(t)\}, \{\hat{x}(t)\}) \tag{28}$$

where

$$\hat{v}(t) = \tilde{v}(t) I_{[\![0,T[\![} + (M - \hat{v}(T) + \tilde{v}(t)) I_{[\![T,+\infty[\![}, \tag{29}$$

$$\hat{u}(t) = \tilde{u}(t) I_{[\![0,T[\![} + \tilde{u}(t)\left[\frac{\tilde{v}(T) - \tilde{v}(T-)}{M - \tilde{v}(T-)} I_{[T,+\infty[\times\{\tilde{v}(T)<M\}} + I_{[T,+\infty[\times\{\tilde{v}(T)=M\}}\right], \tag{30}$$

is optimal. Moreover, $\hat{v}(T) = M$, and $\{\hat{u}(t), \hat{v}(t)\}$ is a $\{\hat{\mathcal{F}}_t^{\hat{W}}\}$-progressively measurable process.

*Proof*. From the definition of $\{\hat{u}(t)\}$, and $\{\hat{v}(t)\}$, we have that $\hat{v}(T) = M$, and

$$(\forall t \in [0, T[), \quad \hat{u}(t) = \tilde{u}(t), \quad \hat{v}(t) = \tilde{v}(t), \quad \text{and} \quad \hat{u}(T)\Delta\hat{v}(T) = \tilde{u}(T)\Delta\tilde{v}(T), \tag{31}$$

consequently, we obtain that $\hat{x}(T) = \tilde{x}(T)$, and $\int_{[0,T]} |\hat{u}(t)| \mathrm{d}\hat{v}(t) = \int_{[0,T]} |\tilde{u}(t)| \mathrm{d}\tilde{v}(t)$, showing that $J[\hat{C}] = J[\tilde{C}]$. Clearly, $\{\hat{u}(t), \hat{v}(t)\}$ is a $B_1(K) \times \mathbb{R}_+$-valued, *corlol*, $\{\hat{\mathcal{F}}_t^{\hat{W}}\}$-progressively measurable process implying that $\hat{C}$ is optimal. □

*Remark* 1. Now, we will work with the optimal control $\hat{C}$ for technical reasons. However, by using Propositions 4.1 and 5.4, a general stochastic maximum principle will be derived in terms of the optimal control $\tilde{C}$ giving the full generality to our result (see Theorem 5.5).

PROPOSITIONS 4.2. Denote the process $\{t + \hat{v}(t)\}$ by $\{\hat{\Gamma}(t)\}$. Let $\{\eta^*(t)\}$ be the right inverse of $\{\hat{\Gamma}(t)\}$. Then, $\{\eta^*(t)\}$ is a continuous time change such that the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_{\eta^*(t)}^{\hat{W}}\})$ satisfies the usual hypotheses. Moreover, there exists a [0,1]-valued,

$\{\hat{\mathcal{F}}_t^{\tilde{W}}\}$-progressively measurable process $\{\hat{z}(t)\}$ such that

$$\hat{v}(t) = \int_{[0,t]} \hat{z}(s)\mathrm{d}\hat{\Gamma}(s). \tag{32}$$

Define the $B_1(K) \times [0,1]$-valued, $\{\hat{\mathcal{F}}^{\hat{W}}_{\eta^*(t)}\}$-progressively measurable process $\{(\alpha^*(t), \theta^*(t))\}$ by

$$\alpha^*(t) = \hat{u}(\eta^*(t)) \quad \text{and} \quad \theta^*(t) = \hat{z}(\eta^*(t)). \tag{33}$$

Then

$$\eta^*(t) = \int_0^t (1 - \theta^*(s))\mathrm{d}s, \tag{34}$$

$$J[\hat{C}] = E_{\hat{P}}[G(\mu^*(T+M), \xi^*(T+M))], \tag{35}$$

$$E_{\hat{P}}[N(\eta^*(T+M))] = 0, \tag{36}$$

where the processes $\{\xi^*(t)\}$ and $\{\mu^*(t)\}$ are solution of the following equations

$$\xi^*(t) \doteq \zeta + \int_0^t A(\eta^*(s), \xi^*(s))(1 - \theta^*(s))\mathrm{d}s + \int_0^t \theta^*(s)B(\eta^*(s), \xi^*(s))\alpha^*(s)\mathrm{d}s$$
$$+ \int_0^t D(\eta^*(s), \xi^*(s))\mathrm{d}\hat{W}(\eta^*(s)), \tag{37}$$

$$\mu^*(t) \doteq \int_0^t |\alpha^*(s)|\theta^*(s)\mathrm{d}s. \tag{38}$$

Moreover,

$$\hat{x}(t) = \xi^*(\hat{\Gamma}(t)) \quad \text{and} \quad \mu^*(T+M) = \int_{[0,T]} |\hat{u}(s)|\mathrm{d}\hat{v}(s) \tag{39}$$

*Proof.* By hypothesis, $\{\hat{\Gamma}(t)\}$ is a $\{\hat{\mathcal{F}}^{\hat{W}}_{\eta^*(t)}\}$-progressively measurable process. Consequently, it follows that $\{\eta^*(t)\}$ is a time change on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_t^{\hat{W}}\})$, from Proposition 1.1 in Ref. [52, Chapter V]. Clearly, the filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}^{\hat{W}}_{\eta^*(t)}\})$ satisfies the usual hypotheses. Following Proposition 3.1 in Ref. [2], it is easy to show the existence of a [0,1]-valued, $\{\hat{\mathcal{F}}_t^{\hat{W}}\}$-progressively measurable process $\{\hat{z}(t)\}$ satisfying equation (32). From proposition 1.4 in Ref. [52, Chapter V], $\{\alpha^*(t), \theta^*(t)\}$ is a $\{\hat{\mathcal{F}}^{\hat{W}}_{\eta^*(t)}\}$-progressively measurable process. By using Proposition 3.2 in Ref. [2], equation (34) follows. Combining equation (32) and Proposition 4.9 in Ref. [52, page 8], we have

$$\hat{v}(t) = \int_0^{\hat{\Gamma}_t} \theta^*(s)\mathrm{d}s, \tag{40}$$

and

$$\int_{[0,t]} |\hat{u}(s)|\mathrm{d}\hat{v}(s) = \int_0^{\hat{\Gamma}(t)} |\alpha^*(s)|\theta^*(s)\mathrm{d}s. \tag{41}$$

Using equations (34) and (40), it follows that $\eta^*(T+M) = T + M - \int_0^{\hat{\Gamma}(T)} \theta^*(s)\mathrm{d}s = T$ since $\hat{v}(T) = M$. It is easy to show that on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}^{\hat{W}}_{\eta^*(t)}\})$ the stochastic differential equations (37)–(38) admit a unique solution. Now, let us show that $\hat{x}(t) = \xi^*(\hat{\Gamma}(t))$. By using Proposition (1.4) in Ref. [52, p. 180], and Proposition 4.8 in Ref. [53], it follows easily that

$$\int_0^{\hat{\Gamma}(t)} A(\eta^*(s), \xi^*(s))(1 - \theta^*(s))\mathrm{d}s = \int_0^t A(s, \xi^*(\hat{\Gamma}(s))\mathrm{d}s, \tag{42}$$

and

$$\int_0^{\hat{\Gamma}(t)} D(\eta^*(s), \xi^*(s))\mathrm{d}\hat{W}(\eta^*(s)) = \int_0^t D(s, \xi^*(\hat{\Gamma}(s))\mathrm{d}\hat{W}(s) \tag{43}$$

Moreover, defining $\mathcal{D} \doteq [0, T+M] \times \Omega - \bigcup_{n=1}^{\infty}[[\hat{\Gamma}(\tau_n -), \hat{\Gamma}(\tau_n)]]$, and

$$\gamma^*(t) \doteq \int_0^t I_{\mathcal{D}}(s, \omega)\theta^*(s)\mathrm{d}s, \tag{44}$$

we obtain that

$$\begin{aligned}\int_0^{\hat{\Gamma}(t)} \theta^*(s)B(\eta^*(s), \xi^*(s))\alpha^*(s)\mathrm{d}s &= \int_{\hat{\Gamma}(0-)}^{\hat{\Gamma}(t)} \theta^*(s)B(\eta^*(s), \xi^*(s))\alpha^*(s)\mathrm{d}s \\ &= \int_{\hat{\Gamma}(0-)}^{\hat{\Gamma}(t)} B(\eta^*(s), \xi^*(s))\alpha^*(s)\mathrm{d}\gamma^*(s) \\ &\quad + \sum_{n=1}^{\infty} \int_{\hat{\Gamma}(\tau_n-)}^{\hat{\Gamma}(\tau_n)} B(\eta^*(s), \xi^*(s))\alpha^*(s)\mathrm{d}s I_{\{\tau_n \leq t\}}\end{aligned}$$

By definition $\{\gamma^*(t)\}$ is clearly a $\{\hat{\Gamma}(t)\}$-continuous process. Consequently, by using Proposition (1.4) in Ref. [52, p. 180], we obtain

$$\begin{aligned}\int_0^{\hat{\Gamma}(t)} \theta^*(s)B(\eta^*(s), \xi^*(s))\alpha^*(s)\mathrm{d}s &= \int_0^t B(s, \xi^*(\hat{\Gamma}(s))\alpha^*(\hat{\Gamma}(s))\mathrm{d}\gamma^*(\hat{\Gamma}(s)) \\ &\quad + \sum_{n=1}^{\infty} \int_{\hat{\Gamma}(\tau_n-)}^{\hat{\Gamma}(\tau_n)} B(\eta^*(s), \xi^*(s))\alpha^*(s)\mathrm{d}s I_{\{\tau_n \leq t\}}\end{aligned}$$

However, from equation (32) we have $\alpha^*(\hat{\Gamma}(s)) = \hat{u}(s)$. Moreover by using equation (40) we obtain $\gamma^*(\hat{\Gamma}(t)) = \hat{v}^c(t)$. Consequently, it follows that

$$\begin{aligned}\int_0^{\hat{\Gamma}(t)} \theta^*(s)B(\eta^*(s), \xi^*(s))\alpha^*(s)\mathrm{d}s &= \int_0^t B(s, \xi^*(\hat{\Gamma}(s))\hat{u}(s)\mathrm{d}\hat{v}^c(s) \\ &\quad + \sum_{n=1}^{\infty} \int_{\hat{\Gamma}(\tau_n-)}^{\hat{\Gamma}(\tau_n)} B(\eta^*(s), \xi^*(s))\alpha^*(s)\mathrm{d}s I_{\{\tau_n \leq t\}}.\end{aligned} \tag{45}$$

Now combining equations (37), and (42)–(45), we have that

$$\xi^*(\hat\Gamma(t)) = \zeta + \int_0^t A(s, \xi^*(\hat\Gamma(s))\mathrm{d}s + \int_0^t B(s, \xi^*(\hat\Gamma(s))\hat u(s)\mathrm{d}\hat v^c(s) + \sum_{n=1}^{\infty}\int_{\hat\Gamma(\tau_n-)}^{\hat\Gamma(\tau_n)} B(\eta^*(s), \xi^*(s))\alpha^*(s)\mathrm{d}s I_{\{\tau_n\le t\}} + \int_0^t D(s, \xi^*(\hat\Gamma(s))\mathrm{d}\hat W(s). \quad (46)$$

Now, from equation (37), we have that for all $(t,\omega) \in [0, T+M]\times\Omega \cap [[\hat\Gamma(\tau_n-), \hat\Gamma(\tau_n)]]$,

$$\xi^*(t) = \xi^*(\hat\Gamma(\tau_n-)) + \int_{\hat\Gamma(\tau_n-)}^t B(\eta^*(s), \xi^*(s))\alpha^*(s)\mathrm{d}s = \xi^*(\hat\Gamma(\tau_n-)) + \int_{\hat\Gamma(\tau_n-)}^t B(\tau_n, \xi^*(s))\hat u(\tau_n)\mathrm{d}s.$$

However, using equations (6), and (10) in Theorem 2.1, we obtain that

$$\xi^*(t) = \xi^*(\hat\Gamma(\tau_n-)) + \Psi(\tau_n, \xi^*(\hat\Gamma(\tau_n-)), (t-\hat\Gamma(\tau_n-))\hat u(\tau_n)),$$

for all $(t,\omega) \in [0, T+M]\times\Omega \cap [[\hat\Gamma(\tau_n-), \hat\Gamma(\tau_n)]]$ implying that

$$\int_{\hat\Gamma(\tau_n-)}^{\hat\Gamma(\tau_n)} B(\tau_n, \xi^*(s))\hat u(\tau_n)\mathrm{d}s = \Psi(\tau_n, \xi^*(\hat\Gamma(\tau_n-)), \hat u(\tau_n)\Delta\hat v(\tau_n)) \quad (47)$$

Now, combining equations (46), (47), it follows that $\{\xi^*(\hat\Gamma(t))\}$ is a solution of equation (25) and uniqueness, it follows that $\hat x(t) = \xi^*(\hat\Gamma(t))$.

Moreover, using equation (41), we obtain $\mu^*(T+M) = \int_{[0,T]} |\hat u(s)|\mathrm{d}\hat v(s)$, giving the result. □

Let $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde P, \{\tilde{\mathcal{F}}_t\})$ be a filtered probability space satisfying the usual hypotheses and supporting a standard $m$-dimensional Brownian motion $\{\tilde V_t\}$. Define by $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$, the usual augmentation of the filtered probability space $\{\hat\Omega\times\tilde\Omega, \hat{\mathcal{F}}\otimes\tilde{\mathcal{F}}, \hat P\otimes\tilde P, \hat{\mathcal{F}}^{\hat W}_{\eta^*(t)}\otimes\tilde{\mathcal{F}}_t\}$.

A random variable $\hat X$ defined on $(\hat\Omega, \hat{\mathcal{F}}, \hat P)$ may be viewed as defined on $(\Omega, \mathcal{G}, Q)$ by setting $\bar X(\hat\omega, \tilde\omega) = \hat x(\hat\omega)$ for $(\hat\omega, \tilde\omega) \in \hat\Omega\times\tilde\Omega$ and similarly for a random variable defined on $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde P)$. Consequently, let us introduce on $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$ the following processes:

$$\begin{cases} \bar\alpha(t,\hat\omega,\tilde\omega) \doteq \alpha^*(t,\hat\omega), & \bar\theta(t,\hat\omega,\tilde\omega) \doteq \theta^*(t,\hat\omega), \quad \bar\eta(t,\hat\omega,\tilde\omega) \doteq \eta^*(t,\hat\omega), \\ \bar\xi(t,\hat\omega,\tilde\omega) \doteq \xi^*(t,\hat\omega), & \bar\mu(t,\hat\omega,\tilde\omega) \doteq \mu^*(t,\hat\omega), \quad \bar W(t,\hat\omega,\tilde\omega) \doteq \hat W(t,\hat\omega), \\ & \tilde W(t,\hat\omega,\tilde\omega) \doteq \tilde V(t,\tilde\omega). \end{cases} \quad (48)$$

Proposition 4.3. On $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$, the process $\{\bar V(t)\}$ defined by

$$\bar V(t) \doteq \int_0^t \sqrt{(1-\bar\theta(s))^+}\mathrm{d}\bar W(\bar\eta(s)) + \int_0^t \sqrt{1-(1-\bar\theta(s))(1-\bar\theta(s))^+}\mathrm{d}\tilde W(s) \quad (49)$$

is a standard $m$-dimensional $\{\mathcal{G}_t\}$-Brownian motion. The process $\{\bar{\xi}(t), \bar{\eta}(t), \bar{\mu}(t)\}$ is the unique solution of the following equations

$$\bar{\xi}(t) = \zeta + \int_0^t A(\bar{\eta}(s), \bar{\xi}(s))[1 - \bar{\theta}(s)]\mathrm{d}s + \int_0^t \bar{\theta}(s)B(\bar{\eta}(s), \bar{\xi}(s))\bar{\alpha}(s)\mathrm{d}s + \int_0^t D(\bar{\eta}(s), \bar{\xi}(s))\sqrt{1 - \bar{\theta}(s)}\mathrm{d}\bar{V}(s), \tag{50}$$

$$\bar{\eta}(t) = \int_0^t [1 - \bar{\theta}(s)]\mathrm{d}s, \tag{51}$$

$$\bar{\mu}(t) = \int_0^t |\bar{\alpha}(s)|\bar{\theta}(s)\mathrm{d}s, \tag{52}$$

Moreover,

$$J[\bar{C}] = E_Q\big[G\big(\bar{\mu}(T+M), \bar{\xi}(T+M)\big)\big], \tag{53}$$

$$E_Q\big[N(\bar{\eta}(T+M))\big] = 0_2. \tag{54}$$

*Proof*. From Theorem 2.75 in Ref. [54], it follows that $\{\bar{V}(t)\}$ is a standard $m$-dimensional $\{\mathcal{G}_t\}$-Brownian motion. Using Theorem 6 in Ref. [55, page 194], we obtain that the solution $\{\xi(t), \eta(t), \mu(t))\}$ of equations (50)–(52) exists and is unique. From the definition of $\{\bar{V}(t)\}$, it is easy to show that $\{\xi(t)\}$ is the unique solution of the following equation:

$$\xi(t) \doteq \zeta + \int_0^t A(\bar{\eta}(s), \xi(s))(1 - \bar{\theta}(s))\mathrm{d}s + \int_0^t \bar{\theta}(s)B(\bar{\eta}(s), \bar{\xi}(s))\bar{\alpha}(s)\mathrm{d}s + \int_0^t D(\bar{\eta}(s), \xi(s))\sqrt{(1 - \bar{\theta}(s))(1 - \bar{\theta}(s))^+}\mathrm{d}\hat{W}(\bar{\eta}(s)). \tag{55}$$

However, combining the fact that the process $\{\xi^*(t)\}$ is solution of equation (37) and Proposition 10.46 in Ref. [54], it follows that $\{\bar{\xi}(t)\}$ satisfies equation (50). Therefore, $\{\bar{\xi}(t)\}$ is the unique solution of equation (50). Moreover, it is clear from their definitions that the processes $\{\mu(t)\}$ and $\{\bar{\mu}(t)\}$ (respectively $\{\eta(t)\}$ and $\{\bar{\eta}(t)\}$) are indistinguishable. Finally, equations (53) and (54) follow easily from equations (35) and (36) and the definition of the probability $Q$. □

On the probability space $(\Omega, \mathcal{G}, Q)$, define the filtration $\mathcal{J}_t \doteq \hat{\mathcal{F}}^{\hat{W}}_{\eta^*(t)} \otimes \{\emptyset, \tilde{\Omega}\}$.

The set of auxiliary control $\mathcal{E}$ is the set of $\{\mathcal{J}_t\}$-progressively measurable processes defined on $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$ and taking their value in $B_1(K) \times [0, 1]$. For any $\{(\alpha(t), \theta(t))\}$ in $\mathcal{E}$, the auxiliary state process $(\eta(t), \xi(t), \mu(t))$ is defined on $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$ by

$$\eta(t) \doteq \int_0^t (1 - \theta(s))\mathrm{d}s, \tag{56}$$

$$\xi(t) \doteq \zeta + \int_0^t A(\eta(s), \xi(s))(1 - \theta(s))\mathrm{d}s + \int_0^t \theta(s)B(\eta(s), \xi(s))\alpha(s)\theta(s)\mathrm{d}s$$
$$+ \int_0^t D(\eta(s), \xi(s))\sqrt{(1 - \theta(s))}\mathrm{d}\bar{V}(s), \tag{57}$$

$$\mu(t) \doteq \int_0^t |\alpha(s)|\theta(s)\mathrm{d}s. \tag{58}$$

Note that for any $\{(\alpha(t), \theta(t))\}$ in $\mathcal{E}$, the previous system admits a unique solution. Moreover, we have $E_Q[G(\mu(T+M), \xi(T+M))] < \infty$.

The associated cost functional is defined by

$$\mathcal{M}[\alpha, \theta] \doteq E_Q[G(\mu(T+M), \xi(T+M))]. \tag{59}$$

Definition 4.4. The set of admissible auxiliary control $\mathcal{E}_{\mathrm{ad}}$ is defined by the set of processes $\{(\alpha(t), \theta(t))\} \in \mathcal{E}$ such that the corresponding auxiliary state process $\{(\eta(t), \xi(t), \mu(t))\}$ satisfies the following constraint

$$E_Q\big[N(\eta(T+M))\big] = 0_2. \tag{60}$$

The auxiliary control problem is to minimize the cost (61) over $\mathcal{E}_{\mathrm{ad}}$.

Proposition 4.5. The auxiliary control process $\{(\bar{\alpha}(t), \bar{\theta}(t))\}$ defined by equation (48) is optimal. Moreover, $\{(\bar{\alpha}(t), \bar{\theta}(t))\}$ and the corresponding optimal auxiliary state $\{(\bar{\eta}(t), \bar{\xi}(t), \bar{\mu}(t))\}$ are $\{\mathcal{J}_t\}$-progressively measurable processes.

*Proof*. Using Proposition 4.3 and Theorem 4.6 in Ref. [2], the result follows easily. □

For the rest of the paper, we shall use the following notation for the different filtered probability space under consideration:

$$\hat{\mathbb{F}}^{\hat{W}}_{\eta^*} \doteq \left(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}^{\hat{W}}_{\eta^*(t)}\}\right), \quad \mathbb{G} \doteq (\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\}), \quad \mathbb{J} \doteq (\Omega, \mathcal{G}, Q, \{\mathcal{J}_t\}).$$

We will need the following result in the proof of the next Theorem.

Lemma 4.6. For all $(t, x, u, z) \in \mathbb{R} \times \mathbb{R}^n \times B_1(K) \times [0, 1]$, $(p^1, p^2, p^3) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ and $(q^1, q^2, q^3) \in \mathbb{R}^{1\times m} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{1\times m}$,

$$\mathcal{H}_t(t, x, u, z, p, q) = [A_t(t, x)(1 - z) + zB_t(t, x)u]^\top p^2 + \sum_{j=1}^m D_{jt}(t, x)^\top q_j^2\sqrt{1 - z},$$

$$\mathcal{H}_x(t, x, u, z, p, q) = \left[A_x(t, x)(1 - z) + \sum_{j=1}^r zB_{jx}(t, x)u_j\right]^\top p^2 + \sum_{j=1}^m D_{jx}(t, x)^\top q_j^2\sqrt{1 - z},$$

where

$$p = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix}.$$

*Proof*. One may easily show that these two equalities follow directly from the definition of $\mathcal{H}$ (equation (3)). □

With the previous results, we can now obtain in the following Theorem a crucial property given by equation (65) of the solution of the backward stochastic differential equations (61)–(64) which renders possible a time change.

THEOREM 4.7. On the filtered probability space $\mathbb{G}$, the system of backward stochastic differential equations

$$\mathrm{d}\bar{p}(t) = -\begin{pmatrix} \mathcal{H}_t(\bar{\eta}(t), \bar{\xi}(t), \bar{\alpha}(t), \bar{\theta}(t), \bar{p}(t), \bar{q}(t)) \\ \mathcal{H}_x(\bar{\eta}(t), \bar{\xi}(t), \bar{\alpha}(t), \bar{\theta}(t), \bar{p}(t), \bar{q}(t)) \\ 0 \end{pmatrix} \mathrm{d}t + \bar{q}(t)\mathrm{d}\bar{V}(t), \tag{61}$$

with

$$\bar{p}(T+M) = -\begin{pmatrix} 0 \\ G_x(\bar{\mu}(T+M), \bar{\xi}(T+M)) \\ G_w(\bar{\mu}(T+M), \bar{\xi}(T+M)), \end{pmatrix} \tag{62}$$

and

$$\begin{aligned} \mathrm{d}\bar{P}(t) = &- A_x(\bar{\eta}(t), \bar{\xi}(t))^\top \bar{P}(t)(1-\bar{\theta}(t))\mathrm{d}t - \bar{P}(t)A_x(\bar{\eta}(t), \bar{\xi}(t))(1-\bar{\theta}(t))\mathrm{d}t \\ &- \sum_{j=1}^{r} ([B_{jx}(\bar{\eta}(t), \bar{\xi}(t))]^\top \bar{P}(t) + \bar{P}(t)B_{jx}(\bar{\eta}(t), \bar{\xi}(t)))\bar{\alpha}_j(t)\bar{\theta}(t)\mathrm{d}t \\ &- \sum_{j=1}^{m} [D_{jx}(\bar{\eta}(t), \bar{\xi}(t))]^\top \bar{P}(t)D_{jx}(\bar{\eta}(t), \bar{\xi}(t))(1-\bar{\theta}(t))\mathrm{d}t \\ &- \sum_{j=1}^{m} ([D_{jx}(\bar{\eta}(t), \bar{\xi}(t))]^\top \bar{Q}^j(t) + \bar{Q}^j(t)D_{jx}(\bar{\eta}(t), \bar{\xi}(t)))\sqrt{1-\bar{\theta}(t)}\mathrm{d}t \\ &- \mathcal{H}_{xx}(\bar{\eta}(t), \bar{\xi}(t), \bar{\alpha}(t), \bar{\theta}(t), \bar{p}(t), \bar{q}(t))\mathrm{d}t + \sum_{j=1}^{m} \bar{Q}^j(t)\mathrm{d}\bar{V}_j(t), \end{aligned} \tag{63}$$

with

$$\bar{P}(T+M) = -G_{xx}(\bar{\mu}(T+M), \bar{\xi}(T+M)), \tag{64}$$

admits a unique solution in the following class of processes

$$\{\bar{p}(t), \bar{q}(t), \bar{P}(t), (\bar{Q}^j(t))_{j\in\mathbb{N}_m}\} \in L^2(\mathbb{G}; [0, T+M]; \mathbb{R}^{2+n} \times \mathbb{R}^{(2+n)\times m} \times \mathcal{S}^n \times [\mathcal{S}^n]^m)$$

with $\{\bar{p}(t), \bar{P}(t)\}$ continuous.

Moreover,

$$\begin{cases} \bar{p}(t,\hat{\omega},\tilde{\omega}) = p^*(t,\hat{\omega}), & \bar{q}(t,\hat{\omega},\tilde{\omega}) = \sqrt{1-\theta^*(t,\hat{\omega})}\,q^*(t,\hat{\omega}), \\ \bar{P}(t,\hat{\omega},\tilde{\omega}) = P^*(t,\hat{\omega}), & \bar{Q}^j(t,\hat{\omega},\tilde{\omega}) = \sqrt{1-\theta^*(t,\hat{\omega})}\,Q^{*j}(t,\hat{\omega}), \end{cases} \tag{65}$$

for all $(\hat{\omega},\tilde{\omega}) \in \hat{\Omega}\times\tilde{\Omega}$ and for all $j \in \mathbb{N}_m$, where $\{p^*(t), P^*(t), q^*(t), (Q^{*j}(t))_{j\in\mathbb{N}_m}\}$ is the unique solution in the following class of processes

$$\{p^*(t), P^*(t)\} \in L^2\left(\hat{\mathbb{F}}^{\hat{W}}_{\eta^*}; [0, T+M]; \mathbb{R}^{2+n}\times\mathcal{S}^n\right)$$

$$\{q^*(t), (Q^{*j}(t))_{j\in\mathbb{N}_m}\} \in L^2_{\theta^*}\left(\hat{\mathbb{F}}^{\hat{W}}_{\eta^*}; [0, T+M]; \mathbb{R}^{(2+n)\times m}\times[\mathcal{S}^n]^m\right)$$

with $\{p^*(t), P^*(t)\}$ continuous of the system of backward stochastic differential equations defined on the filtered probability space $\hat{\mathbb{F}}^{\hat{W}}_{\eta^*}$:

$$\mathrm{d}p^*(t) = -\begin{pmatrix} \mathcal{H}_t(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^*(t), \sqrt{1-\theta^*(t)}\,q^*(t)) \\ \mathcal{H}_x(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^*(t), \sqrt{1-\theta^*(t)}\,q^*(t)) \\ 0 \end{pmatrix}\mathrm{d}t$$

$$+\, q^*(t)\,\mathrm{d}\hat{W}(\eta^*(t)) \tag{66}$$

with

$$p^*(T+M) = -\begin{pmatrix} 0 \\ G_x(\mu^*(T+M), \xi^*((T+M)) \\ G_w(\mu^*(T+M), \xi^*(T+M)) \end{pmatrix} \tag{67}$$

and

$$\begin{aligned} \mathrm{d}P^*(t) = &- A_x(\eta^*(t), \xi^*(t))^\top P^*(t)(1-\theta^*(t))\mathrm{d}t - P^*(t)A_x(\eta^*(t), \xi^*(t))(1-\theta^*(t))\mathrm{d}t \\ &- \sum_{j=1}^{r} ([B_{jx}(\eta^*(t), \xi^*(t))]^\top P^*(t) + P^*(t)B_{jx}(\eta^*(t), \xi^*(t)))\alpha^*_j(t)\theta^*(t)\mathrm{d}t \\ &- \sum_{j=1}^{m} [D_{jx}(\eta^*(t), \xi^*(t))]^\top P^*(t) D_{jx}(\eta^*(t), \xi^*(t))(1-\theta^*(t))\mathrm{d}t \\ &- \sum_{j=1}^{m} \big([D_{jx}(\eta^*(t), \xi^*(t))]^\top Q^{*j}(t) + Q^{*j}(t)D_{jx}(\eta^*(t), \xi^*(t))\big)(1-\theta^*(t))\mathrm{d}t \\ &- \mathcal{H}_{xx}(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^*(t), \sqrt{1-\theta^*(t)}\,q^*(t))\mathrm{d}t \\ &+ \sum_{j=1}^{m} Q^{*j}(t)\,\mathrm{d}\hat{W}(\eta^*(t)) \end{aligned} \tag{68}$$

with

$$P^*(T+M) = -G_{xx}(\mu^*(T+M), \xi^*(T+M)). \tag{69}$$

*Proof*. Using Lemma 4.6, equation (61) can be written in the form

$$\bar{p}(t) = -\bar{p}(T+M) - \int_t^{T+M} g(s, \bar{p}(s), \sqrt{1-\bar{\theta}(s)}\bar{q}(s))\mathrm{d}s - \int_t^{T+M} \bar{q}(s)\mathrm{d}\bar{V}_s$$

where the $\mathbb{R}^{n+2}$-valued function $g(t, p, q)$ is defined by

$$-\begin{pmatrix} [A_t(\bar{\eta}(t), \bar{\xi}(t))(1-\bar{\theta}(t)) + \bar{\theta}(t)B_t(\bar{\eta}(t))\bar{\alpha}(t)]^\top p^2 + \sum_{j=1}^m D_{jt}(\bar{\eta}(t), \bar{\xi}(t))^\top q_j^2 \\ \left[A_x(\bar{\eta}(t), \bar{\xi}(t))(1-\bar{\theta}(t)) + \sum_{j=1}^r \bar{\theta}(t)B_{jx}(\bar{\eta}(t), \bar{\xi}(t))\bar{\alpha}_j(t)\right]^\top p^2 + \sum_{j=1}^m D_{jx}(\bar{\eta}(t), \bar{\xi}(t))^\top q_j^2 \\ 0 \end{pmatrix},$$

for

$$p = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix}$$

with $p^1 \in \mathbb{R}$, $p^2 \in \mathbb{R}^n$ and $p^3 \in \mathbb{R}$ and for

$$q = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix}$$

where $q^1 \in \mathbb{R}^{1\times n}$, $q^2 \in \mathbb{R}^{n\times n}$ and $q^3 \in \mathbb{R}^{1\times n}$. Moreover, from equation (62), it follows that $\bar{p}(T+M) \in L^2(\Omega, \mathcal{J}_{T+M}, Q; \mathbb{R}^k)$. Now, using the hypotheses A1, A3 and A4 on the data $A$, $B$ and $D$, it is easy to show that equation (61) satisfies the hypotheses of Corollary 4.2 in Ref. [1], implying the existence and the uniqueness for equation (61). Using similar arguments, the existence and the uniqueness results can be obtained for equation (63). By using Proposition 4.1 in Ref. [1], it can be shown that equations (66)–(69) admit a unique solution. The last part of the Theorem (see equation (65)) follows easily from Corollary 4.2 in Ref. [1]. □

Now we give the maximum principle for the auxiliary control problem.

THEOREM 4.8. There exists $\psi \in S_1(\mathbb{R}^3)$ such that $(\forall(\alpha, \theta) \in B_1(K) \times [0,1])$

$$\begin{aligned} &H(\eta^*(t), \xi^*(t), \alpha, \theta, p^{*1}(t), p^{*2}(t), p^{*3}(t), r^*(t), P^*(t), \psi) \\ &\quad \leq H(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^{*1}(t), p^{*2}(t), p^{*3}(t), r^*(t), P^*(t), \psi) \end{aligned} \tag{70}$$

$\lambda \otimes \hat{P} -$ a.s. on $[0, T+M] \times \hat{\Omega}$, with $\{p^{*1}(t)\}$ (respectively, $\{p^{*2}(t)\}$, $\{p^{*3}(t)\}$) an $\mathbb{R}$ (respectively $\mathbb{R}^n$, $\mathbb{R}$)-valued process, and $\{q^{*1}(t)\}$ (respectively, $\{q^{*2}(t)\}$, $\{q^{*3}(t)\}$) an $\mathbb{R}^{1\times m}$

(respectively $\mathbb{R}^{n\times m}$, $\mathbb{R}^{1\times m}$)-valued process where

$$\{p^*(t)\} \doteq \left\{\begin{pmatrix} p^{*1}(t) \\ p^{*2}(t) \\ p^{*3}(t) \end{pmatrix}\right\}, \quad \{q^*(t)\} \doteq \left\{\begin{pmatrix} q^{*1}(t) \\ q^{*2}(t) \\ q^{*3}(t) \end{pmatrix}\right\}, \quad \{P^*(t)\}, \quad (\{Q^{*j}\})_{j\in\mathbb{N}_m}$$

are the unique solutions, in the following class of processes

$$\{p^*(t), P^*(t)\} \in L^2\left(\hat{\mathbb{F}}^{\hat{W}}_{\eta^*}; [0, T+M]; \mathbb{R}^{2+n} \times \mathcal{S}^n\right)$$

$$\left\{q^*(t), (Q^{*j}(t))_{j\in\mathbb{N}_m}\right\} \in L^2_{\theta^*}\left(\hat{\mathbb{F}}^{\hat{W}}_{\eta^*}; [0, T+M]; \mathbb{R}^{(2+n)\times m} \times [\mathcal{S}^n]^m\right)$$

with $\{p^*(t), P^*(t)\}$ corlol, of the system of backward stochastic differential equations defined by equations (75)–(80), and where $\{r^*(t)\}$ is defined by

$$r^*(t) \doteq [q^{*2}(t) - P^*(t)D(\eta^*(t), \xi^*(t))]\sqrt{1-\theta^*(t)}. \tag{71}$$

*Proof*. The idea to obtain the result is to apply the maximum principle (see for example Theorem 5 in Ref. [56] or Theorem 6.1 in Ref. [57]) for the control process $(\bar{\theta}(t), \bar{\alpha}(t))$ which is optimal for the auxiliary control problem by using Proposition 4.5. From assumptions A1, A3 and A4, the hypothesis (3) of Theorem 5 in Ref. [56] are satisfied. However, one hypothesis of Theorem 5 in Ref. [56] is not satisfied here in the sense that we do not require the control processes $\{\bar{\alpha}(t)\}$, $\{\bar{\theta}(t)\}$ to be $\{\mathcal{F}^{\bar{V}}_t\}$-adapted. As pointed out in Ref. [57] (see page 114 and the top of page 116), this hypothesis may appear crucial in order to ensure the existence of the adjoint variables. However, although we required the processes to be $\{\mathcal{J}_t\}$-adapted, the proof of Theorem 5 in Ref. [56] remains unchanged.

Due to the particular block structure of the matrix $\mathcal{D}$ and from Theorem 5 in Ref. [56], it follows that there exist $\psi \in S_1(\mathbb{R}^3)$ such that $(\forall(\alpha,\theta) \in B_1(K)\times[0,1])$ the variational inequality can be written in the following form

$$\begin{aligned}
&(1-\theta)\bar{p}^{1\psi}(t) + (1-\theta)A(\bar{\eta}(t), \bar{\xi}(t))^\top \bar{p}^{2\psi}(t) + \theta[B(\bar{\eta}(t), \bar{\xi}(t))\alpha]^\top \bar{p}^{2\psi}(t) + \theta|u|\bar{p}^{3\psi}(t) \\
&\quad + \sqrt{1-\theta}\mathrm{tr}[D(\bar{\eta}(t), \bar{\xi}(t))^\top \bar{r}^{\psi}(t)] + (1-\theta)\mathrm{tr}[D(\bar{\eta}(t), \bar{\xi}(t))^\top \bar{P}^{\psi}(t) D(\bar{\eta}(t), \bar{\xi}(t))] \\
&\leq (1-\bar{\theta}(t))\bar{p}^{1\psi}(t) + (1-\bar{\theta}(t))A(\bar{\eta}(t), \bar{\xi}(t))^\top \bar{p}^{2\psi}(t) + \bar{\theta}(t) \\
&\quad \times [B(\bar{\eta}(t), \bar{\xi}(t))\bar{\alpha}(t)]^\top \bar{p}^{2\psi}(t) + \bar{\theta}(t)|\bar{\alpha}(t)|\bar{p}^{3\psi}(t) + \sqrt{1-\bar{\theta}(t)}\mathrm{tr}[D(\bar{\eta}(t), \bar{\xi}(t))^\top \bar{r}^{\psi}(t)] \\
&\quad + (1-\bar{\theta}(t))\mathrm{tr}[D(\bar{\eta}(t), \bar{\xi}(t))^\top \bar{P}^{\psi}(t) D(\bar{\eta}(t), \bar{\xi}(t))]
\end{aligned} \tag{72}$$

$\lambda \otimes P -$ a.s. on $[0, T+M]\times\Omega$ with

$$\bar{r}^{\psi}(t) = \bar{q}^{2\psi}(t) - \bar{P}^{\psi}(t)D(\bar{\eta}(t), \bar{\xi}(t))\sqrt{1-\bar{\theta}(t)}, \tag{73}$$

and where $\{\bar{p}^{1\psi}(t)\}$ (respectively, $\{\bar{p}^{2\psi}(t)\}$, $\{\bar{p}^{3\psi}(t)\}$) is an $\mathbb{R}$ (respectively $\mathbb{R}^n$, $\mathbb{R}$)-valued process, and $\{\bar{q}^{1\psi}(t)\}$ (respectively, $\{\bar{q}^{2\psi}(t)\}$, $\{\bar{p}^{3\psi}(t)\}$) is an $\mathbb{R}^{1\times m}$ (respectively $\mathbb{R}^{n\times m}$, $\mathbb{R}^{1\times m}$)-

valued process such that

$$\left(\left\{\begin{pmatrix}\bar p^{1\psi}(t)\\ \bar p^{2\psi}(t)\\ \bar p^{3\psi}(t)\end{pmatrix}\right\},\left\{\begin{pmatrix}\bar q^{1\psi}(t)\\ \bar q^{2\psi}(t)\\ \bar q^{3\psi}(t)\end{pmatrix}\right\}\right)$$

are the first-order adjoint variables solution of the following backward stochastic differential equation (see equation (21) in Ref. [56]):

$$\mathrm{d}\bar p^{\psi}(t) = -\begin{pmatrix}\mathcal{H}_t(\bar\eta(t),\bar\xi(t),\bar\alpha(t),\bar\theta(t),\bar p^{\psi}(t),\bar q^{\psi}(t))\\ \mathcal{H}_x(\bar\eta(t),\bar\xi(t),\bar\alpha(t),\bar\theta(t),\bar p^{\psi}(t),\bar q^{\psi}(t))\\ 0\end{pmatrix}\mathrm{d}t + \bar q^{\psi}(t)\mathrm{d}\bar V(t) \tag{74}$$

with

$$\bar p^{\psi}(T+M) = -\begin{pmatrix}\psi_1 N_{1t}(\bar\eta(T+M)) + \psi_3 N_{2t}(\bar\eta(T+M))\\ \psi_2 G_x(\bar\mu(T+M),\bar\xi(T+M))\\ \psi_2 G_w(\bar\mu(T+M),\bar\xi(T+M))\end{pmatrix} \tag{75}$$

and where the second-order adjoint variable is solution of the following equation (see equation (22) in Ref. [56]):

$$\begin{aligned}\mathrm{d}\bar P^{\psi}(t) = &- A_x(\bar\eta(t),\bar\xi(t))^{\top}\bar P^{\psi}(t)(1-\bar\theta(t))\mathrm{d}t\\ &- \bar P^{\psi}(t)A_x(\bar\eta(t),\bar\xi(t))(1-\bar\theta(t))\mathrm{d}t\\ &- \sum_{j=1}^{r}([B_{jx}(\bar\eta(t),\bar\xi(t))]^{\top}\bar P^{\psi}(t) + \bar P^{\psi}(t)B_{jx}(\bar\eta(t),\bar\xi(t)))\bar\alpha_j(t)\bar\theta(t)\mathrm{d}t\\ &- \sum_{j=1}^{m}[D_{jx}(\bar\eta(t),\bar\xi(t))]^{\top}\bar P^{\psi}(t)D_{jx}(\bar\eta(t),\bar\xi(t))(1-\bar\theta(t))\mathrm{d}t\\ &- \sum_{j=1}^{m}([D_{jx}(\bar\eta(t),\bar\xi(t))]^{\top}\bar Q^{\psi,j}(t) + \bar Q^{\psi,j}(t)D_{jx}(\bar\eta(t),\bar\xi(t)))\sqrt{1-\bar\theta(t)}\mathrm{d}t\\ &- \mathcal{H}_{xx}(\bar\eta(t),\bar\xi(t),\bar\alpha(t),\bar\theta(t),\bar p^{\psi}(t),\bar q^{\psi}(t))\mathrm{d}t + \sum_{j=1}^{m}\bar Q^{\psi,j}(t)\mathrm{d}\bar V_j(t)\end{aligned} \tag{76}$$

with

$$\bar P^{\psi}(T+M) = -\psi_2\, G_{xx}(\bar\mu(T+M),\bar\xi(T+M)). \tag{77}$$

From Theorem 3.1 in Ref. [58], the solutions of equations (74)–(77) exist and are unique.

Since $E_P[N(\bar{\eta}(T+M))] = 0_2$, we have $N_{1t}(\bar{\eta}(T+M)) = 1$, and $N_{2t}(\bar{\eta}(T+M)) = 2T$, (see the definition of $N$). Therefore, the terminal condition (75) is given by

$$\bar{p}^{\psi}(T+M) = -\begin{pmatrix} \psi_1 + 2T\psi_3 \\ \psi_2 G_x(\bar{\mu}(T+M), \bar{\xi}(T+M)) \\ \psi_2 G_w(\bar{\mu}(T+M), \bar{\xi}(T+M)) \end{pmatrix} \tag{78}$$

Simple calculations show that

$$\left( \left\{ \psi_2 \bar{p}(t) + \begin{pmatrix} -\psi_1 - 2T\psi_3 \\ 0 \\ 0 \end{pmatrix} \right\}, \{\psi_2 \bar{q}(t)\} \right)$$

is solution of equation (74) with the terminal condition (78). By uniqueness, we obtain that

$$\bar{p}^{\psi}(t) = \psi_2 \bar{p}(t) + \begin{pmatrix} -\psi_1 - 2T\psi_3 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{q}^{\psi}(t) = \psi_2 \bar{q}(t). \tag{79}$$

Moreover, combining equation (79) and the definition of $\mathcal{H}$ (equation (3)), it is easy to obtain that

$$\mathcal{H}_{xx}(\bar{\eta}(t), \bar{\xi}(t), \bar{\alpha}(t), \bar{\theta}(t), \bar{p}^{\psi}(t), \bar{q}^{\psi}(t)) = \psi_2 \mathcal{H}_{xx}(\bar{\eta}(t), \bar{\xi}(t), \bar{\alpha}(t), \bar{\theta}(t), \bar{p}(t), \bar{q}(t)).$$

Consequently, it gives that $(\{\psi_2 \bar{P}(t)\}, (\{\psi_2 \bar{Q}^j(t)\})_{j\in\mathbb{N}_m})$ is solution of equation (76) with the terminal condition (77). By uniqueness, we obtain that

$$\bar{P}^{\psi}(t) = \psi_2 \bar{P}(t), \quad \bar{Q}^{\psi,j}(t) = \psi_2 \bar{Q}^j(t). \tag{80}$$

Now using Theorem 4.7 (equation (65)) and equations (79), (80), it follows that

$$\bar{p}^{\psi}(t,\hat{\omega},\tilde{\omega}) = \psi_2 p^*(t,\hat{\omega}) + \begin{pmatrix} -\psi_1 - 2T\psi_3 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{P}^{\psi}(t,\hat{\omega},\tilde{\omega}) = \psi_2 P^*(t,\hat{\omega}), \tag{81}$$

$$\bar{q}^{\psi}(t,\hat{\omega},\tilde{\omega}) = \psi_2\sqrt{1-\theta^*(t,\hat{\omega})}\,q^*(t,\hat{\omega}), \quad \bar{Q}^{\psi,j}(t,\hat{\omega},\tilde{\omega}) = \psi_2\sqrt{1-\theta^*(t,\hat{\omega})}\,Q^{*j}(t,\hat{\omega}), \tag{82}$$

and from equation (73) it implies

$$\bar{r}^{\psi}(t,\hat{\omega},\tilde{\omega}) = \psi_2\sqrt{1-\theta^*(t,\hat{\omega})}\left[q^*(t,\hat{\omega}) - P^*(t,\hat{\omega})D(\eta^*(t,\hat{\omega}),\xi^*(t,\hat{\omega}))\right] = \psi_2 r^*(t,\hat{\omega}) \tag{83}$$

where we have used the definitions of $\{\bar{\eta}(t)\}$, $\{\bar{\xi}(t)\}$ (equation (48)).

Now, using the definitions of the processes $\{\bar{\alpha}(t)\}$ and $\{\bar{\theta}(t)\}$, the definition of $H$ (see equation (5)), and equations (81)–(83), it follows that the variational inequality (72) can be written on the filtered probability space $\hat{\mathbb{F}}^{\hat{W}}_{\eta^*}$ as $(\forall(\alpha,\theta)\in B_1(K)\times[0,1])$

$$\begin{aligned} &H(\eta^*(t),\xi^*(t),\alpha,\theta,p^{*1}(t),p^{*2}(t),p^{*3}(t),r^*(t),P^*(t),\psi) \\ &\quad \le H(\eta^*(t),\xi^*(t),\alpha^*(t),\theta^*(t),p^{*1}(t),p^{*2}(t),p^{*3}(t),r^*(t),P^*(t),\psi), \end{aligned}$$

$\lambda\otimes\hat{P}-$a.s. on $[0,T+M]\times\hat{\Omega}$. showing the result. □

## 5. The singular maximum principle and adjoint variables

In this section, the adjoint variables for the original control problem are characterized. The interesting feature of these adjoint variables is that they are solutions of singular backward equations (see equations (84)–(87)). It is shown in Theorem 5.2 that by using a time change technique the adjoint variables for the original control problem can be derived from the adjoint variables of the auxiliary control problem (see equations (66)–(69)). As in Section 4, an important difference with respect to the approach used previously by the authors in Ref. [1] is that now the gain of the auxiliary control variables (the map $B$) in the backward stochastic differential equations defining the auxiliary adjoint variables (see equations (66)–(69)) depends on the auxiliary state variables. In particular, as shown in the proof of Theorem 5.2 the results of Theorem 2.1 are again essential for applying our time change approach. Finally, before presenting the stochastic maximum principle for the original singular control problem (see Theorem 5.5) some technical results are derived.

In the following definition, we introduce the backward stochastic differential equations that will be satisfied by the adjoint variables for the original control problem. Note the special form of the right hand side of these equation that gives a singular part for these adjoint variables.

Definition 5.1. Let $C \in \mathfrak{C}^a$ be a singular control

$$C \doteq (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, \{u(t), v(t)\}, \{W(t)\}, \{x(t)\})$$

such that $\{u(t), v(t)\}$ is $\{\mathcal{F}_t^W\}$-progressively measurable. If the system of backward stochastic differential equations

$$\begin{aligned} p^1(t) = {} & \int_t^T A_t(s, x(s))^\top p^2(s)\mathrm{d}s + \sum_{j=1}^m \int_t^T D_{jt}(s, x(s))^\top q_j^2(s)\mathrm{d}s - \int_t^T q^1(s)\mathrm{d}W(s) \\ & + \int_t^T p^2(s)^\top B_t(s, x(s))u(s)\mathrm{d}v^c(s) \\ & + \sum_{n=1}^\infty p^2(\tau_n)^\top \Psi_t(\tau_n, x(\tau_n-), u(\tau_n)\Delta v(\tau_n)) I_{\{t<\tau_n\}}, \end{aligned} \tag{84}$$

$$\begin{aligned} p^2(t) = {} & -G_x\left(\int_{[0,T]} |u(s)|\mathrm{d}v(s), x(T)\right) + \int_t^T A_x(s, x(s))^\top p^2(s)\mathrm{d}s \\ & + \sum_{j=1}^m \int_t^T D_{jx}(s, x(s))^\top q_j^2(s)\mathrm{d}s + \sum_{j=1}^r \int_t^T B_{jx}(s, x(s))^\top p^2(s)u_j(s)\mathrm{d}v^c(s) \\ & + \sum_{n=1}^\infty \Psi_x(\tau_n, x(\tau_n-), u(\tau_n)\Delta v(\tau_n))^\top p^2(\tau_n) I_{\{t<\tau_n\}} - \int_t^T q^2(s)\mathrm{d}W(s), \end{aligned} \tag{85}$$

$$p^3(t) = -G_w\left(\int_0^T |u(s)|\mathrm{d}v(s), x(T)\right) - \int_t^T q^3(s)\mathrm{d}W(s), \tag{86}$$

and

$$
\begin{aligned}
P(t) = & - G_{xx}\left(\int_{[0,T]} |u(s)| \mathrm{d}v(s), x(T)\right) + \int_t^T [A_x(s,x(s))^\top P(s) + P(s)A_x(s,x(s))]\mathrm{d}s \\
& + \sum_{j=1}^{r} \int_t^T [B_{jx}(s,x(s))^\top P(s) + P(s)B_{jx}(s,x(s))]u_j(s)\mathrm{d}v^c(s) \\
& + \sum_{j=1}^{m} \int_t^T \left[D_{jx}(s,x(s))^\top P(s)D_{jx}(s,x(s)) + D_{jx}(s,x(s))^\top Q^j(s) + Q^j(s)D_{jx}(s,x(s))]\mathrm{d}s\right] \\
& + \sum_{j=1}^{n} \int_t^T \left[A_{jxx}(s,x(s))p_j^2(s) + \sum_{i=1}^{m} D_{jixx}(s,x(s))q_{ji}^2(s)\right]\mathrm{d}s \\
& + \sum_{j=1}^{n}\sum_{i=1}^{r} \int_t^T p_j^2(s)B_{jixx}(s,x(s))u_i(s)\mathrm{d}v^c(s) - \int_t^T \sum_{j=1}^{m} Q^j(s)\mathrm{d}W(s) \\
& + \sum_{n=1}^{\infty} [I_n + \Psi_x(\tau_n, x(\tau_n -), u(\tau_n)\Delta v(\tau_n))]^\top P(\tau_n)[I_n + \Psi_x(\tau_n, x(\tau_n -), u(\tau_n)\Delta v(\tau_n))] I_{\{t<\tau_n\}} \\
& + \sum_{n=1}^{\infty}\sum_{j=1}^{n} \left[p_j^2(\tau_n)\Psi_{jxx}(\tau_n, x(\tau_n -), u(\tau_n)\Delta v(\tau_n)) - P(\tau_n)\right] I_{\{t<\tau_n\}}
\end{aligned} \tag{87}
$$

admits a solution in the following class of processes

$$
\begin{aligned}
&\{p^1(t), p^2(t), p^3(t)\} \in L^2(\mathbb{F}^W; [0,T]; \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}), \\
&\{q^1(t), q^2(t), q^3(t)\} \in L^2(\mathbb{F}^W; [0,T]; \mathbb{R}^{1\times m}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{1\times m}), \\
&\left(\{P(t)\}, (\{Q^j(t)\})_{j\in\mathbb{N}_m}\right) \in L^2(\mathbb{F}^W; [0,T]; \mathcal{S}^n)\times[L^2(\mathbb{F}^W; [0,T]; \mathcal{S}^n)]^m,
\end{aligned}
$$

with $\{p^1(t), p^2(t), p^3(t), P(t)\}$ corlol.

Then $((\{p^i(t)\}, \{q^i(t)\})_{i\in\mathbb{N}_3}, \{P(t)\}, (\{Q^j(t)\})_{j\in\mathbb{N}_m})$ are called the adjoint variables associated to the control $C$. The solution is said unique if the solution of the previous system is unique in this class of processes.

By using a time transformation, we show that from $(\{p^*(t)\}, \{q^*(t)\}, \{P^*(t)\}, (\{Q^{*j}(t)\})_{j\in\mathbb{N}_m})$ we can obtain the adjoint variables for the original optimal control.

THEOREM 5.5. Define the following processes:

$$
\hat{p}(t) \doteq p^*(\hat{\Gamma}(t)),\quad \hat{q}(t) \doteq q^*(\hat{\Gamma}(t)),\quad \hat{P}(t) \doteq P^*(\hat{\Gamma}(t)),\ \text{and for } j\in\mathbb{N}_m\quad \hat{Q}^j(t) \doteq Q^{*j}(\hat{\Gamma}(t)), \tag{88}
$$

where $(\{p^*(t)\}, \{q^*(t)\}, \{P^*(t)\}, (\{Q^{*j}\})_{j\in\mathbb{N}_m})$ are solutions of equations (66)–(69). Write $\hat{p}(t)$ in the form

$$
\hat{p}(t) = \begin{pmatrix} \hat{p}^1(t) \\ \hat{p}^2(t) \\ \hat{p}^3(t) \end{pmatrix}
$$

where $\hat{p}^1(t) \in \mathbb{R}$, $\hat{p}^2(t) \in \mathbb{R}^n$ and $\hat{p}^3(t) \in \mathbb{R}$ and similarly

$$\hat{q}(t) = \begin{pmatrix} \hat{q}^1(t) \\ \hat{q}^2(t) \\ \hat{q}^3(t) \end{pmatrix}$$

where $\hat{q}^1(t) \in \mathbb{R}^{1\times n}$, $\hat{q}^2(t) \in \mathbb{R}^{n\times n}$ and $\hat{q}^3(t) \in \mathbb{R}^{1\times n}$.

Then $((\{\hat{p}^i(t)\}, \{\hat{q}^i(t)\})_{i\in\mathbb{N}_3}, \{\hat{P}(t)\}, (\{\hat{Q}^j(t)\})_{j\in\mathbb{N}_m})$ are the unique adjoint variables associated to the control $\hat{C}$.

*Proof*. Combining equations (66), and (34), we obtain

$$p^{*2}(t) = -G_x(\mu^*(T+M), \xi^*((T+M)) + \int_t^{T+M} A_x(\eta^*(s), \xi^*(s))^\top p^{*2}(s)\mathrm{d}\eta^*(s)$$
$$+ \sum_{j=1}^m \int_t^{T+M} D_{jx}(\eta^*(s), \xi^*(s))^\top q_j^{*2}(s)\mathrm{d}\eta^*(s) - \int_t^{T+M} q^{*2}(s)\mathrm{d}\hat{W}(\eta^*(s))$$
$$+ \sum_{j=1}^r \int_t^{T+M} B_{jx}(\eta^*(s), \xi^*(s))^\top p^{*2}(s)\alpha_j^*(s)\theta^*(s)\mathrm{d}s \tag{89}$$

and using the fact that $\hat{\Gamma}(T) = T+M$, and equation (39), it gives

$$p^{*2}(\hat{\Gamma}(t)) = -G_x\left(\int_{[0,T]} |\hat{u}(s)|\mathrm{d}\hat{v}(s), \hat{x}(T)\right) + \int_{\hat{\Gamma}(t)}^{\hat{\Gamma}(T)} A_x(\eta^*(s), \xi^*(s))^\top p^{*2}(s)\mathrm{d}\eta^*(s)$$
$$+ \sum_{j=1}^m \int_{\hat{\Gamma}(t)}^{\hat{\Gamma}(T)} D_{jx}(\eta^*(s), \xi^*(s))^\top q_j^{*2}(s)\mathrm{d}\eta^*(s) - \int_{\hat{\Gamma}(t)}^{\hat{\Gamma}(T)} q^{*2}(s)\mathrm{d}\hat{W}(\eta^*(s))$$
$$+ \sum_{j=1}^r \int_{\hat{\Gamma}(t)}^{\hat{\Gamma}(T)} B_{jx}(\eta^*(s), \xi^*(s))^\top p^{*2}(s)\alpha_j^*(s)\theta^*(s)\mathrm{d}s \tag{90}$$

According to Definition (1.3) in Ref. [52, p.180], $\{\eta^*(t)\}$ is clearly a $\{\hat{\Gamma}(t)\}$-continuous process since $\{\eta^*(t)\}$ is the right inverse of $\{\hat{\Gamma}(t)\}$. Recalling $\eta^*(\hat{\Gamma}(s)) = s$, $\xi^*(\hat{\Gamma}(s)) = \hat{x}(s)$ and applying Proposition (1.4) in Ref. [52, p.180] to the second and the third terms of the right hand side of equation (90), and applying Proposition (1.5) in Ref. [52, p.181] to the fourth term of the right hand side of this equation it follows that

$$p^{*2}(\hat{\Gamma}(t)) = -G_x\left(\int_{[0,T]} |\hat{u}(s)|\mathrm{d}\hat{v}(s), \hat{x}(T)\right) + \int_t^T A_x(s, \hat{x}(s))^\top p^{*2}(\hat{\Gamma}(s))\mathrm{d}s$$
$$+ \sum_{j=1}^m \int_t^T D_{jx}(s, \hat{x}(s))^\top q_j^{*2}(\hat{\Gamma}(s))\mathrm{d}s - \int_t^T q^{*2}(\hat{\Gamma}(s))\mathrm{d}\hat{W}(s)$$
$$+ \sum_{j=1}^r \int_{\hat{\Gamma}(t)}^{\hat{\Gamma}(T)} B_{jx}(\eta^*(s), \xi^*(s))^\top p^{*2}(s)\alpha_j^*(s)\theta^*(s)\mathrm{d}s \tag{91}$$

Now, we obtain for the last term of the previous equation

$$\sum_{j=1}^{r}\int_{\hat\Gamma(t)}^{\hat\Gamma(T)} B_{jx}(\eta^*(s),\xi^*(s))^\top p^{*2}(s)\alpha_j^*(s)\theta^*(s)\mathrm{d}s$$

$$=\sum_{j=1}^{r}\int_{\hat\Gamma(t)}^{\hat\Gamma(T)} B_{jx}(\eta^*(s),\xi^*(s))^\top p^{*2}(s)\alpha_j^*(s)\mathrm{d}\gamma^*(s)$$

$$+\sum_{n=1}^{\infty}\sum_{j=1}^{r}\int_{\hat\Gamma(\tau_n-)}^{\hat\Gamma(\tau_n)} B_{jx}(\eta^*(s),\xi^*(s))^\top p^{*2}(s)\alpha_j^*(s)\mathrm{d}sI_{\{t<\tau_n\}}, \qquad (92)$$

where the process $\{\gamma^*(t)\}$ has been defined in equation (44). Using Proposition (1.4) in Ref. [52, p.180], we obtain

$$\sum_{j=1}^{r}\int_{\hat\Gamma(t)}^{\hat\Gamma(T)} B_{jx}(\eta^*(s),\xi^*(s))^\top p^{*2}(s)\alpha_j^*(s)\mathrm{d}\gamma^*(s)$$

$$=\sum_{j=1}^{r}\int_{t}^{T} B_{jx}(s,\hat x(s))^\top p^{*2}(\hat\Gamma(s))\hat u_j(s)\mathrm{d}\gamma^*(\hat\Gamma(s)), \qquad (93)$$

since $\{\gamma^*(t)\}$ is a $\{\hat\Gamma(t)\}$-continuous process. However, $\gamma^*(\hat\Gamma(t))=v^c(t)$, consequently combining equations (91)–(93) we have

$$p^{*2}(\hat\Gamma(t)) = -G_x\left(\int_{[0,T]}|\hat u(s)|\mathrm{d}\hat v(s),\hat x(T)\right)+\int_t^T A_x(s,\hat x(s))^\top p^{*2}(\hat\Gamma(s))\mathrm{d}s$$

$$+\sum_{j=1}^{m}\int_t^T D_{jx}(s,\hat x(s))^\top q_j^{*2}(\hat\Gamma(s))\mathrm{d}s-\int_t^T q^{*2}(\hat\Gamma(s))\mathrm{d}\hat W(s)$$

$$+\sum_{j=1}^{r}\int_t^T B_{jx}(s,\hat x(s))^\top p^{*2}(\hat\Gamma(s))\hat u_j(s)\mathrm{d}v^c(s)$$

$$+\sum_{n=1}^{\infty}\sum_{j=1}^{r}\int_{\hat\Gamma(\tau_n-)}^{\hat\Gamma(\tau_n)} B_{jx}(\eta^*(s),\xi^*(s))^\top p^{*2}(s)\alpha_j^*(s)\mathrm{d}sI_{\{t<\tau_n\}}. \qquad (94)$$

Now, from equation (89), we have that for all $(t,\omega)\in[0,T+M]\times\Omega\cap[[\hat\Gamma(\tau_n-),\hat\Gamma(\tau_n)]]$,

$$p^{*2}(t)=p^{*2}(\hat\Gamma(\tau_n))+\sum_{j=1}^{r}\int_t^{\hat\Gamma(\tau_n)} B_{jx}(\eta^*(s),\xi^*(s))^\top p^{*2}(s)\alpha_j^*(s)\mathrm{d}s$$

$$=p^{*2}(\hat\Gamma(\tau_n))+\sum_{j=1}^{r}\int_t^{\hat\Gamma(\tau_n)} B_{jx}(\tau_n,\xi^*(s))^\top p^{*2}(s)\hat u j(\tau_n)\mathrm{d}s$$

and using equations (7), and (11) in Theorem 2.1, it follows that

$$\sum_{j=1}^{r}\int_{\hat\Gamma(\tau_n-)}^{\hat\Gamma(\tau_n)} B_{jx}(\tau_n,\xi^*(s))^\top p^{*2}(s)\hat u_j(\tau_n)\mathrm{d}s=\Psi_x(\tau_n,\hat x(\tau_n-),\hat u(\tau_n)\Delta\hat v(\tau_n))^\top p^{*2}(\hat\Gamma(\tau_n)). \qquad (95)$$

Finally, combining equations (94) and (95), we obtain that $\{p^{*2}(\hat{\Gamma}(t))\}$ is solution of equation (85).

Using similar arguments, it is easy to obtain that

$$\begin{aligned} p^{*1}(\hat{\Gamma}(t)) = {} & \int_t^T A_t(s,\hat{x}(s))^\top p^{*2}(\hat{\Gamma}(s))\mathrm{d}s + \sum_{j=1}^{m}\int_t^T D_{jt}(s,\hat{x}(s))^\top q_j^{*2}(\hat{\Gamma}(s))\mathrm{d}s \\ & - \int_t^T q^{*1}(\hat{\Gamma}(s))\mathrm{d}\hat{W}(s) + \int_t^T p^{*2}(\hat{\Gamma}(s))^\top B_t(s,\hat{x}(s))\hat{u}(s)\mathrm{d}v^c(s) \\ & + \sum_{n=1}^{\infty}\int_{\hat{\Gamma}(\tau_n-)}^{\hat{\Gamma}(\tau_n)} p^{*2}(s)^\top B_t(\tau_n,\xi^*(s))\hat{u}(\tau_n)\mathrm{d}s I_{\{t<\tau_n\}}, \end{aligned} \tag{96}$$

and using equations (8), (12) in Theorem 2.1,

$$\int_{\hat{\Gamma}(\tau_n-)}^{\hat{\Gamma}(\tau_n)} p^{*2}(s)^\top B_t(\tau_n,\xi^*(s))\hat{u}(\tau_n)\mathrm{d}s = p^{*2}(\hat{\Gamma}(\tau_n))^\top \Psi_t(\tau_n,\hat{x}(\tau_n-),\hat{u}(\tau_n)\Delta\hat{v}(\tau_n)). \tag{97}$$

Therefore, it gives that $\{p^{*1}(\hat{\Gamma}(t))\}$ is solution of equation (84).

Similarly, it follows that

$$\begin{aligned} P^*(\hat{\Gamma}(t)) = {} & \int_{\hat{\Gamma}(t)}^{\hat{\Gamma}(T)} [A_x(\eta^*(s),\xi^*(s))^\top P^*(s) + P^*(s)A_x(\eta^*(s),\xi^*(s))](1-\theta^*(s))\mathrm{d}s \\ & + \sum_{j=1}^{r}\int_{\hat{\Gamma}(t)}^{\hat{\Gamma}(T)} [B_{jx}(\eta^*(s),\xi^*(s))^\top P^*(s) + P^*(s)B_{jx}(\eta^*(s),\xi^*(s))]\alpha_j^*(s)\theta^*(s)\mathrm{d}s \\ & + \sum_{j=1}^{m}\int_{\hat{\Gamma}(t)}^{\hat{\Gamma}(T)} D_{jx}(\eta^*(s),\xi^*(s))^\top P^*(s)D_{jx}(\eta^*(s),\xi^*(s))(1-\theta^*(s))\mathrm{d}s \\ & + \sum_{j=1}^{m}\int_{\hat{\Gamma}(t)}^{\hat{\Gamma}(T)} D_{jx}(\eta^*(s),\xi^*(s))^\top Q^{*j}(s) + Q^{*j}(s)D_{jx}(\eta^*(s),\xi^*(s))](1-\theta^*(s))\mathrm{d}s \\ & + \int_{\hat{\Gamma}(t)}^{\hat{\Gamma}(T)} J_{xx}(\eta^*(s),\xi^*(s),\alpha^*(s),\theta^*(s),p^{*2}(s),\sqrt{1-\theta^*(s)}q^{*2}(s))\mathrm{d}s \\ & - \sum_{j=1}^{m}\int_{\hat{\Gamma}(t)}^{\hat{\Gamma}(T)} Q^{*j}(s)\mathrm{d}\hat{W}(\eta^*(s)) - G_{xx}\left(\int_{[0,T]}|\hat{u}(s)|\mathrm{d}\hat{v}(s),\hat{x}(T)\right), \end{aligned}$$

implying that

$$\begin{aligned}
P^*(\hat\Gamma(t)) &= \int_t^T [A_x(s,\hat x(s))^\top P^*(\hat\Gamma(s)) + P^*(\hat\Gamma(s))A_x(s,\hat x(s))]\mathrm{d}s \\
&\quad + \sum_{j=1}^{r}\int_t^T B_{jx}(s,\hat x(s))^\top P^*(\hat\Gamma(s)) + P^*(\hat\Gamma(s))B_{jx}(s,\hat x(s))\hat u_j(s)\mathrm{d}\hat v^c(s) \\
&\quad + \sum_{n=1}^{\infty}\sum_{j=1}^{r}\int_{\hat\Gamma(\tau_n-)}^{\hat\Gamma(\tau_n)} [B_{jx}(\eta^*(s),\xi^*(s))^\top P^*(s) + P^*(s)B_{jx}(\eta^*(s),\xi^*(s))]\alpha_j^*(s)\mathrm{d}s I_{\{t<\tau_n\}} \\
&\quad + \sum_{j=1}^{m}\int_t^T D_{jx}(s,\hat x(s))^\top P^*(\hat\Gamma(s))D_{jx}(s,\hat x(s))\mathrm{d}s \\
&\quad + \sum_{j=1}^{m}\int_t^T D_{jx}(s,\hat x(s))^\top Q^{*j}(\hat\Gamma(s)) + Q^{*j}(\hat\Gamma(s))D_{jx}(s,\hat x(s))]\mathrm{d}s \\
&\quad + \sum_{j=1}^{n}\int_t^T \left[A_{jxx}(s,\hat x(s))p_j^{*2}(\hat\Gamma(s)) + \sum_{i=1}^{m} D_{jixx}(s,\hat x(s))q_{ji}^{*2}(\hat\Gamma(s))\right]\mathrm{d}s \\
&\quad + \sum_{j=1}^{n}\sum_{i=1}^{r}\int_t^T p_j^{*2}(\hat\Gamma(s))B_{jixx}(s,\hat x(s))\hat u_i(s)\mathrm{d}\hat v^c(s) \\
&\quad + \sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{i=1}^{r}\int_{\hat\Gamma(\tau_n-)}^{\hat\Gamma(\tau_n)} p_j^{*2}(s)B_{jixx}(\eta^*(s),\xi^*(s))\alpha_i^*(s)\mathrm{d}s I_{\{t<\tau_n\}} \\
&\quad - \sum_{j=1}^{m}\int_t^T Q^{*j}(\hat\Gamma(s))\mathrm{d}\hat W(s) - G_{xx}\left(\int_{[0,T]}|\hat u(s)|\mathrm{d}\hat v(s), \hat x(T)\right). \qquad (98)
\end{aligned}$$

Now, from equation (68), we have that for all $(t,\omega)\in[0,T+M]\times\Omega\cap[[\hat\Gamma(\tau_n-),\hat\Gamma(\tau_n)]]$,

$$\begin{aligned}
P^*(t) &= P^*(\hat\Gamma(\tau_n)) + \sum_{j=1}^{r}\int_t^{\hat\Gamma(\tau_n)}\left[B_{jx}(\eta^*(s),\xi^*(s))^\top P^*(s) + P^*(s)B_{jx}(\eta^*(s),\xi^*(s))\right]\alpha_j^*(s)\mathrm{d}s \\
&\quad + \sum_{j=1}^{n}\sum_{i=1}^{r}\int_t^{\hat\Gamma(\tau_n)} p_j^{*2}(s)B_{jixx}(\eta^*(s),\xi^*(s))\alpha_i^*(s)\mathrm{d}s,
\end{aligned}$$

and from equations (9), (13) in Theorem 2.1, it gives that

$$\begin{aligned}
&\sum_{j=1}^{r}\int_{\hat\Gamma(\tau_n-)}^{\hat\Gamma(\tau_n)}\left[B_{jx}(\eta^*(s),\xi^*(s))^\top P^*(s) + P^*(s)B_{jx}(\eta^*(s),\xi^*(s))\right]\alpha_j^*(s)\mathrm{d}s \\
&\quad + \sum_{j=1}^{n}\sum_{i=1}^{r}\int_{\hat\Gamma(\tau_n-)}^{\hat\Gamma(\tau_n)} p_j^{*2}(s)B_{jixx}(\eta^*(s),\xi^*(s))\alpha_i^*(s)\mathrm{d}s \\
&= \left[I_n + \Psi_x(\tau_n,\hat x(\tau_n-),\hat u(\tau_n)\Delta\hat v(\tau_n))\right]^\top P^*(\hat\Gamma(\tau_n))\left[I_n + \Psi_x(\tau_n,\hat x(\tau_n-),\hat u(\tau_n)\Delta\hat v(\tau_n))\right] \\
&\quad - P^*(\hat\Gamma(\tau_n)) + \sum_{j=1}^{n} p_j^{*2}(\hat\Gamma(\tau_n))\Psi_{jxx}(\tau_n,\hat x(\tau_n-),\hat u(\tau_n)\Delta\hat v(\tau_n)). \qquad (99)
\end{aligned}$$

Combining equations (98) and (99), we obtain that $\{P^*(\hat\Gamma(t))\}$ is solution of equation (87).

Moreover, it is easy to show that

$$p^{*3}(\hat{\Gamma}(t)) = -G_w\left(\int_{[0,T]} |\hat{u}(s)|\mathrm{d}\hat{v}(s), \hat{x}(T)\right) - \int_t^T q^3(\hat{\Gamma}(s))\mathrm{d}\hat{W}(s)$$

Finally, it follows that $((\{p^{*i}(\hat{\Gamma}(t))\}, \{q^{*i}(\hat{\Gamma}(t))\})_{i\in\mathbb{N}_3}, \{P(\hat{\Gamma}(t))\}, (\{Q^{*j}(\hat{\Gamma}(t))\})_{j\in\mathbb{N}_m})$ are the adjoint variables associated to the control $\hat{C}$. By using the fact that $|\Psi_{xx}|$ is bounded and equation (21), it can be shown that the system of equations given by $\{p^{*2}(\hat{\Gamma}(t)), P(\hat{\Gamma}(t))\}$ satisfies the hypotheses of Proposition 2.4 showing the uniqueness for $\{p^{*2}(\hat{\Gamma}(t)), P(\hat{\Gamma}(t))\}$. This implies the uniqueness for the adjoint variables associated to the control $\hat{C}$, showing the result □

LEMMA 5.3. For $\psi \in \mathbb{R}^3$, and $(\alpha, \theta) \in B_1(K) \times [0, 1]$, write

$$\begin{aligned} L(t, \alpha, \theta) =& H(\eta^*(t), \xi^*(t), \alpha, \theta, p^{*1}(t), p^{*2}(t), p^{*3}(t), r^*(t), P^*(t), \psi) \\ &- H(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^{*1}(t), p^{*2}(t), p^{*3}(t), r^*(t), P^*(t), \psi). \end{aligned} \tag{100}$$

Then, for all $t \in [\hat{\Gamma}(\tau_i-), \hat{\Gamma}(\tau_i)]$

$$\begin{aligned} L(t, \alpha, \theta) =& \psi_2\left[B(\tau_i, \xi^*(t))(\theta\alpha - \hat{u}(\tau_i))\right]^\top p^{*2}(t) + \psi_2(\theta|\alpha| - |\hat{u}(\tau_i)|)\hat{p}^3(\tau_i) \\ &+ (1-\theta)\Bigg\{-\psi_1 - 2T\psi_3 + \psi_2\Bigg[p^{*1}(t) + A(\tau_i, \xi^*(t))^\top p^{*2}(t) \\ &+ \frac{1}{2}\mathrm{tr}\left[D(\tau_i, \xi^*(t))^\top P^*(t) D(\tau_i, \xi^*(t))\right]\Bigg]\Bigg\}, \end{aligned} \tag{101}$$

with

$$\xi^*(t) = \hat{x}(\tau_i-) + \Psi(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-))), \tag{102}$$

$$\begin{aligned} p^{*1}(t) =& -\Psi_t(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top \left[I_n + \Psi_x(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top\right]^{-1} \\ &\times \hat{p}^2(\tau_i-) + \hat{p}^1(\tau_i-), \end{aligned} \tag{103}$$

$$p^{*2}(t) = \left[I_n + \Psi_x(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top\right]^{-1} \hat{p}^2(\tau_i-), \tag{104}$$

and

$$\begin{aligned} P^*(t) =& \left[I_n + \Psi_x(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top\right]^{-1} \\ &\left[\hat{P}(\tau_i-) - \sum_{i=1}^n \Psi_{ixx}(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top\right] \\ &\times \left[I_n + \Psi_x(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top\right]^{-1}, \end{aligned} \tag{105}$$

$\hat{P}-$a.s. on $\{\tau_i \le T\}$.

Moreover, for $(\alpha, \theta) \in B_1(K) \times [0, 1]$,

$$(\forall t \in [\hat{\Gamma}(\tau_i-), \hat{\Gamma}(\tau_i)]), \quad L(t, \alpha, \theta) \leq 0 \quad \hat{P} - \text{a.s. on } \{\tau_i \leq T\}. \tag{106}$$

$$L(\hat{\Gamma}(t), \alpha, \theta) \leq 0, \quad \mathcal{M}_{\hat{\Gamma}} - \text{a.s. on } [0, T] \times \hat{\Omega}. \tag{107}$$

*Proof*. It is easy to show that $(\forall t \in [\hat{\Gamma}(\tau_i-), \hat{\Gamma}(\tau_i)])$,

$$\eta^*(t) = \tau_i, \quad \theta^*(t) = 1, \quad \alpha^*(t) = \hat{u}(\tau_i), \text{ and } r^*(t) = 0, \tag{108}$$

$\hat{P}-$a.s. on $\{\tau_i \leq T\}$. Combining equations (3), (5) and (108), we obtain that $(\forall t \in [\hat{\Gamma}(\tau_i-), \hat{\Gamma}(\tau_i)])$,

$$\begin{aligned} H(\eta^*(t), \xi^*(t), \alpha, \theta, p^*(t), r^*(t), P^*(t)) = {}& \theta[B(\tau_i, \xi^*(t))\alpha]^\top p^{*2}(t) + \theta|\alpha| p^{*3}(t) \\ & + (1-\theta)\Big[p^{*1}(t) + A(\tau_i, \xi^*(t))^\top p^{*2}(t) \\ & + \frac{1}{2}\mathrm{tr}\big[D(\tau_i, \xi^*(t))^\top P^*(t) D(\tau_i, \xi^*(t))\big]\Big], \end{aligned} \tag{109}$$

$\hat{P}-$a.s. on $\{\tau_i \leq T\}$. From equations (37), and (108) it follows that for all $t \in [\hat{\Gamma}(\tau_i-), \hat{\Gamma}(\tau_i)]$,

$$\xi^*(t) = \xi^*(\hat{\Gamma}(\tau_i-)) - \int_{\hat{\Gamma}(\tau_i-)}^{t} B(\tau_i, \xi^*(s))\hat{u}(\tau_i)\mathrm{d}s. \tag{110}$$

Note that from equation (39), we have that $\xi^*(\hat{\Gamma}(\tau_i-)) = \hat{x}(\tau_i-)$, and so by using Theorem 2.1 it gives equation (102).

For $\theta = 1$, simple calculations shows that $\forall(\eta, \xi, \alpha, p, q) \in \mathbb{R} \times \mathbb{R}^n \times B_1(K) \times \mathbb{R}^{2+n} \times \mathbb{R}^{(2+n)\times m}$,

$$\mathcal{H}_x(\eta, \xi, \alpha, \theta, p, q) = \left(0 \middle| \left[\sum_{j=1}^{r} B_{jx}(\eta, \xi)\alpha_j\right]^\top \middle| 0\right) p, \text{ and}$$

$$\mathcal{H}_t(\eta, \xi, \alpha, \theta, p, q) = (0|[B_t(\eta, \xi)\alpha]^\top|0)p.$$

Write $p^*(t)$ in the form

$$p^*(t) = \begin{pmatrix} p^{*1}(t) \\ p^{*2}(t) \\ p^{*3}(t) \end{pmatrix}$$

where $p^{*1}(t) \in \mathbb{R}$, $p^{*2}(t) \in \mathbb{R}^n$ and $p^{*3}(t) \in \mathbb{R}$. Moreover, from equation (108) we have $(\forall t \in [\hat{\Gamma}(\tau_i-), \hat{\Gamma}(\tau_i)])$

$$\int_{\hat{\Gamma}(\tau_i-)}^{t} q^*(s)\mathrm{d}W(\eta^*(s)) = 0 \quad \hat{P} - \text{a.s. on } \{\tau_i \leq T\}. \tag{111}$$

Moreover, note that from the definition of $\{\hat{p}(t)\}$ (see Theorem 5.2), we have $p^*(\hat{\Gamma}(\tau_i-)) = \hat{p}(\tau_i-)$. Therefore, it follows from equations (66), (108) and (111) that

$\hat{P}$-a.s. on $\{\tau_i \leq T\}$,

$$p^*(t) = \begin{pmatrix} \hat{p}^1(\tau_i-) - \int_{\hat{\Gamma}(\tau_i-)}^t [B_t(\tau_i, \xi^*(s))\hat{u}(\tau_i)]^\top p^{*2}(s)\mathrm{d}s \\ \hat{p}^2(\tau_i-) - \sum_{j=1}^r \int_{\hat{\Gamma}(\tau_i-)}^t [B_{jx}(\tau_i, \xi^*(s))\hat{u}_j(\tau_i)]^\top p^{*2}(s)\mathrm{d}s \\ \hat{p}^3(\tau_i) \end{pmatrix}$$

for all $t$ in $[\hat{\Gamma}(\tau_i-), \hat{\Gamma}(\tau_i)]$. By using Theorem 2.1, we obtain that

$$p^{*2}(t) = \Big[I_n + \Psi_x(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top\Big]^{-1}\hat{p}^2(\tau_i-),$$

and

$$p^{*1}(t) = \hat{p}^1(\tau_i-) - \Psi_t(\tau_i, \hat{x}(\tau_i-), \hat{u}(\tau_i)(t - \hat{\Gamma}(\tau_i-)))^\top p^{*2}(t),$$

for all $t$ in $[\hat{\Gamma}(\tau_i-), \hat{\Gamma}(\tau_i)]$, showing equations (103) and (104). Using similar arguments, we obtain equation (105).

Finally, combining equations (108) and (109), and the fact that $p^{*3}(t) = \hat{p}^3(\tau_i)$ for all $t$ in $[\hat{\Gamma}(\tau_i-), \hat{\Gamma}(\tau_i)]$, the first part of the result follows.

From equations (101)–(104) we have that $L(t, \alpha, \theta)$ is continuous on $[\hat{\Gamma}(\tau_i-), \hat{\Gamma}(\tau_i)]$, $\hat{P}-$a.s. on $\{\tau_i \leq T\}$. Combining this fact with Theorem 4.8, we obtain equation (106). Finally, by using Corollary 5.5 in Ref. [1], the last assertion (equation (107)) can be easily obtained. □

The next Proposition presents some relations between the optimal singular control $\tilde{C}$ and $\hat{C}$. This result is important as it gives the full generality to our maximum principle.

PROPOSITION 5.4. Assume the existence of an optimal singular control denoted by

$$\tilde{C} \doteq \Big(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_t\}, \{\tilde{u}(t), \tilde{v}(t)\}, \{\hat{W}(t)\}, \{\tilde{x}(t)\}\Big).$$

such that $\{\tilde{u}(t), \tilde{v}(t)\}$ is $\{\hat{\mathcal{F}}_t^{\hat{W}}\}$-progressively measurable.

Then, the optimal control

$$\hat{C} \doteq \Big(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_t\}, \{\hat{u}(t), \hat{v}(t)\}, \{\hat{W}(t)\}, \{\hat{x}(t)\}\Big).$$

defined by equations (29), and (30) is such that on the filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_t\})$ the processes $\{\tilde{x}(t)\}$ and $\{\hat{x}(t)\}$ are indistinguishable, and

$$\mathcal{M}_{\tilde{v}^s} \ll \mathcal{M}_{\hat{v}^s}, \quad \mathcal{M}_{\tilde{v}^{\mathrm{ac}}} = \mathcal{M}_{\hat{v}^{\mathrm{ac}}}, \tag{112}$$

$$\tilde{u}(t) = \tilde{u}(t), \quad \text{and} \quad \tilde{v}(t) = \hat{v}(t), \quad \mathcal{M}_{\tilde{v}^{\mathrm{ac}}} - \text{a.s. on } [0,T]\times\hat{\Omega}. \tag{113}$$

The adjoint variables $((\{\tilde{p}^i(t)\}, \{\tilde{q}^i(t)\})_{i\in\mathbb{N}_3}, \{\tilde{P}(t)\}, (\{\tilde{Q}^j(t)\})_{j\in\mathbb{N}_m})$ associated to $\tilde{C}$ exist, and are unique. Moreover, for $i \in \mathbb{N}_3$, $j \in \mathbb{N}_m$,

$$\tilde{p}^i(t) = \hat{p}^i(t), \quad \tilde{q}^i(t) = \tilde{q}^i(t), \quad \tilde{P}(t) = \hat{P}(t) \quad \text{and} \quad \tilde{Q}^j(t) = \hat{Q}^j(t), \tag{114}$$

where $((\{\hat{p}^i(t)\}, \{\hat{q}^i(t)\})_{i\in\mathbb{N}_3}, \{\hat{P}(t)\}, (\{\hat{Q}^j(t)\})_{j\in\mathbb{N}_m})$ are the adjoint variables associated to $\hat{C}$.

*Proof.* From equation (31), we have that $(\forall t \in [0,T])$, $\hat{v}^c(t) = \tilde{v}^c(t)$, and $\hat{u}(t)\Delta\hat{v}^c(t) = \tilde{u}(t)\tilde{v}^c(t)$. Moreover, it is very important to remark that the adjoint variables associated to a singular control $(u(t),\ v(t))$ satisfy backward stochastic differential equations (see equations (84)–(87)) that depend only on the control through the terms $v^c(t)$ and $u(t)\Delta v(t)$. By using these key properties, the result follows as in proposition 5.8 in Ref. [1]. □

We can now establish a necessary condition for any control $\tilde{C} \in \mathfrak{C}^a$ to be optimal. In fact similarly to the maximum principle for non-singular control problem, see for example [56,57] and the references therein, we show that if a control $\tilde{C} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_t\}, \{\tilde{u}(t), \tilde{v}(t)\}, \{\hat{W}(t)\}, \{\tilde{x}(t)\})$ is optimal it maximizes a certain Hamiltonian almost surely with respect to the measure of Doleans-Dade generated by the optimal control $\{\tilde{v}(t)\}$.

Let $\{v(t)\}$ be an $\mathbb{R}_+$-valued, *corlol*, progressively measurable increasing process defined on a filtered probability space $(\Omega, \mathcal{F}, P)$. The process $\{v'(t)\}$ defined by

$$v'(t) = \begin{cases} \limsup\limits_{\epsilon \downarrow 0} \frac{v(t+\epsilon)-v(t)}{\epsilon} & \text{if the limit exists in } \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$

is an $\mathbb{R}_+$-valued, $\{\mathcal{F}_t\}$-progressively measurable process (see Lemma 5.6 in Ref. [1]).

Theorem 5.5. Assume the existence of an optimal singular control denoted by

$$\tilde{C} \doteq (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_t\}, \{\tilde{u}(t), \tilde{v}(t)\}, \{\hat{W}(t)\}, \{\tilde{x}(t)\}).$$

such that $\{\tilde{u}(t), \tilde{v}(t)\}$ is $\{\hat{\mathcal{F}}_t^{\hat{W}}\}$-progressively measurable.

Denote by $((\{\tilde{p}^i(t)\}, \{\tilde{q}^i(t)\})_{i \in \mathbb{N}_3}, \{\tilde{P}(t)\}, (\{\tilde{Q}^j(t)\})_{j \in \mathbb{N}_m})$ the adjoint variables associated $\tilde{C}$ and let $\{\tilde{\tau}_i\}_{i \in \mathbb{N}^*}$ the sequence of $\{\hat{\mathcal{F}}_t^{\hat{W}}\}$-stopping times which exhausts the jumps of $\{\tilde{v}(t)\}$. Then, there exists $\psi \in S_1(\mathbb{R}^3)$ such that for all $(u, z) \in B_1(K) \times [0, 1]$

$$\begin{aligned} &[\tilde{z}(t) - z]\left\{-\psi_1 - 2T\psi_3 + \psi_2\left[\tilde{p}^1(t) + \frac{1}{2}\mathrm{tr}[D(t,\tilde{x}(t))^\top \tilde{P}(t) D(t,\tilde{x}(t))] + A(t,\tilde{x}(t))^\top \tilde{p}^2(t)\right]\right\} \\ &\quad + \psi_2\big[[B(t)[zu - \tilde{z}(t)\tilde{u}(t)]]^\top \tilde{p}^2(t) + [z|u| - \tilde{z}(t)|\tilde{u}(t)|]\tilde{p}^3(t) \\ &\quad + \mathrm{tr}[D(t,\tilde{x}(t))^\top \tilde{r}(t)](\sqrt{1-z} - \sqrt{1-\tilde{z}(t)})\big] \le 0 \end{aligned} \tag{115}$$

$\mathcal{M}_{\tilde{v}^{ac}}$ − a.s. on $[0, T] \times \hat{\Omega}$, where

$$\tilde{r}(t) \doteq \left[\tilde{q}^2(t) - \tilde{P}(t) D(t,\tilde{x}(t))\right]\sqrt{1-\tilde{z}(t)}, \quad \tilde{z}(t) \doteq \frac{\tilde{v}(t)}{1+\tilde{v}(t)}, \tag{116}$$

and

$$\begin{aligned} &\psi_2\big[[zu - \tilde{u}(t)]^\top B(t)^\top \tilde{p}^2(t) + [z|u| - |\tilde{u}(t)|]\tilde{p}^3(t)\big] \\ &\quad + (1-z)\left\{-\psi_1 - 2T\psi_3 + \psi_2\left[\tilde{p}^1(t) + \frac{1}{2}\mathrm{tr}\big[D(t,\tilde{x}(t))^\top \tilde{P}(t) D(t,\tilde{x}(t))\big] + A(t,\tilde{x}(t))^\top \tilde{p}^2(t)\right]\right\} \le 0 \end{aligned} \tag{117}$$

$\mathcal{M}_{\tilde{v}^s}-$a.s. on $[0,T]\times\hat{\Omega}$, and for all $i\in\mathbb{N}$ and $\gamma\in[0,1]$

$$\psi_2\Big[\Big[B(\tilde{\tau}_i,\tilde{x}^{\gamma}(\tau_i))(zu-\tilde{u}(\tilde{\tau}_i))\Big]^{\top}\tilde{p}^{\gamma 2}(\tilde{\tau}_i)+(z|u|-|\tilde{u}(\tilde{\tau}_i)|)\tilde{p}^{3}(\tilde{\tau}_i)\Big]$$
$$+(1-z)\Big\{-\psi_1-2T\psi_3+\psi_2\Big[\tilde{p}^{\gamma 1}(\tilde{\tau}_i-)+A\big(\tilde{\tau}_i,\tilde{x}^{\gamma}(\tilde{\tau}_i)\big)^{\top}\tilde{\gamma p}^{2}(\tilde{\tau}_i)$$
$$+\frac{1}{2}\mathrm{tr}\Big[D\big(\tilde{\tau}_i,\tilde{x}^{\gamma}(\tilde{\tau}_i)\big)^{\top}\tilde{P}^{\gamma}(\tilde{\tau}_i)D\big(\tilde{\tau}_i,\tilde{x}^{\gamma}(\tilde{\tau}_i)\big)\Big]\Big]\leq 0\Big\}, \tag{118}$$

$\hat{P}-$a.s. on $\{\tilde{\tau}_i\leq T\}$, with

$$\tilde{x}^{\gamma}(\tau_i)=\tilde{x}(\tau_i-)+\Psi\big(\tau_i,\tilde{x}(\tau_i-),\gamma\tilde{u}(\tau_i)\Delta\tilde{v}(\tau_i)\big), \tag{119}$$

$$\tilde{p}^{\gamma 1}(\tau_i)=\tilde{p}^{1}(\tau_i-)-\Psi_t\big(\tau_i,\tilde{x}(\tau_i-),\gamma\tilde{u}(\tau_i)\Delta\tilde{v}(\tau_i)\big)^{\top}$$
$$\times\Big[I_n+\Psi_x\big(\tau_i,\tilde{x}(\tau_i-),\gamma\tilde{u}(\tau_i)\Delta\tilde{v}(\tau_i)\big)^{\top}\Big]^{-1}\tilde{p}^{2}(\tau_i-), \tag{120}$$

$$\tilde{p}^{\gamma 2}(\tau_i)=\Big[I_n+\Psi_x\big(\tau_i,\tilde{x}(\tau_i-),\gamma\tilde{u}(\tau_i)\Delta\tilde{v}(\tau_i)\big)^{\top}\Big]^{-1}\tilde{p}^{2}(\tau_i-), \tag{121}$$

and

$$\tilde{P}^{\gamma}(\tau_i)=\Big[I_n+\Psi_x\big(\tau_i,\tilde{x}(\tau_i-),\gamma\tilde{u}(\tau_i)\Delta\tilde{v}(\tau_i)\big)^{\top}\Big]^{-1}\Big[\tilde{P}(\tau_i-)-\sum_{i=1}^{n}\Psi_{ixx}\big(\tau_i,\tilde{x}(\tau_i-),\gamma\tilde{u}(\tau_i)\Delta\tilde{v}(\tau_i)\big)^{\top}\Big]$$
$$\times\Big[I_n+\Psi_x\big(\tau_i,\tilde{x}(\tau_i-),\gamma\tilde{u}(\tau_i)\Delta\tilde{v}(\tau_i)\big)^{\top}\Big]^{-1}. \tag{122}$$

*Proof*. Combining Lemma 5.3 and Proposition 5.4, the results can be obtained by using the same arguments as those of Theorem 5.9 in Ref. [1]. □

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