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An almost sure functional limit theorem at zero for a class of Lévy processes normed by the square root function, and applications

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Abstract A recent result of Bertoin, Doney and Maller (Ann. Prob., 2007) gives an integral condition to characterize the class of Lévy processes $X(t)$ for which $\limsup_{t\downarrow 0} |X(t)|/\sqrt{t} \in (0, \infty)$ occurs almost surely (a.s.). For such processes we have a kind of almost sure "iterated logarithm" result, but without the logs. In the present paper we prove a functional version of this result, which then opens the way to various interesting applications obtained via a continuous mapping theorem. We set these out in a rigorous framework, including a characterisation of the existence of an a.s. cluster set for the interpolated process, appropriate to the continuous time situation. The applications relate to functional laws for the supremum, reflected and a variety of other processes, including a class of stochastic differential equations, where we aim to give as informative a description as we can of the functional limit sets.

Keywords Lévy process · Local behaviour · Almost sure convergence · Strassen's functional LIL · Iterated logarithm laws

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1 Introduction and functional law

Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a Lévy process X with $X(0) = 0$ and canonical triplet (γ, σ^2, Π), where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, and Π is a nonnegative measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}\setminus\{0\}}(x^2 \wedge 1)\Pi(dx) < \infty$.

The finite dimensional distributions of X are determined by the distribution of $X(1)$ via the celebrated Lévy–Khintchine formula, which states that $Ee^{i\theta X(t)} = e^{it\Psi(\theta)}$, where

$$\Psi(\theta) = i\gamma\theta - \frac{\sigma^2\theta^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{i\theta x} - 1 - i\theta x 1_{|x|\leq 1})\, \Pi(dx), \tag{1}$$

for $t > 0$ and $\theta \in \mathbb{R}$. Without loss of generality, we may assume throughout that, for all $\omega \in \Omega$, the sample paths $t \mapsto X(t, \omega)$ are càdlàg functions, i.e., right-continuous with left limits. See Bertoin [2] and Sato [15] for basic properties of Lévy processes.

A recent paper by Bertoin, Doney and Maller [3] characterizes when a Lévy process $X(t)$ crosses boundaries like t^κ, $\kappa > 0$, in a one or two-sided sense, for small times t; thus, necessary and sufficient conditions are given for the almost sure (a.s.) finiteness or otherwise of $\limsup_{t\downarrow 0} |X(t)|/t^\kappa$ (and similarly of $\limsup_{t\downarrow 0} X(t)/t^\kappa$ and $\liminf_{t\downarrow 0} X(t)/t^\kappa$) for each value of $\kappa > 0$. As a case of particular interest, when $\sigma^2 = 0$ an integral test is given to decide which value in $[0, \infty]$ is taken by $\limsup_{t\downarrow 0} |X(t)|/\sqrt{t}$, a.s.

In particular, [3] show that it is possible to have $\limsup_{t\downarrow 0} |X(t)|/\sqrt{t}$ in $(0, \infty)$ a.s., in which case we have a kind of almost sure "iterated logarithm" result, but without the logs. This is a curious situation since we usually think in terms of the square root norming giving convergence to a normal distribution (even as $t \downarrow 0$) with an extra log–log needed for the a.s. result. Our aim is to explore this effect further, in particular, by proving a functional generalisation of it. This opens the way to various interesting applications which are discussed below.

We proceed by stating in this section (in Theorem 2) the functional law for X. The applications which follow from it relate to functional laws for the supremum, reflected and a variety of other processes, including a class of stochastic differential equations. These are given in Sect. 3. We take pains in Sect. 3 to give *as informative a description as we can* of the functional limit sets, so this section is not just a routine, tautological, application of the continuous mapping theorem. Nevertheless, the bridge between Sects. 1 and 3 is indeed a continuous mapping theorem, and in Sect. 2 we provide a rigorous outline of the topological concepts that we need for Sect. 3, including a characterisation of the existence of an a.s. cluster set for the interpolated process. All proofs are in Sects. 4–6.

As a preliminary to stating the functional law, it is helpful to cite here Theorem 2 of [3] in full. First note that, if $\sigma^2 > 0$ in (1), X contains a Brownian component, which will dominate the process near zero. In particular, we then have $\limsup_{t\downarrow 0} X(t)/\sqrt{t} = \infty$ a.s. (cf. the discussion in [3]). Throughout, we will therefore assume that $\sigma^2 = 0$. Define a function $V(x)$ for $x > 0$ by setting

$$V(x) = \int_{0<|y|\le x} y^2\,\Pi(dy). \tag{2}$$

We assume throughout that $V(x) > 0$ for all $x > 0$; if $V(x_0) = 0$ for some $x_0 > 0$ then X is a compound Poisson process with a deterministic drift and local behaviour of X will not be an issue.

Theorem 1 [Bertoin, Doney, Maller] *Put*

$$I(a) = \int_0^1 \exp\left(-\frac{a^2}{2V(x)}\right)\frac{\mathrm{d}x}{x}, \quad a > 0, \tag{3}$$

and let

$$\lambda_X^* := \inf\{a > 0 : I(a) < \infty\} \in [0,\infty] \tag{4}$$

(*with the convention, throughout, that the inf of the empty set is* $+\infty$). *Then, a.s.,*

$$-\liminf_{t\downarrow 0}\frac{X(t)}{\sqrt{t}} = \limsup_{t\downarrow 0}\frac{X(t)}{\sqrt{t}} = \limsup_{t\downarrow 0}\frac{|X(t)|}{\sqrt{t}} = \lambda_X^*. \tag{5}$$

Examples are given in [3] to show that λ_X^* may take any value in $[0,\infty]$.

Our aim in this section is to give a functional version of Theorem 1. Denote by $D_0[0,1]$ the Banach space of càdlàg functions $g : [0,1] \to \mathbb{R}$, with $g(0) = 0$, endowed with the uniform norm $\|g\| = \sup_{0\le y\le 1}|g(y)|$. As in Strassen's functional law of the iterated logarithm [16], we consider the compact subset $\mathcal{K} \subseteq D_0[0,1]$ of absolutely continuous functions $g : [0,1] \to \mathbb{R}$ with $\int_0^1(\dot g(u))^2du \le 1$ (here $\dot g$ denotes the derivative of $g \in \mathcal{K}$, which is well defined on a set of Lebesgue measure 1 in $[0,1]$).

Given a Lévy process X for which $\lambda_X^* \in (0,\infty)$, define interpolated processes

$$(Z_t(y,\omega))_{0\le y\le 1} := \left(\frac{X(ty,\omega)}{\lambda_X^*\sqrt{t}}\right)_{0\le y\le 1}, \quad t > 0, \quad \omega \in \Omega. \tag{6}$$

Corresponding sample paths $y \mapsto Z_t(y,\omega)$ are functions in $D_0[0,1]$ indexed by $t > 0$, which we denote by $Z_t(\omega)$ or just by Z_t in the sequel.

Next follows our main result in this section. See Sect. 5 for its proof.

Theorem 2 *Suppose that* (5) *holds with* $\lambda_X^* \in (0,\infty)$. *Then there exists an* $\widetilde{\Omega} \in \mathcal{F}$ *with* $P(\widetilde{\Omega}) = 1$ *such that, for all* $\omega \in \widetilde{\Omega}$,

$$\liminf_{t\downarrow 0}\inf_{g\in\mathcal{K}}\|Z_t(\omega) - g\| = 0, \quad \text{and} \quad \liminf_{t\downarrow 0}\|Z_t(\omega) - f\| = 0 \quad \text{for all} \quad f \in \mathcal{K}. \tag{7}$$

Remark 1 (i) The first part of (7) says that the uniform distance $d_u(Z_t, \mathcal{K})$ between function Z_t and set $\mathcal{K}$ tends to 0 a.s. as $t \downarrow 0$. Thus, all accumulation points of $(Z_t)_{t>0}$ are in $\mathcal{K}$ ($\mathcal{K}$ is closed). The second part of (7) says that any f in $\mathcal{K}$ is obtained as the limit (in norm) of a subsequence of $(Z_t)_{t>0}$. $\mathcal{K}$ is thus the *cluster set* of $(Z_t)_{t>0}$. In the next section we make this and the other topological concepts precise.

(ii) It is easy to deduce from (7) that, for a fixed $y \in [0, 1]$, $-\sqrt{y} = \liminf_{t\downarrow 0} Z_t(y)$ and $\limsup_{t\downarrow 0} Z_t(y) = \sqrt{y}$, a.s.; in fact, the a.s. accumulation points of $(Z_t(y))_{t>0}$, as $t \downarrow 0$, for a fixed $y \in [0, 1]$, fill up the interval $[-\sqrt{y}, \sqrt{y}]$, a.s. This follows from more general results obtained in Sect. 3.

(iii) Suppose (7) holds for Z_t as defined in (6), for some $\lambda_X^* \in (0, \infty)$. Then $\limsup_{t\downarrow 0} |Z_t(1)| = \limsup_{t\downarrow 0} |X(t)|/(\lambda_X^* \sqrt{t}) = 1$ a.s., as just argued in (ii), so (5) holds for this λ_X^*. Theorem 1 then shows that $I(a)$ is finite (infinite) for a large (small), and in fact the point of division must be λ_X^*, by (5). Thus the converse of Theorem 2 is true in this sense.

(iv) As far as we know, "small time" iterated log-related results for Lévy processes remain relatively unexplored. A small time version of Khintchine's log–log law for Brownian motion is in Gantert [10]. This carries over immediately to any Lévy process with $\sigma^2 > 0$, in fact, [3] give

$$-\liminf_{t\downarrow 0} \frac{X(t)}{\sqrt{2t \log|\log t|}} = \limsup_{t\downarrow 0} \frac{X(t)}{\sqrt{2t \log|\log t|}} = \sigma, \text{ a.s.} \tag{8}$$

Gantert [10] gives a functional version of this; see also Deuschel and Stroock [9]. For related literature we refer also to Baldi [1] and Caramellino [5], who consider functional laws for some diffusions other than Brownian motion. de Acosta [8] gives some large time functional LILs using a formulation similar to ours in Theorem 2, in relation to some investigations regarding refinements of the approximation for elements in the interior of $\mathcal{K}$. See also Csáki [6].

(v) Many applications of weak and strong functional limit theorems for random walks have been made and published in various areas. These are "large time" applications and they have obvious analogues for Lévy processes at large times. Similar "small time" applications for Lévy processes are certainly of interest, and, to our knowledge, remain unexplored. We present some results like this in Sect. 3. To prepare the way for these we set out some relevant topological concepts in Sect. 2.

2 Continuous mappings

Functional limit theorems provide a basis for transferring limit results from an arena where they may be relatively easily derived to a quite different and seemingly much more complex class of processes via a continuous mapping theorem. Thus the result of Theorem 2, which is proved making extensive use of very specialised properties of the Lévy process X (independent and stationary increments, small and large jump decomposition, etc.), can easily be transferred for example to the supremum process

$S(t) := \sup_{0<s\le t} X(s)$, and even much more complex functionals of X, which have no such helpful properties. Some surprising and colourful examples appear in Strassen's pioneering paper [16] for random walks. As another example of the possibilities see Csáki et al. [7]. In the next section we examine a variety of results like this for Lévy processes. Complications in the present context are that we want to work in the non-separable space $D_0[0, 1]$, and within this space to deal with families of processes in continuous time. The following discussion clarifies some important points.

Definition 2.1 (Almost sure cluster set) Let $Z = (Z_t)_{t>0}$ be a family of mappings from a probability space $(\Omega, \mathcal{F}, P)$ to a metric space (E, d). For each $\omega \in \Omega$ we define the cluster set $\mathcal{C}_0((Z_t(\omega))_{t>0})$ as the collection of all accumulation points of $(Z_t(\omega))_{t>0}$ as $t \downarrow 0$; that is,

$$\mathcal{C}_0((Z_t(\omega))_{t>0}) = \bigcap_{t>0} \overline{\{Z_s(\omega) : s \le t\}},$$

where the closure is taken with respect to the metric d.

If there exists a nonrandom set $\widetilde{\mathcal{C}}_0(Z)$ and $\widetilde{\Omega} \in \mathcal{F}$ with $P(\widetilde{\Omega}) = 1$ such that $\widetilde{\mathcal{C}}_0(Z) = \mathcal{C}_0((Z_t(\omega))_{t>0})$ for all $\omega \in \widetilde{\Omega}$, we call $\widetilde{\mathcal{C}}_0(Z)$ an *a.s. cluster set* of $Z = (Z_t)_{t>0}$ for $t \downarrow 0$. Since all a.s. cluster sets are equal a.s. we call any one of them *the* a.s. cluster set, $\mathcal{C}_0(Z)$.

The next theorem shows that the existence of an a.s. cluster set $\mathcal{C}_0(Z)$ is equivalent to a weak form of the Blumenthal zero-one law. Say that a nonempty closed set E_0 *attracts* $(Z_t)_{t>0}$, a.s., as $t \downarrow 0$, if $\lim_{t\downarrow 0} d(Z_t, E_0) = 0$, a.s.

Theorem 3 *Let $Z = (Z_t)_{t>0}$ be a family of mappings from a probability space $(\Omega, \mathcal{F}, P)$ to a metric space (E, d) and let $E_0 \subseteq E$ be a nonempty countable set attracting Z for $t \downarrow 0$ a.s. Then* (i) *is equivalent to* (ii)*, where*: (i) *the a.s. cluster set $\mathcal{C}_0(Z)$ exists*; (ii) *for all $\varepsilon > 0$ and $e \in E_0$, either $I_{e,\delta} = 0$ a.s. for all $\delta \ge \varepsilon$ or $I_{e,\delta} > 0$ a.s. for all $\delta \in (0, \varepsilon)$, respectively, where, for $e \in E$ and $\delta > 0$, we set*

$$I_{e,\delta}(\omega) = \liminf_{t\downarrow 0} d(Z_t(\omega), B_\delta(e)), \qquad \omega \in \Omega,$$

and $B_\delta(e)$ denotes the open ball with center $e \in E$ and radius $\delta > 0$ in E.

Note that (i) of Theorem 3 does not preclude the possibility that $\mathcal{C}_0(Z)$ exists but is empty. The next corollary addresses this.

Corollary 1 *Let $Z = (Z_t)_{t>0}$ be a family of mappings from a probability space $(\Omega, \mathcal{F}, P)$ to a complete metric space (E, d) which satisfies* (ii) *in Theorem 3 and is a.s. sequentially relatively compact at zero, that is, is such that for all sequences $t_n \downarrow 0$ the set $\overline{\{Z_{t_n} : n \in \mathbb{N}\}}$ is compact, a.s.*

If $Z = (Z_t)_{t>0}$ is a.s. attracted by a countable subset, E_0, say, of E, for $t \downarrow 0$, or E is separable, then $\mathcal{C}_0(Z)$ exists, is nonempty, compact and attracts $(Z_t)_{t>0}$, a.s.

The next corollary uses the Blumenthal zero-one law to give sufficient conditions for (ii) of Theorem 3 to hold.

Corollary 2 *Let $Z = (Z_t)_{t>0}$ be a family of random elements from a probability space $(\Omega, \mathcal{F}, P)$ to a metric space (E, d) endowed with the σ-algebra $\mathcal{B}_B(E)$ generated by the open balls on E.*

Suppose that, for all $\varepsilon > 0$, $e \in E_0$ and $0 < \alpha < \beta$, there exists a countable subset $J \subseteq [\alpha, \beta] \times B_\varepsilon(e)$ such that, a.s.,

$$\inf_{\alpha \le t \le \beta} d(Z_t, B_\varepsilon(e)) = \inf_{(t,f) \in J} d(Z_t, f). \tag{9}$$

Let $\mathcal{F}_{0+} = \cap_{t>0} \mathcal{F}_t$, where $\mathcal{F}_t$ is the σ-algebra generated by $\{Z_u, 0 < u \le t\}$. If the strict Blumenthal zero-one law is satisfied, i.e., $P(B) \in \{0, 1\}$ for all $B \in \mathcal{F}_{0+}$, then (ii) *in Theorem 3 is satisfied.*

It follows from Corollary 1 that, if in addition Z in Corollary 2 is a.s. attracted to a countable subset for $t \downarrow 0$, then the a.s. cluster set $\mathcal{C}_0(Z)$ exists and is nonempty.

Remark 2 (i) (Continuous mappings) Let $Z = (Z_t)_{t>0}$ be a family of random processes with values in a complete metric space (E_1, d_1). Suppose that $Z = (Z_t)_{t>0}$ is a.s. sequentially relatively compact. It is straightforward to show that if Ψ is a continuous mapping from (E_1, d_1) to another metric space (E_2, d_2) and the a.s. cluster set $\mathcal{C}_0(Z)$ exists, then the a.s. cluster set $\mathcal{C}_0(\Psi \circ Z)$ exists and we have

$$\mathcal{C}_0(\Psi(Z)) = \Psi \circ \mathcal{C}_0(Z). \tag{10}$$

Properties such as a.s. attracting or a.s. sequentially relatively compact are also maintained under Ψ.

(ii) $(D_0[0, 1], d_u)$ is a complete (but not separable) metric space with d_u as the uniform metric on $D_0[0, 1]$. Going back to our original Lévy process setup, in the situation of Theorem 2 suppose that (5) is satisfied for a $\lambda_X^* \in (0, \infty)$. Then there exists $\widetilde{\Omega} \in \mathcal{F}$ with $P(\widetilde{\Omega}) = 1$ such that, for all $\omega \in \widetilde{\Omega}$, (7) is satisfied with $Z_t(\omega)$ as defined in (6). Since $\mathcal{K}$ is closed, it follows from the continuous mapping theorem that the cluster set $\mathcal{C}_0((Z_t(\omega))_{t>0})$ exists and equals $\mathcal{K}$ for all $\omega \in \widetilde{\Omega}$. Since $\mathcal{K}$ is nonrandom, it is the a.s. cluster set at 0 of $(Z_t)_{t>0}$, according to Definition 2.1. As $\mathcal{K}$ is also compact and attracts $(Z_t(\omega))_{t>0}$, the family $(Z_t(\omega))_{t>0}$ is sequentially relatively compact for $t \downarrow 0$ and all $\omega \in \widetilde{\Omega}$. Thus we can apply the continuous mapping theorem to $(Z_t)_{t>0}$. Note that in this case the existence of the a.s. cluster set is established by direct construction in the proof of Theorem 2.

(iii) It might be enquired why we do not assume the a.s. relative compactness of $\{Z_s : 0 < s \le t\}$ at $t > 0$, in the above formulation. Our setup is more general in that almost sure sequential relative compactness of a family $(Z_t(\omega))_{t>0}$ at zero does not in general imply the a.s. relative compactness of $\{Z_s : 0 < s \le t\}$ for any $t > 0$. To see this, consider the situation in the previous paragraph. Fix $\omega \in \widetilde{\Omega}$. Suppose that there exists $t = t(\omega)$ such that $\overline{\{Z_t(\omega) : 0 < s \le t\}}$ is compact in $(D_0[0, 1], d_u)$. This would imply the existence of a sequence $t_n = t_n(\omega) \uparrow t$ and $g = g(\omega) \in D_0[0, 1]$ such that $Z_{t_n}(\omega) \to g$ uniformly and, thus, pointwise.

Consequently, $g(y) = X(ty-)/(\lambda_X^* \sqrt{t})$ for all $0 \le y \le 1$. But since X is a pure jump process, g is not right continuous. This contradicts $g \in D_0[0, 1]$ a.s.

(iv) In the sequel, we will only consider mappings $f : D_0[0, 1] \to D_0[0, 1]$ which are continuous in the uniform metric d_u. We could, alternatively, consider other metrics d on $D_0[0, 1]$. In view of the continuous mapping theorem, compact sets of (a.s.) accumulation points will not differ whenever the identity mapping, as a mapping from $(D_0[0, 1], d_u)$ to $(D_0[0, 1], d)$, remains continuous. Particularly, this concerns any metric generating the Skohorod topology on $D_0[0, 1]$ (cf. [4, Chap. 3]).

Theorem 3 and its corollaries are a kind of generalisation of Theorem 1 of Kesten [13] to small time behaviour of a continuous time process. Kesten considers the accumulation points in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ of a centered, normalised random walk, $(Z_t)_{t=1,2,\ldots}$, say, as $t \to \infty$, and shows that the cluster set is a.s. a deterministic subset of $\overline{\mathbb{R}}$. With only notational changes, we can take $t \uparrow \infty$ in Definition 2.1, and speak of the a.s. cluster set, $\mathcal{C}_\infty((Z_t)_{t>0})$, for $t \uparrow \infty$, of a family of random processes. If sequential relative compactness at 0 is replaced by sequential relative compactness at ∞, and $\mathcal{C}_0$ is replaced by $\mathcal{C}_\infty$, then Theorem 3 and its corollaries have obvious counterparts. With these understandings, in the next section we will drop the subscript on $\mathcal{C}(Z)$ except where it is essential or needed for emphasis.

3 Applications

Throughout this section let $X = (X(y, \omega))_{y \ge 0}$, $\omega \in \Omega$, be a stochastic process taking values in $\mathbb{R}$ with $X(0) = 0$ and càdlàg paths. We suppose $b(t) > 0$ is a nonstochastic function on $[0, \infty)$ and consider the process

$$(Z_t^X(y, \omega))_{0 \le y \le 1} := \left(\frac{X(ty, \omega)}{b(t)} \right)_{0 \le y \le 1}, \quad \omega \in \Omega, \tag{11}$$

indexed by $t > 0$. Throughout we consider the uniform norm on $D_0[0, 1]$ and assume that $Z^X = (Z_t^X)_{t>0}$ is a.s. sequentially relatively compact with an a.s. cluster set $\mathcal{C}(Z^X) = \mathcal{K}$ in $D_0[0, 1]$. This is the case for example when $b(t) = \lambda_X^* \sqrt{t}$ and Theorem 2 obtains; $\mathcal{K}$ is then the cluster set of Z_t^X as $t \downarrow 0$. But with the general norming $b(t)$, and without specifying the limiting value of t, we allow much more general situations. We mention for example that (M. Savov, personal communication, 2006) has recently generalised Theorem 1 to allow norming by a function which is regularly varying with index $1/2$ as $t \downarrow 0$.

The following functionals are of importance in a variety of applications:

Supremum and Infimum: $S(t) = \sup_{0 \le s \le t} X(s)$; $i(t) = \inf_{0 \le s \le t} X(s)$.

Reflection in Sup and Inf: $R(t) = S(t) - X(t)$; $r(t) = X(t) - i(t)$.

Reflected Suprema: $Q(t) = \sup_{0 \le s \le t} R(s)$; $q(t) = \sup_{0 \le s \le t} r(s)$.

These operations define continuous functionals on $D_0[0, 1]$, and to each corresponds a family of processes: $Z^S = \left(Z_t^S(y)\right)_{0<y\le 1} = (S(ty)/b(t))_{0<y\le 1}$, indexed by $t > 0$, and similarly for Z^R, Z^Q, etc. Thus it follows immediately from the continuous mapping theorem that, under our assumption $\mathcal{C}(Z) = \mathcal{K}$, we have $\mathcal{C}(Z^S) = \mathcal{K}^S$ and $\mathcal{C}(Z^R) = \mathcal{K}^R$, where

$$\mathcal{K}^S = \{s : \exists f \in \mathcal{K} : s(v) = \sup_{0 \le u \le v} f(u),\ 0 \le v \le 1\}, \tag{12}$$

and

$$\mathcal{K}^R = \{r : \exists f \in \mathcal{K} : r(v) = \sup_{0 \le u \le v} f(u) - f(v),\ 0 \le v \le 1\}. \tag{13}$$

These results are tautological in that they follow directly from the continuous mapping theorem. What we would like, instead, is a simple description of the derived function classes. The next theorem gives such a description for the joint triple (Z^X, Z^S, Z^R) as a family of mappings (in t) from the underlying probability space into $D_0^3[0, 1]$ (for $m \in \mathbb{N}$, we let $D_0^m[0, 1]$ denote the Banach space of càdlàg functions $g : [0, 1] \to \mathbb{R}^m$, with $g(0) = 0$, endowed with the uniform norm). Let $\mathcal{A}$ be the set of absolutely continuous functions $f : [0, 1] \to \mathbb{R}$ with $f(0) = 0$. The nondecreasing and nonnegative functions in $\mathcal{A}$ are denoted by $\mathcal{A}^{\le}$ and $\mathcal{A}^{+}$, respectively. The proof of the following theorem is in Sect. 6.1.

Theorem 4 *If Z^X is a.s. sequentially relatively compact with $\mathcal{C}(Z^X) = \mathcal{K}$ in $D_0[0, 1]$, then (Z^X, Z^S, Z^R) is a.s. sequentially relatively compact in $D_0^3[0, 1]$ with*

$$\mathcal{C}(Z^X, Z^S, Z^R) = \Bigg\{(s - r, s, r) \in D_0^3[0, 1] \ :\ s \in \mathcal{A}^{\le},\quad r \in \mathcal{A}^{+},\ \int_0^1 r(u)\,\dot{s}(u)\,du = 0,\quad \int_0^1 \dot{r}^2(u)\,du + \int_0^1 \dot{s}^2(u)\,du \le 1\Bigg\}.$$

In particular, Z^S and Z^R are a.s. sequentially relatively compact in $D_0[0, 1]$ with

$$\mathcal{C}(Z^S) = \mathcal{K}^S = \{f \in \mathcal{K} : f \text{ nondecreasing}\},$$

and

$$\mathcal{C}(Z^R) = \mathcal{K}^R = \{f \in \mathcal{K} : f \ge 0\}.$$

One can iterate these procedures; let $R^{(0)} = X$, $S^{(0)} = S$, and, for $n = 0, 1, 2, \ldots$, set $R^{(n+1)} = S^{(n)} - R^{(n)}$ and $S^{(n+1)}(t) = \sup_{0 \le u \le t} R^{(n+1)}(u)$ (so that $Q(t)$ is $S^{(1)}$). Interestingly, the cluster sets of the marginals of the processes correspond to smaller and smaller cluster sets as the order of iteration increases. We make this precise in the next theorem (see Subsect. 6.2 for a proof).

Theorem 5 *Suppose that Z is a.s. sequentially relatively compact in $D_0[0, 1]$ with $\mathcal{C}(Z) = \mathcal{K}$. Then, for $n \in \mathbb{N}$, both $(R^{(n)}(t)/b(t))_{t>0}$ and $(S^{(n)}(t)/b(t))_{t>0}$ are sequentially relatively compact in $\mathbb{R}$ with*

$$\mathcal{C}\left(\left(R^{(n)}(t)/b(t)\right)_{t>0}\right) = \mathcal{C}\left(\left(S^{(n)}(t)/b(t)\right)_{t>0}\right) = [0, 1/n]. \tag{14}$$

Going back to our Lévy process $X(t)$, assume (5) holds for a $\lambda_X^* \in (0, \infty)$ and take, for $t > 0, 0 < y \le 1$, $Z_t(y)$ as defined in (6). We can then read off from Theorem 4 that the a.s. cluster set at 0 of the processes $\sup_{0<s\le t} Z_s(y) = \sup_{0<s\le t} X(sy)/(\lambda_X^* \sqrt{t})$, for example, is $\mathcal{K}^S = \{$the nondecreasing functions in $\mathcal{K}\}$; and correspondingly for X reflected in its supremum or infimum, the running suprema of the reflected processes, etc. Similarly, as immediate corollaries, we also get one-dimensional results such as $0 = \liminf_{t\downarrow 0} S(t)/\sqrt{t} < \limsup_{t\downarrow 0} S(t)/\sqrt{t} = \lambda_X^*$, a.s., where $S(t) = \sup_{0<s\le t} X(s)$; and, further, the a.s. cluster set (in $\mathbb{R}$) of $\left(S(t)/\sqrt{t}\right)_{t>0}$, as $t \downarrow 0$, is the closed interval $[0, \lambda_X^*]$. These statements remain true if S is replaced by R. The a.s. cluster set of $\left(X(t)/\sqrt{t}\right)_{t>0}$ is the closed interval $[-\lambda_X^*, \lambda_X^*]$ (cf. Remark 1 (ii)).

Remark 3 (i) There are analogous results to Theorems 4 and 5 for other variants of the reflected and supremum or infimum processes, such as $i(t)$, $r(t)$, $q(t)$, etc., and iterates of them, which we leave to the interested reader. Also, there are applications of these objects; for example, the *local score* as used in genetics would, in the present notation, be $\sup_{0\le s\le t} r(s) = \sup_{0\le s\le t}(X(s) - \inf_{0\le y\le s} X(y))$.

(ii) We can apply the same methods to the large time LIL for a Brownian motion, $W(t)$, with variance rate $\sigma^2 > 0$, which satisfies (8) with $t \uparrow \infty$. Then if, for $0 \le y \le 1, t > e$,

$$U_t(y) := \frac{W(ty)}{\sqrt{2\sigma t \log|\log t|}},$$

we get that the cluster set of $(U_t)_{t>e}$ as $t \uparrow \infty$, i.e., $\mathcal{C}_\infty(U)$, is $\mathcal{K}$. We can then proceed to write down the cluster sets, as $t \uparrow \infty$, of the supremum of W, its reflected process, etc., by the same methods. Similarly for $\mathcal{C}_0(U)$, the cluster set of U as $t \downarrow 0$.

Similarly, we can deal for example with the other functionals considered by Strassen [16], who gave only the one-dimensional results.

(iii) In the Abstract we introduced the result (5) as a kind of "iterated logarithm" result, but without the logs. The reference here is of course to the classical large time LIL as discussed in the previous paragraph. As is well known, this classical result is closely related to normal approximation, in the case of random walk, or to properties of the normal distribution itself, as in the case of Brownian motion. It's less obvious, but made clear in our proof of Theorem 2 (see, e.g., Lemma 1 in Sect. 5), that (5) also comes about by normal approximation. The square root norming is even more natural in our context, in this sense.

Now we turn to some applications to stochastic differential equations (SDEs). Consider the process $L(u)$ satisfying

$$L(u) = \int_0^u B(L(v))\,dv + X(u), \quad 0 \le u \le 1, \tag{15}$$

for a function $B : \mathbb{R} \to \mathbb{R}$. This is the unique finite pathwise solution to the corresponding SDE, provided B is Lipschitz. Let $\left(Z_t^L(y)\right)_{0<y\leq 1} = (L(ty)/b(t))_{0<y\leq 1}$, for $t > 0$, where $b(t) > 0$. The proof of the next theorem is found in Sect. 6.3.

Theorem 6 *Let B be a Lipschitz function. Suppose that Z^X is a.s. sequentially relatively compact as $t \downarrow 0$ with $\mathcal{C}_0(Z) = \mathcal{K}$, and that $b(t) \to 0$ and $t/b(t) \to 0$ as $t \downarrow 0$. Then (Z^X, Z^L) is a.s. sequentially relatively compact as $t \downarrow 0$ in $D_0^2[0,1]$ with $\mathcal{C}_0((Z^X, Z^L)) = \{(f, f) \in D_0^2[0,1] : \ f \in \mathcal{K}\}$, a.s., as $t \downarrow 0$. Suppose in addition that B is $n+1$ times continuously differentiable in a neighborhood of zero with derivatives $B^{(k)}(0) = 0$, $0 \leq k \leq n-1$, $B^{(n)}(0) \neq 0$, for $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Then the a.s. cluster set as $t \downarrow 0$ of*

$$\frac{n!}{B^{(n)}(0)} \left(\frac{L(t\cdot) - X(t\cdot)}{tb^n(t)}\right)_{t>0}$$

is the set of continuously differentiable functions f with $f(0) = 0$ satisfying the following properties: if $n = 0$ then $\dot{f} \equiv 1$. If $n > 0$ is even then $\dot{f} \geq 0$ and $(\dot{f})^{1/n} \in \mathcal{K}$. If n is odd then $(\dot{f})^{1/n} \in \mathcal{K}$.

Remark 4 (i) Let $B(u) = -\gamma u$ where $\gamma \neq 0$. Then L is the Ornstein-Uhlenbeck process driven by X with mean reversion parameter γ. In the situation of Theorem 6, we find that $(L(t\cdot) - X(t\cdot))/(tb(t))$ has as its almost sure cluster set for $t \downarrow 0$ the class of continuously differentiable functions f, where $\dot{f}/|\gamma| \in \mathcal{K}$.

(ii) We restricted ourselves to uncentered processes in the preceding, as this is all that's needed in Theorem 2. More generally, we could also center X at a nonstochastic function. We will not go into details here.

4 Proofs for Sect. 2

Proof of Theorem 3 Assume throughout that E_0 is nonempty, countable, and attracts $(Z_t)_{t>0}$, a.s. First, (i) $\Rightarrow$ (ii) is immediate from Definition 2.1. Indeed: suppose $\mathcal{C}_0(Z)$ exists, and let $e \in E_0$ and $\delta > 0$. Then $\mathcal{C}_0(Z) \cap B_\varepsilon(e) \neq \emptyset$ implies $I_{e,\delta} = 0$ a.s. for all $\delta \geq \varepsilon$, whereas $\mathcal{C}_0(Z) \cap B_\varepsilon(e) = \emptyset$ implies $I_{e,\delta} > 0$ a.s. for all $\delta < \varepsilon$.

Next, to show (ii)$\Rightarrow$(i), assume (ii), let $(\Omega, \mathcal{F}^*, P^*)$ be the completion of $(\Omega, \mathcal{F}, P)$, and set $\mathcal{C}_0(Z(\omega)) := \mathcal{C}_0((Z_t(\omega))_{t>0})$, $\omega \in \Omega$. As E_0 attracts $(Z_t)_{t>0}$ a.s. for $t \downarrow 0$, there exists $\Omega_1 \in \mathcal{F}^*$ with $P^*(\Omega_1) = 1$ such that

$$\lim_{t\downarrow 0} d(Z_t(\omega), E_0) = 0, \qquad \omega \in \Omega_1 . \tag{16}$$

It follows easily that $\mathcal{C}_0(Z(\omega)) \subseteq \overline{E_0}$ for all $\omega \in \Omega_1$.

Now set

$$E_{1,n} = \left\{e \in E_0 \ : \ P^*(I_{e,1/n} = 0) = 1\right\}, \qquad n \in \mathbb{N} . \tag{17}$$

First suppose that there exists $N \in \mathbb{N}$ with $E_{1,N} = \emptyset$, and set

$$\Omega_2 = \bigcap_{e \in E_0} \left\{\omega \in \Omega_1 : \ I_{e,1/(2N)}(\omega) > 0\right\}. \tag{18}$$

(Recall that E_0 is countable.) In view of (ii), we have $\Omega_2 \in \mathcal{F}^*$ with $P^*(\Omega_2) = 1$. Suppose there exist $\omega \in \Omega_2$ and $c \in \mathcal{C}_0(Z(\omega))$. As $\omega \in \Omega_1$, recall that $\mathcal{C}_0(Z(\omega)) \subseteq \overline{E_0}$. Consequently, there exists $e \in E_0$ with $d(e, c) < 1/(2N)$. But then $I_{e,1/(2N)}(\omega) = 0$, as $c \in \mathcal{C}_0(Z(\omega))$, contradicting $\omega \in \Omega_2$. To summarize, if there exists $N \in \mathbb{N}$ such that $E_{1,N}$ is empty then the a.s. cluster set $\mathcal{C}_0(Z)$ exists and equals the empty set.

Now suppose that, for all $n \in \mathbb{N}$, $E_{1,n}$ is nonempty, and consider

$$\begin{aligned}\Omega_3 = &\bigcap_{n \in \mathbb{N}} \bigcap_{e \in E_{1,n}} \left\{\omega \in \Omega_1 : \ I_{e,1/n}(\omega) = 0\right\} \\ &\cap \bigcap_{n \in \mathbb{N}} \bigcap_{e \in E_0 \setminus E_{1,n}} \left\{\omega \in \Omega_1 : \ I_{e,1/(2n)}(\omega) > 0\right\},\end{aligned}$$

where, for $n \in \mathbb{N}$ with $E_{1,n} = E_0$, we make use of the convention

$$\bigcap_{e \in E_0 \setminus E_{1,n}} \left\{\omega \in \Omega_1 : \ I_{e,1/(2n)}(\omega) > 0\right\} := \Omega.$$

As E_0 is countable and (ii) is satisfied, we observe $\Omega_3 \in \mathcal{F}^*$ and $P^*(\Omega_3) = 1$. Define

$$M = \bigcap_{n \in \mathbb{N}} \bigcup_{e \in E_{1,n}} B_{1/n}(e).$$

We shall show that $M = \mathcal{C}_0(Z(\omega))$ for all $\omega \in \Omega_3$. In this case, the a.s. cluster set $\mathcal{C}_0(Z)$ exists and equals M.

To see that $M \subseteq \mathcal{C}_0(Z(\omega))$, let $c \in M$ and $\omega \in \Omega_3$. For $n \in \mathbb{N}$, there exists $e_n \in E_{1,n}$ with $d(c, e_n) < 1/n$. Note that $B_{1/n}(e_n) \subseteq B_{2/n}(c)$. As $\omega \in \Omega_3$, for all $n \in \mathbb{N}$, this implies

$$\liminf_{t \downarrow 0} d(Z_t(\omega), B_{2/n}(c)) \leq \liminf_{t \downarrow 0} d(Z_t(\omega), B_{1/n}(e_n)) = I_{e_n,1/n}(\omega) = 0,$$

thus, $c \in \mathcal{C}_0(Z(\omega))$.

To see the remaining inclusion, fix $\omega \in \Omega_3$ and let $c \in \mathcal{C}_0(Z(\omega))$. As we also have $\omega \in \Omega_1$, recall that $c \in \mathcal{C}_0(Z(\omega)) \subseteq \overline{E_0}$. Thus, for all $n \in \mathbb{N}$, there exists $e_n = e_n(\omega) \in E_0$ with $d(c, e_n) < 1/(2n)$. Since $c \in \mathcal{C}_0(Z(\omega))$, we get $I_{e_n,1/(2n)}(\omega) = 0$. Consequently, for all $n \in \mathbb{N}$, we must have $e_n \in E_{1,n}$ and, thus, $c \in M$. This completes the proof of (ii)⇒(i). □

Proof of Corollary 1 Assume Z and (E, d) as specified, with (E, d) complete. If E is separable then there is a countable subset $E_0 \subseteq E$ with $E = \overline{E_0}$. Clearly, Z is then a.s. attracted by E_0 as $t \downarrow 0$; thus, this holds in either case.

As Z is a.s. sequentially relatively compact as $t \downarrow 0$, we find a set $\Omega_1 \in \mathcal{F}$ with $P(\Omega_1) = 1$ such that, for all $\omega \in \Omega_1$, $(Z_t(\omega))_{t>0}$ is sequentially relatively compact for $t \downarrow 0$. Take $\omega \in \Omega_1$. Recall that in complete metric spaces compactness of a set K is equivalent to the Bolzano–Weierstrass property, i.e., for all $(k_n)_{n\in\mathbb{N}} \subseteq K$ there exists $k \in K$ and a subsequence n' such that $k_{n'} \to k$ in E. In view of this, $\overline{\{Z_{t_n}(\omega) : n \in \mathbb{N}\}}$ is nonempty for any $t_n \downarrow 0$, and as $\mathcal{C}_0((Z_t(\omega))_{t>0})$ contains the accumulation points of $Z_{t_n}(\omega)$ for any $t_n \downarrow 0, \mathcal{C}_0((Z_t(\omega))_{t>0})$ is nonempty too. Given a sequence $(g_n(\omega)) \subseteq \mathcal{C}_0((Z_t(\omega))_{t>0})$, we can find $t_n \downarrow 0$ such that $d(g_n(\omega), Z_{t_n}(\omega)) \to 0$ for $n \to \infty$. Then there exists a subsequence $n' \to \infty$ and $Z(\omega) \in \overline{\{Z_{t_n}(\omega) : n \in \mathbb{N}\}} \subseteq \mathcal{C}_0((Z_t(\omega))_{t>0})$ such that $Z_{t_{n'}}(\omega) \to Z(\omega)$ and, thus, $g_{n'}(\omega) \to Z(\omega)$ by the triangle inequality. Consequently, $\mathcal{C}_0((Z_t(\omega))_{t>0})$ is compact. In order to show that $\mathcal{C}_0((Z_t(\omega))_{t>0})$ attracts $(Z_t(\omega))_{t>0}$ for $t \downarrow 0$, suppose that there are $t_n \downarrow 0$ and $\varepsilon > 0$ such that $\inf_{g\in\mathcal{C}_0((Z_t(\omega))_{t>0})} d(Z_{t_n}(\omega), g(\omega)) \geq \varepsilon$. But there exist a subsequence $n' \to \infty$ and $Z(\omega) \in \overline{\{Z_{t_n}(\omega) : n \in \mathbb{N}\}} \subseteq \mathcal{C}_0((Z_t(\omega))_{t>0})$ such that $d(Z_{t_{n'}}(\omega), Z(\omega)) \to 0$ for $n' \to \infty$. This yields the contradiction

$$0 < \varepsilon \leq \liminf_{n\to\infty} \inf_{g\in\mathcal{C}_0((Z_t(\omega))_{t>0})} d(g(\omega), Z_{t_n}(\omega)) \leq \limsup_{n'\to\infty} d(Z(\omega), Z_{t_{n'}}(\omega)) = 0.$$

Thus $\mathcal{C}_0((Z_t(\omega))_{t>0})$ is nonempty and compact and attracts $(Z_t(\omega))_{t>0}$ as $t \downarrow 0$. In view of Theorem 3, there exists $\Omega_2 \in \mathcal{F}$ with $\Omega_2 \subseteq \Omega_1$ and $P(\Omega_2) = 1$ and a nonrandom set $M \subseteq E$ such that $\mathcal{C}_0((Z_t(\omega))_{t>0}) = M$ for all $\omega \in \Omega_2$. Clearly, M must be nonempty and compact and, for all $\omega \in \Omega_2$, $(Z_t(\omega))_{t>0}$ is attracted by M as $t \downarrow 0$. □

Proof of Corollary 2 Let $e \in E_0$ and $\varepsilon > 0$. In view of (9), for $n \in \mathbb{N}$ there exists $\Omega_n \in \mathcal{F}$ with $P(\Omega_n) = 1$ and a countable set $J_n \subseteq [2^{-n-1}, 2^{-n}] \times B_\varepsilon(e)$ such that

$$\inf_{2^{-n-1}\leq t\leq 2^{-n}} d(Z_t(\omega), B_\varepsilon(e)) = \inf_{(t,f)\in J_n} d(Z_t(\omega), f), \qquad \omega \in \Omega_n .$$

For $\omega \in \Omega$ set $V_n(\omega) = \inf_{(t,f)\in J_n} d(Z_t(\omega), f)$. As J_n is countable, V_n is $\mathcal{F}_{2^{-n}}$-measurable and, thus, $V_\infty := \liminf_{n\to\infty} V_n$ is $\mathcal{F}_{0+}$-measurable. Note that

$$I_{e,\varepsilon}(\omega) = \liminf_{t\downarrow 0} d(Z_t(\omega), B_\varepsilon(e)) = V_\infty(\omega), \qquad \omega \in \bigcap_{n\in\mathbb{N}} \Omega_n ,$$

where $I_{e,\varepsilon}$ is the quantity as defined in Theorem 3.

Let $(\Omega, \mathcal{F}^*_{0+}, P^*)$ be the completion of $(\Omega, \mathcal{F}_{0+}, P)$ with respect to P. Then the identity in the last display implies that $I_{e,\varepsilon}$ is $\mathcal{F}^*_{0+}$-measurable. It follows from the Blumenthal zero-one law, that $P^*(F) \in \{0, 1\}$ for all $F \in \mathcal{F}^*_{0+}$ and, in particular, $P^*(I_{e,\varepsilon} = 0) \in \{0, 1\}$, giving (ii) in Theorem 3. □

5 Proof of Theorem 2

We follow the proof of Theorem 2 of [3] as far as possible but substantial modification is needed to accommodate the functional elements. We handle these using a method of Finkelstein [11]. Throughout, we assume that $\sigma^2 = 0$, $V(x) > 0$ for all $x > 0$, and (5) holds for a $\lambda_X^* \in (0, \infty)$.

Let $p \in \mathbb{N}$ and let $\mathcal{L}_p \subseteq D_0[0, 1]$ be the subspace of continuous functions which are linear on subintervals $[0, 1/p], \ldots, [(p-1)/p, 1]$. Define a mapping $L_p : D_0[0, 1] \to \mathcal{L}_p$, where $L_p f = L_p[f] \in \mathcal{L}_p$ denotes the (unique and continuous) function interpolating $f \in D_0[0, 1]$ on the grid $0, 1/p, \ldots, (p-1)/p, 1$, and define a linear bijection $\nu_p : \mathcal{L}_p \to \mathbb{R}^p$, by setting

$$\nu_p(f) = (f(1/p), f(2/p) - f(1/p), \ldots, f(1) - f((p-1)/p)) , \quad f \in \mathcal{L}_p .$$

The Euclidean norm of $x \in \mathbb{R}^p$ is denoted by $|x|_p$ and the sphere of radius $1/\sqrt{p}$ is

$$\mathcal{S}_p = \{x \in \mathbb{R}^p : |x|_p \le p^{-1/2}\}.$$

Suppose we can establish the following statements for Z_t as defined in (6):

(1) For all $p \in \mathbb{N}$,

$$P\left(\limsup_{t\downarrow 0} \|Z_t - L_p[Z_t]\| \le 4p^{-1/2}\right) = 1; \tag{19}$$

(2) for all $p \in \mathbb{N}$,

$$P\left(\limsup_{t\downarrow 0} \left|\nu_p L_p[Z_t]\right|_p \le \frac{1}{\sqrt{p}}\right) = 1; \tag{20}$$

(3) for all $p \in \mathbb{N}$ and $z \in \mathcal{S}_p$,

$$P\left(\liminf_{t\downarrow 0} \left|z - \nu_p L_p[Z_t]\right|_p = 0\right) = 1. \tag{21}$$

Then Theorem 2 will be proved. To see this, let $\widetilde{\Omega}$ be an element of $\mathcal{F}$ with $P(\widetilde{\Omega}) = 1$, contained in the intersection of the events in the brackets in (19)–(21) taken over all $p \in \mathbb{N}$ and all $z \in \mathbb{R}^p$ from a countable dense subset of $\mathcal{S}_p$. Let $\omega \in \widetilde{\Omega}$ and $\varepsilon > 0$. Choose $p \in \mathbb{N}$ satisfying $6p^{-1/2} < \varepsilon$.

If (20) and (21) hold, we will have

$$\lim_{t\downarrow 0} \inf_{y\in\mathcal{S}_p} |y - \nu_p L_p[Z_t(\omega)]|_p = 0, \quad \text{and} \quad \liminf_{t\downarrow 0} |z - \nu_p L_p[Z_t(\omega)]| = 0 \quad \forall z \in \mathcal{S}_p. \tag{22}$$

The map $\nu_p^{-1} : \mathbb{R}^p \to \mathcal{L}_p$ is continuous, $\nu_p^{-1}\mathcal{S}_p = \mathcal{L}_p \cap \mathcal{K}$ and $\mathcal{S}_p$ is compact, so we get from (22) that

$$\lim_{t\downarrow 0} \inf_{h\in\mathcal{L}_p\cap\mathcal{K}} \|h - L_p[Z_t(\omega)]\| = 0, \quad \text{and} \quad \liminf_{t\downarrow 0} \|g - L_p[Z_t(\omega)]\| = 0 \quad \forall g \in \mathcal{L}_p \cap \mathcal{K}. \tag{23}$$

As shown by Finkelstein ([11, p. 612]), for all $g \in \mathcal{K}$, we have $L_p g \in \mathcal{L}_p \cap \mathcal{K}$ together with the following upper bound:

$$\|g - L_p g\| \le \frac{2}{\sqrt{p}}. \tag{24}$$

Particularly, note that $\inf_{g\in\mathcal{K}} \|g - L_p[Z_t(\omega)]\| \le \inf_{g\in\mathcal{L}_p\cap\mathcal{K}} \|g - L_p[Z_t(\omega)]\|$, and thus, by means of (19) and the first part of (23), we get

$$\begin{aligned}&\lim_{t\downarrow 0} \inf_{g\in\mathcal{K}} \|g - Z_t(\omega)\| \\ &\le \limsup_{t\downarrow 0} \|Z_t(\omega) - L_p[Z_t(\omega)]\| + \lim_{t\downarrow 0} \inf_{g\in\mathcal{L}_p\cap\mathcal{K}} \|g - L_p[Z_t(\omega)]\| < \varepsilon,\end{aligned}$$

giving the first part of (7).

For the second part of (7), take $f \in \mathcal{K}$. Then $L_p f \in \mathcal{L}_p \cap \mathcal{K}$. By the second part of (23), we can find a sequence $t_n = t_n(p, \omega) \downarrow 0$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \|L_p f - L_p[Z_{t_n}(\omega)]\| = 0.$$

Then, by means of (19) and (24),

$$\begin{aligned}\liminf_{t\downarrow 0} \|f - Z_t(\omega)\| &\le \limsup_{n\to\infty} \|f - Z_{t_n}(\omega)\| \\ &\le \limsup_{t\downarrow 0} \|Z_t(\omega) - L_p[Z_t(\omega)]\| + 2p^{-1/2} + \lim_{n\to\infty} \|L_p f - L_p[Z_{t_n}]\| < \varepsilon,\end{aligned}$$

giving the second part of (7), and completing the proof of Theorem 2.

Thus, we proceed by establishing (19)–(21). In order to prepare for this, take $0 < b \le 1$ and $i = 1, 2$, and define Lévy measures $\Pi_i^{(b)}$ by setting

$$\Pi_1^{(b)}(B) = \Pi([-b, b] \cap B) \quad \text{and} \quad \Pi_2^{(b)}(B) = \Pi(B \setminus [-b, b]), \qquad B \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$. Then we may write

$$X(t) = Y^{(b)}(t) + W^{(b)}(t), \quad t \ge 0, \quad 0 < b \le 1, \tag{25}$$

where $Y^{(b)}$ and $W^{(b)}$ are independent Lévy processes with triplets $(0, 0, \Pi_1^{(b)})$ and $(\gamma, 0, \Pi_2^{(b)})$, respectively. In particular, jumps are smaller than or equal in magnitude to b for $Y^{(b)}$ and bigger in magnitude than b for $W^{(b)}$.

Let $p \in \mathbb{N}$ and $c = (c_1, \ldots, c_p)' \in \mathbb{R}^p$ with $|c|_p = 1$, and set

$$X_{p,c}(t) = \sum_{j=1}^{p} c_j \big[X(jt) - X((j-1)t)\big], \quad t \ge 0. \tag{26}$$

Note that $X_{p,c}$ is in general not a Lévy process.

The representation in (25) implies a corresponding decomposition for $X_{p,c}$, namely (in obvious notation),

$$X_{p,c}(t) = Y^{(b)}_{p,c}(t) + W^{(b)}_{p,c}(t), \quad 0 < b \le 1, \quad p \in \mathbb{N}, \quad c \in \mathbb{R}^p . \tag{27}$$

We first deal with the process $Y^{(b)}_{p,c}$ in the following lemma.

Lemma 1 *Let $y > 0$, $0 < r < 1$, $a > 0$, $p \in \mathbb{N}$, and $c \in \mathbb{R}^p$ with $|c|_p = 1$. Then*

$$\sum_{n\ge1} \left| P\left(Y^{\sqrt{r^n}}_{p,c}(yr^n) > a\sqrt{yr^n} \right) - \overline{\Phi}\left(a/\sqrt{V(\sqrt{r^n})} \right) \right| < \infty , \tag{28}$$

where V is defined in (2) and $\overline{\Phi}$ is the tail of the normal distribution, i.e.,

$$\overline{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-y^2/2)\, dy , \qquad x \in \mathbb{R} .$$

Proof of Lemma 1 In virtue of Lemma 4.3 in [3], there exists a finite universal constant A such that, for all $t > 0$, $x > 0$ and $0 < b \le 1$, the following inequality holds:

$$\left| P\left(Y^{(b)}(t) > x\sqrt{tV(b)} \right) - \overline{\Phi}(x) \right| \le \frac{Am_3(b)}{\sqrt{t}\, V(b)^{3/2}(1+x)^3} ,$$

where $m_3(b) = \int_{|x|\le b} |x|^3\, \Pi(dx)$. Thus, for all $0 < r < 1$, $y > 0$ and $a > 0$,

$$\begin{aligned} &\left| P\left(Y^{(\sqrt{r^n})}(yr^n) > a\sqrt{yr^n} \right) - \overline{\Phi}\left(a/\sqrt{V(\sqrt{r^n})} \right) \right| \\ &\quad \le \frac{Am_3(\sqrt{r^n})}{\sqrt{yr^n}\,(V(\sqrt{r^n}))^{3/2}\left(1 + a/\sqrt{V(\sqrt{r^n})} \right)^3} \le \frac{A}{a^3\sqrt{y}}\, \frac{m_3(\sqrt{r^n})}{\sqrt{r^n}} . \end{aligned} \tag{29}$$

It is shown in Proposition 4.3 in [3] that $\sum_{n\ge1} m_3(\sqrt{r^n})/\sqrt{r^n}$ is convergent for any Lévy process, giving (28) for $p = 1$ and $c = 1$.

To deal with general p and c, fix $p \in \mathbb{N}$ and $c \in \mathbb{R}^p$ with $|c|_p = 1$. For $0 < b \le 1$ define a Lévy measure $\Pi^{\dagger,(b)}_{p,c}$ by setting

$$\Pi^{\dagger,(b)}_{p,c}(B) = \int_B \sum_{c_j \ne 0} 1_{[-b,b]}(x/c_j)\, \Pi(dx/c_j) , \quad B \in \mathcal{B}(\mathbb{R}) .$$

Let $Y^{\dagger,(b)}_{p,c}$ be a Lévy process with triplet $(0, 0, \Pi^{\dagger,(b)}_{p,c})$. By (1)

$$E\exp(i\theta Y^{(b)}_{p,c}(t)) = \prod_{c_j\neq 0} E\exp(i\theta c_j Y^{(b)}(t)) = E\exp(i\theta Y^{\dagger,(b)}_{p,c}(t)),$$

thus, $Y^{\dagger,(b)}_{p,c}(t) \stackrel{\mathcal{D}}{=} Y^{(b)}_{p,c}(t)$ for all $0 < b \le 1$ and $t \ge 0$, where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution.

For $0 < b \le 1$, observe that

$$\int_{|x|\le b} x^2\,\Pi^{\dagger,(b)}_{p,c}(dx) = \sum_{j=1}^{p} c_j^2 \int_{|x|\le b} x^2\,\Pi(dx) = V(b),$$

and

$$\int_{|x|\le b} |x|^3\,\Pi^{\dagger,(b)}_{p,c}(dx) = \sum_{j=1}^{p} |c_j|^3 \int_{|x|\le b} |x|^3\,\Pi(dx) \le \mu_3(b).$$

Replacing $Y^{(\sqrt{r^n})}(yr^n)$ by $Y^{\dagger,(\sqrt{r^n})}_{p,c}(yr^n)$ in (29) completes the proof of (28). □

Next we are concerned with $W^{(b)}_{p,c}$ in the following lemma.

Lemma 2 *Let* $0 < r < 1$, $y > 0$, $\delta > 0$, $p \in \mathbb{N}$, *and* $c \in \mathbb{R}^p$ *with* $|c|_p = 1$. *Then*

$$\sum_{n\ge 1} P\left(\sup_{0\le t\le yr^n} |W^{(\sqrt{r^n})}_{p,c}(t)| > \sqrt{\delta r^n}\right) < \infty. \tag{30}$$

Proof of Lemma 2 Following the proof of Lemma 4.2 in [3], for $0 < b \le 1$ set $\gamma^{(b)} = \gamma - \int_{b<|x|\le 1} x\Pi(dx)$. Then we can write

$$W^{(b)}_{p,c}(t) = \gamma^{(b)} t + Q^{(b)}_{p,c}(t), \quad t \ge 0, \quad 0 < b \le 1,$$

where $Q^{(b)}_{p,c}$ is a compound Poisson process with rate $\overline{\Pi}(b)$. As, for any Lévy measure, $\lim_{x\downarrow 0} x^2\overline{\Pi}(x) \to 0$, it follows from the Cauchy–Schwarz inequality that $\lim_{n\to\infty} r^{n/2}|\gamma^{(\sqrt{r^n})}| = 0$ for all $0 < r < 1$. On the other hand, for all $0 < r < 1$, $y > 0$, $\delta > 0$,

$$\begin{aligned} &P\left(\sup_{0\le s\le yr^n} |Q^{(\sqrt{r^n})}_{p,c}(s)| \ge \sqrt{\delta r^n}\right) \\ &\le P(Q^{(\sqrt{r^n})}(\cdot) \text{ has at least one jump up to time} yr^n) \\ &= 1 - \exp(-yr^n\overline{\Pi}(\sqrt{r^n})) \le yr^n\overline{\Pi}(\sqrt{r^n}), \qquad n \in \mathbb{N}. \end{aligned}$$

Finally, to get (30), just note that the series $\sum_{n\geq 1} r^n \overline{\Pi}(\sqrt{r^n})$ converges for any Lévy measure and $0 < r < 1$, as $\int_{\{|x|\leq 1\}} x^2 \Pi(dx) < \infty$. □

The next lemma will allow some applications of the Borel–Cantelli lemma.

Lemma 3 *Let $y > 0$, $r \in (0,1)$, $a > 0$, $p, m \in \mathbb{N}$ and $c \in \mathbb{R}^p$ with $|c|_p = 1$. Then the following assertions hold.*

(i) *If $a > \lambda_X^*$ then*

$$\sum_{n\geq 1} P\left(|X_{p,c}(yr^n)| > a\sqrt{yr^n}\right) < \infty, \tag{31}$$

and

$$\sum_{n\geq 1} P\left(\sup_{0\leq s\leq yr^n} |X(s)| > a\sqrt{yr^n}\right) < \infty. \tag{32}$$

(ii) *If $0 < \alpha < \beta < \lambda_X^*/\sqrt{m}$, then*

$$\sum_{n\geq 1} \left(P\left(\alpha\sqrt{yr^n} < X_{p,c}(yr^n) \leq \beta\sqrt{yr^n}\right)\right)^{m-1} = \infty. \tag{33}$$

Proof of Lemma 3 We are given that (5) holds for a $\lambda_X^* \in (0,\infty)$. As shown in Proposition 4.3 in [3], for all $a \in (0,\infty)$ we then have $J(a) = \infty$ and $J(a) < \infty$, when $a < \lambda_X^*$ and $a > \lambda_X^*$, respectively, where

$$J(a) := \sum_{n\geq 1} \overline{\Phi}\left(a/\sqrt{V(\sqrt{r^n})}\right), \quad a > 0, \tag{34}$$

and V is as given in (2).

(i) Fix $p \in \mathbb{N}$ and $c \in \mathbb{R}^p$ with $|c|_p = 1$. Let $y > 0$ and $0 < r < 1$. Apply Lemma 1 to $-Y_{p,c}^{(b)}$ as obtained by (25) from the Lévy process $-X$. Since $-X$ has Lévy measure $B \mapsto \Pi(-B)$ and the same V as defined in (2), we observe that, for all $a > 0$,

$$\sum_{n\geq 1} \left| P\left(|Y_{p,c}^{\sqrt{r^n}}(yr^n)| > a\sqrt{yr^n}\right) - 2\overline{\Phi}\left(a/\sqrt{V(\sqrt{r^n})}\right)\right| < \infty. \tag{35}$$

Recall the elementary inequality: for rvs ξ_1, ξ_2, and all $a > 0$, $0 < \varepsilon < 1$,

$$P(|\xi_1 + \xi_2| > a) \leq P(|\xi_1| > \varepsilon a) + P(|\xi_2| > (1-\varepsilon)a),$$

to see that (28) and (30) imply (31) for all $a > \lambda_X^*$, since $J(a)$ is finite in this case.

Next we prove (32). We have the following maximal inequality (cf. the proof of Theorem 2.2 of [3]). For all $t > 0$, $x > 0$ and $0 < b \leq 1$,

$$P\left(\sup_{0\leq s\leq t} |Y^{(b)}(s)| > x\right) \leq 2P\left(|Y^{(b)}(t)| > x - \sqrt{2tV(b)}\right) .$$

Applying this with $b = \sqrt{r^n}, t = yr^n, x = a\sqrt{yr^n}$, and noting $\lim_{x\downarrow 0} V(x) = 0$, we complete the proof of (32) by using (30) and (31).

(ii) Finally we establish (33). Let $m \in \mathbb{N}$ and $0 < \alpha < \beta < \lambda_X^*/\sqrt{m}$. Let $y > 0$, $0 < r < 1$, $p \in \mathbb{N}$ and $c \in \mathbb{R}^p$ with $|c|_p = 1$. Pick $\delta \in (0, (\beta - \alpha)/2)$, and define $\alpha^* = \alpha + \delta$ and $\beta^* = \beta - \delta$. Clearly, then, $\delta > 0$, $0 < \alpha^* < \beta^*$ and $\sqrt{m}\beta^* < \lambda_X^*$.

In view of (30), there exists n_0 such that

$$P\left(\left|W_{p,c}^{(\sqrt{r^n})}(yr^n)\right| \leq \delta\sqrt{yr^n}\right) \geq \frac{1}{2}, \quad n \geq n_0 .$$

The processes $Y^{(b)}$ and $W^{(b)}$ are independent; thus

$$\begin{aligned}
&P(\alpha\sqrt{yr^n} < X_{p,c}(yr^n) \leq \beta\sqrt{yr^n}) \\
&\geq P(\alpha^*\sqrt{yr^n} < Y_{p,c}^{(\sqrt{r^n})}(yr^n) \leq \beta^*\sqrt{yr^n}\,,\ |W_{p,c}^{(\sqrt{r^n})}(yr^n)| \leq \delta) \\
&\geq \frac{1}{2}\, P(\alpha^*\sqrt{yr^n} < Y_{p,c}^{(\sqrt{r^n})}(yr^n) \leq \beta^*\sqrt{yr^n})\,, \qquad n \geq n_0 .
\end{aligned}$$

In virtue of Lemma 1, it thus suffices to show that

$$\sum_{n\geq 1}\left(\overline{\Phi}\left(\alpha^*/\sqrt{V(\sqrt{r^n})}\right) - \overline{\Phi}\left(\beta^*/\sqrt{V(\sqrt{r^n})}\right)\right)^{m-1} = \infty.$$

However, the term on the left hand-side is bounded from below by

$$\left(\frac{(\beta^* - \alpha^*)^2}{2\pi}\right)^{(m-1)/2} \sum_{n\geq 1} \frac{\exp(-(m-1)(\beta^*)^2/2V(\sqrt{r^n}))}{(V(\sqrt{r^n}))^{(m-1)/2}} .$$

Recalling that $\sqrt{m}\beta^* < \lambda_X^*$, so that $J(\beta^*\sqrt{m-1}) = \infty$, and that $\overline{\Phi}(x) \sim (2\pi)^{-1/2} e^{-x^2/2}/x$ as $x \to \infty$, we conclude that the series in the last display is divergent, giving (33). □

Now we are in the position to give the proofs of (19)–(21), thereby completing the proof of Theorem 2.

Proof of (19) Set

$$S(\alpha,\beta) = \sup_{\alpha\le y\le\beta} |X(y) - X(\alpha)|, \quad 0 \le \alpha \le \beta. \tag{36}$$

Note that, for any $f \in D_0[0, 1]$,

$$\|f - L_p f\| \le 2 \max_{1\le j\le p} \sup_{\frac{j-1}{p}\le y\le\frac{j}{p}} |f(y) - f((j-1)/p)|,$$

so

$$\|Z_t - L_p Z_t\| \le \frac{2}{\lambda_X^*\sqrt{t}} \max_{1\le j\le p} S\left(\frac{(j-1)t}{p}, \frac{jt}{p}\right), \quad t > 0.$$

Thus, for $0 < r < 1$ and $n \in \mathbb{N}$,

$$\sup_{r^n\le t\le r^{n-1}} \|Z_t - L_p[Z_t]\| \le \frac{4}{\lambda_X^*\sqrt{r^n}} \max_{1\le j\le p} S\left(\frac{(j-1)r^n}{p}, \frac{jr^{n-1}}{p}\right), \quad n \in \mathbb{N}. \tag{37}$$

However, for $1 \le j \le p$, $0 < r < 1$, $n \in \mathbb{N}$, we have

$$S\left(\frac{(j-1)r^n}{p}, \frac{jr^{n-1}}{p}\right) \stackrel{\mathcal{D}}{=} S\left(0, \frac{r^n}{p}\left[1 + \frac{1-r}{r}j\right]\right) \le S\left(0, \frac{r^n}{p}\left[1 + \frac{1-r}{r}p\right]\right).$$

It follows from this and (32) and the Borel–Cantelli lemma applied to (37), that for all $r \in (0, 1)$ and $p \in \mathbb{N}$, there exists $\widetilde{\Omega}(p, r) \in \mathcal{F}$ with $P(\widetilde{\Omega}(p, r)) = 1$ such that, on $\widetilde{\Omega}(p, r)$, we have

$$\begin{aligned}\limsup_{t\downarrow 0} \|Z_t - L_p Z_t\| &\le \limsup_{n\to\infty} \frac{4}{\lambda_X^*\sqrt{r^n}} \max_{1\le j\le p} S\left(\frac{(j-1)r^n}{p}, \frac{jr^{n-1}}{p}\right) \\ &\le \frac{4}{\sqrt{p}}\left[1 + \frac{1-r}{r}p\right]^{1/2} \quad \text{a.s.}\end{aligned}$$

Take $r \uparrow 1$ in the last display. This gives (19). □

Proof of (20) As in the proof of (19), we find for all $p \in \mathbb{N}$,

$$\begin{aligned}&\limsup_{t\downarrow 0} |\nu_p L_p Z_t|_p \\ &\le 2p^{1/2} \max_{1\le k\le p} \limsup_{t\downarrow 0} \frac{|X(kt/p) - X((k-1)/p)|}{\lambda_X^*\sqrt{t}} \le 2 \quad \text{a.s.}\end{aligned} \tag{38}$$

Next we improve on this inequality. Observe that

$$c'\nu_p L_p Z_t = \frac{X_{p,c}(t/p)}{\lambda_X^*\sqrt{t}}, \quad t > 0, \quad p \in \mathbb{N}, \quad c \in \mathbb{R}^p, \tag{39}$$

where $X_{p,c}$ is the process defined in (26) and $c'x$ denotes the Euclidean product of $c, x \in \mathbb{R}^p$. Note that $\sum_{i=1}^{p} |c_i| \le p^{1/2}$ for all $p \in \mathbb{N}$ and $c \in \mathbb{R}^p$ with $|c|_p = 1$. As before, we may apply (31) and (32) together with the Borel–Cantelli lemma, for all $0 < r < 1$, $p \in \mathbb{N}$ and $c \in \mathbb{R}^p$ with $|c|_p = 1$, to get

$$\begin{aligned}
&\limsup_{t\to\infty} \left| \frac{X_{p,c}(t/p)}{\lambda_X^* \sqrt{t}} \right| \\
&\le \limsup_{n\to\infty} \left| \frac{X_{p,c}(r^n/p)}{\lambda_X^* \sqrt{r^n}} \right| + 2\sqrt{p} \max_{1\le j\le p} \limsup_{n\to\infty} \frac{S(jr^n/p, jr^{n-1}/p)}{\lambda_X^* \sqrt{r^n}} \\
&\le 1/\sqrt{p} + 2\,((1-r)/r)^{1/2} \quad \text{a.s.,}
\end{aligned} \tag{40}$$

with $S(\alpha, \beta)$ as given in (36). Take $r \uparrow 1$ in (40); together with (38) this completes the proof of (20). □

Proof of (21) It suffices to show that, for all $p \in \mathbb{N}$, $c \in \mathbb{R}^p$ with $|c|_p = 1$ and $0 < |\eta| < 1$,

$$\liminf_{t\to\infty} \left| \frac{\eta}{\sqrt{p}} - \frac{X_{p,c}(t/p)}{\lambda_X^* \sqrt{t}} \right| = 0 \text{ a.s.} \tag{41}$$

Fix $p \in \mathbb{N}$ and $c \in \mathbb{R}^p$ with $|c|_p$=1. Let $0 < |\eta| < 1$. As the same arguments apply to $-X$ we may assume that $\eta > 0$.

Let $m \in \mathbb{N}$ with $m > \max\{2, 2\eta/(1-\eta)\}$ and choose $r = 1/(pm)$. For $1 \le j \le p$ and $1 \le q \le m$ define random variables

$$\Delta(q, j, n) = X((q/m + j-1)r^n/p) - X(((q-1)/m + j-1)r^n/p).$$

Observe that

$$\sum_{j=1}^{p} c_j \Delta(q, j, n) \stackrel{\mathcal{D}}{=} X_{p,c}(r^n/(pm)), \quad 1 \le q \le m, \quad n \in \mathbb{N}. \tag{42}$$

In view of the Borel–Cantelli lemma, and (31) and (42), we get

$$\limsup_{n\to\infty} \left| \frac{\sum_{j=1}^{p} c_j \Delta(1, j, n)}{\lambda_X^* \sqrt{r^n}} \right| \le \frac{1}{\sqrt{pm}} \qquad \text{a.s..}$$

Now set

$$M_n = \max_{2\le q\le m} \left| \frac{\sum_{j=1}^{p} c_j \Delta(q, j, n)}{\lambda_X^* \sqrt{r^n}} - \frac{\eta}{(m-1)\sqrt{p}} \right|, \quad n \in \mathbb{N}.$$

For all $n \geq \mathbb{N}$, observe that $(\sum_{j=1}^{p} c_j \Delta(q, j, n))_{2 \leq q \leq m}$ is a vector with independent and identically distributed entries. Thus, noting (42),

$$
\begin{aligned}
&P\left(M_n \leq \frac{\eta}{\sqrt{p}m(m-1)}\right)\\
&= \left(P\left(\frac{\eta}{\sqrt{m}}\lambda_X^* \sqrt{r^n/pm} < X_{p,c}(r^n/pm) \leq \frac{m+1}{m-1}\frac{\eta}{\sqrt{m}}\lambda_X^*\sqrt{r^n/pm}\right)\right)^{m-1}\\
&= \left(P\left(\alpha\sqrt{r^n/pm} < X_{p,c}(r^n/pm) \leq \beta\sqrt{r^n/pm}\right)\right)^{m-1},
\end{aligned}
$$

where $\alpha = \eta\lambda_X^*/\sqrt{m}$ and $\beta = (m+1)\eta\lambda_X^*/(\sqrt{m}(m-1))$. Recall that $m > \max\{2, 2\eta/(1-\eta)\}$, so $\beta < \lambda_X^*/\sqrt{m}$. In view of (33), we conclude that

$$
\sum_{n \geq 1} P\left(M_n \leq \frac{\eta}{\sqrt{p}\, m(m-1)}\right) = \infty .
$$

$(M_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables because $r \leq 1/(pm)$. Thus Borel–Cantelli applies to give

$$
\liminf_{n\to\infty} M_n \leq \frac{\eta}{\sqrt{p}\, m(m-1)} \qquad \text{a.s.}
$$

Noting $\sum_{j=1}^{p} |c_j| \leq \sqrt{p}$, we have for all $m > \max\{2, 2\eta/(1-\eta)\}$

$$
\begin{aligned}
&\liminf_{t\downarrow 0}\left|\frac{\eta}{\sqrt{p}} - \frac{X_{p,c}(t/p)}{\lambda_X^*\sqrt{t}}\right|\\
&\leq \liminf_{n\to\infty}\sum_{q=2}^{m}\left|\frac{\sum_{j=1}^{p} c_j\Delta(q, j, n)}{\lambda_X^*\sqrt{r^n}} - \frac{\eta}{(m-1)\sqrt{p}}\right| + \frac{1}{\sqrt{pm}}\\
&\leq (m-1)\liminf_{n\to\infty} M_n + \frac{1}{\sqrt{pm}} \leq \frac{\eta}{\sqrt{p}\, m} + \frac{1}{\sqrt{pm}} \qquad \text{a.s.}
\end{aligned} \tag{43}
$$

Take $m \to \infty$ in (43) to complete the proof of (21). □

6 Proofs for Sect. 3

In what follows we will make use of the following fact which follows from *Skohorod's lemma* (cf. [12, Lemma 3.6.14]): let $(y(t))_{t\geq 0}$ be a continuous function with $y(0) = 0$. The function $k(t) := -\inf_{0\leq s\leq t} y(s)$ is the unique continuous nondecreasing function with $k(0) = 0$ such that $\int_0^\infty \mathbf{1}_{\{y(s)+k(s)>0\}} dk(s) = 0$.

6.1 Proof of Theorem 4

Assume Z a.s. sequentially relatively compact in $D_0[0,1]$ and $\mathcal{C}(Z) = \mathcal{K}$. Define mappings $I, \Psi^S, \Psi^R : D_0[0,1] \to D_0[0,1]$, where I denotes the identity mapping and, for $f \in D_0[0,1]$ and $0 \le u \le 1$, set

$$\Psi^S(f)(u) = \sup_{0\le v\le u} f(v)\,, \quad \Psi^R(f)(u) = \Psi^S(f)(u) - f(u)\,.$$

Since, for $f_1, f_2 \in D_0[0,1]$ and $0 \le u \le 1$, we have

$$|\sup_{0\le v\le u} f_1(v) - \sup_{0\le v\le u} f_2(v)| \le \sup_{0\le v\le u} |f_1(v) - f_2(v)|,$$

Ψ^S and hence Ψ^R are continuous with respect to the uniform metric. Consequently, $f \mapsto (I(f), \Psi^S(f), \Psi^R(f))$ defines a continuous mapping from $D_0[0,1]$ to $D_0^3[0,1]$. Define

$$\mathcal{K}^{X,S,R} = \Big\{(f,s,r) \in D_0^3[0,1] : \; f \in \mathcal{K}\,,\; s = \Psi^S(f)\,,\; r = \Psi^R(f)\Big\}$$

and

$$\widetilde{\mathcal{K}}^{X,S,R} = \Bigg\{(s-r,s,r) \in D_0^3[0,1] \;:\; s \in \mathcal{A}^{\le}\,,\; r \in \mathcal{A}^{+}\,,$$
$$\int_0^1 r(u)\,\dot{s}(u)\,du = 0\,, \int_0^1 \dot{r}^2(u)\,du + \int_0^1 \dot{s}^2(u)\,du \le 1\Bigg\}\,.$$

In view of the continuous mapping theorem and Remark 2(i), (Z^X, Z^S, Z^R) is a.s. sequentially relatively compact in $D_0^3[0,1]$ with $\mathcal{C}(Z^X, Z^S, Z^R) = \mathcal{K}^{X,S,R}$.

Next we show that $\mathcal{K}^{X,S,R} = \widetilde{\mathcal{K}}^{X,S,R}$. To get $\mathcal{K}^{X,S,R} \subseteq \widetilde{\mathcal{K}}^{X,S,R}$ let $f \in \mathcal{K}$ and set $s = \Psi^S(f)$ and $r = \Psi^R(f)$. It's clear that $f = s - r$, s is nondecreasing and r is nonnegative, and that f is differentiable Lebesgue a.e. on $[0,1]$ with derivative $\dot{f}$ satisfying $\int_0^1 \dot{f}^2(u)\,du \le 1$.

Now observe that

$$s(v_2) - s(v_1) \le \int_{v_1}^{v_2} \dot{f}^{+}(u)\,du\,, \tag{44}$$

where $f^{+}(u) = \max\{f(u), 0\}$ denotes the positive part of a function f. In view of the Radon–Nikodym theorem, (44) implies that there exists a Lebesgue measurable function h such that $\int |h(u)|\dot{f}^{+}(u)\,du$ is finite and $s(v) = \int_0^v h(u)\dot{f}^{+}(u)\,du, 0 \le v \le 1$. In particular, we have that s is differentiable Lebesgue-a.e., and for its derivative $\dot{s}$ we have $\dot{s} = h\dot{f}^{+}$ Lebesgue-a.e.

Fix $u \in [0, 1)$ such that both s and f are differentiable at u with derivatives $\dot{s}(u) = h(u)\dot{f}^+(u)$ and $\dot{f}(u)$, respectively. Without loss of generality we may assume that $h(u) = 0$ when $\dot{f}^+(u) = 0$. Suppose $\dot{f}^+(u) > 0$. There are two possible scenarios. Firstly, suppose that there exists a sequence $(v_n) \subseteq [0, 1]$ satisfying $v_n \downarrow u$ for $n \to \infty$ such that $s(v_n) = f(v_n)$ holds for all $n \in \mathbb{N}$. Then we have $s(u) = f(u)$ by continuity, and thus

$$h(u)\dot{f}^+(u) = \dot{s}(u) = \lim_{n\to\infty} \frac{s(v_n) - s(u)}{v_n - u} = \lim_{n\to\infty} \frac{f(v_n) - f(u)}{v_n - u} = \dot{f}^+(u),$$

so, as $\dot{f}^+(u) > 0$, $h(u) = 1$. Alternatively, if there is no such sequence, we can find a $\delta > 0$ such that $s(v) = s(u)$ for all $v \in (u, u + \delta) \cap [0, 1]$. Then

$$h(u)\dot{f}^+(u) = \dot{s}(u) = \lim_{v\downarrow u} \frac{s(v) - s(u)}{v - u} = 0,$$

and since $\dot{f}^+(u) > 0$ we must have $h(u) = 0$.

Summarizing our reasoning, we found that $h \in \{0, 1\}$, Lebesgue almost surely. Clearly, $r = s - f$ is also absolutely continuous. For its derivative $\dot{r}$ we have $\dot{r} = h\dot{f}^+ - \dot{f}$, Lebesgue-a.e. Observe that

$$\int_0^1 \dot{r}(u)\dot{s}(u)\, du = \int_0^1 h(h-1)(\dot{f}^+(u))^2\, du = 0,$$

and

$$\int_0^1 \dot{s}^2(u)\, du + \int_0^1 \dot{r}^2(u)\, du = \int (\dot{s}(u) - \dot{r}(u))^2 = \int \dot{f}^2\, du \le 1 .$$

In view of Skohorod's lemma as cited at the beginning of this section, we have $\int_0^1 r(u)\dot{s}(u)\, du = 0$. Thus, $(f, s, r) \in \widetilde{\mathcal{K}}^{X,S,R}$, giving $\mathcal{K}^{X,S,R} \subseteq \widetilde{\mathcal{K}}^{X,S,R}$.

To get $\widetilde{\mathcal{K}}^{X,S,R} \subseteq \mathcal{K}^{X,S,R}$ let $(f, s, r) \in \widetilde{\mathcal{K}}^{X,S,R}$. Using Skohorod's lemma again, it follows from $\int_0^1 r(u)\, \dot{s}(u)\, du = 0$ that $s = \Psi^S(f)$ and $r = \Psi^R(f)$. It's clear that f is absolutely continuous and, thus, $\int_0^1 \dot{r}(u)\dot{s}(u)\, du = 0$ by the same arguments as above. Consequently, we have $f \in \mathcal{K}$, giving the inclusion $\widetilde{\mathcal{K}}^{X,S,R} \subseteq \mathcal{K}^{X,S,R}$. This completes the proof of the first part:

$$\mathcal{C}(Z^X, Z^S, Z^R) = \mathcal{K}^{X,S,R} = \widetilde{\mathcal{K}}^{X,S,R}.$$

The second part of the theorem is simply obtained using the projection mappings $\hat{\pi}_j : D_0^3[0, 1] \to D_0[0, 1]$, $\hat{\pi}_j(f_1, f_2, f_3) = f_j$, which are continuous with respect to the uniform norm, $1 \le j \le 3$. In view of the continuous mapping theorem and the first part, we get that $Z^S = \hat{\pi}_2(Z^X, Z^S, Z^R)$ is a.s. relatively compact in $D_0[0, 1]$ with

$$\mathcal{C}(Z^S) = \hat{\pi}_2(\mathcal{C}(Z^X, Z^S, Z^R)) = \hat{\pi}_2(\mathcal{K}^{X,S,R}) = \mathcal{K}^S,$$

where $\mathcal{K}^S$ is defined in (12). On the other hand, note that, also,

$$\mathcal{C}(Z^S) = \hat{\pi}_2(\widetilde{\mathcal{K}}^{X,S,R}) = \mathcal{K} \cap \mathcal{A}^{\leq},$$

since $(s, s, 0) \in \widetilde{\mathcal{K}}^{X,S,R}$ for all $s \in \mathcal{K} \cap \mathcal{A}^{\leq}$.

Analogously, we find $(-r, 0, r) \in \mathcal{K}^{X,S,R}$ for all $r \in \mathcal{K} \cap \mathcal{A}^+$ and, thus, a similar argument applies to Z^R, giving the second part of the theorem.

6.2 Proof of Theorem 5

The idea behind the proof is that iteratively reflecting a process in $D_0[0, 1]$ in its maximum sends it to the zero function, as simple diagrams suggest. But to get the precise order of magnitude in (14) takes some analysis.

Recall the set $\mathcal{A}^+$ of all absolutely continuous and nonnegative functions $f : [0, 1] \to \mathbb{R}$ with $f(0) = 0$ defined in Sect. 3, and the mappings $I, \Psi^R, \Psi^S : D_0[0, 1] \to D_0[0, 1]$ defined in the proof of Theorem 4. Let $\Psi_0^R = I$ and, for $n \in \mathbb{N}_0$, define recursively $\Psi_{n+1}^R = \Psi^R \circ \Psi_n^R$. From Theorem 4 we have that $r \in \mathcal{K}^R = \Psi^R \circ \mathcal{K}$ if and only if r is nonnegative and absolutely continuous with $\int_0^1 \dot{r}(u)^2\, du \leq 1$. Iterating the arguments in the proof of Theorem 4, we then see that $\Psi_n^R(\mathcal{A}^+) \subseteq \mathcal{A}^+$ for all $n \in \mathbb{N}_0$.

We prepare for the proof of the theorem with the following two lemmas. Let $\pi_t : D_0[0, 1] \to \mathbb{R}$ be defined by $\pi_t(g) = g(t)$, $t \in [0, 1]$, and, for $n \in \mathbb{N}_0$ and $\varepsilon > 0$, let

$$\mathcal{A}^+(n, \varepsilon) := \left\{ f \in \mathcal{A}^+ : \pi_1 \circ \Psi^S \circ \Psi_n^R(f) \geq \varepsilon \right\}. \tag{45}$$

Lemma 4 *Let $n \in \mathbb{N}_0$, $\varepsilon > 0$, $f \in \mathcal{A}^+$. Then $f \in \mathcal{A}^+(n, \varepsilon)$ if and only if there exist reals $0 = t_0 < t_1 < \cdots < t_{n+1} \leq 1$ such that*

$$(-1)^{i+1}\ (f(t_i) - f(t_{i-1})) \geq \varepsilon\,, \qquad 1 \leq i \leq n+1\,. \tag{46}$$

Proof of Lemma 4 This is shown by induction. Fix $\varepsilon > 0$. For $n \in \mathbb{N}_0$, let $\widetilde{\mathcal{A}}^+(n, \varepsilon)$ be the set of all $f \in \mathcal{A}^+$ such there exist $0 = t_0 < t_1 < \cdots < t_{n+1} \leq 1$ satisfying (46).

It is clear that $\mathcal{A}^+(0, \varepsilon) = \widetilde{\mathcal{A}}^+(0, \varepsilon)$. Suppose that $\mathcal{A}^+(n, \varepsilon) = \widetilde{\mathcal{A}}^+(n, \varepsilon)$. Pick $f \in \mathcal{A}^+(n+1, \varepsilon)$ and set $s = \Psi^S(f)$ and $r = \Psi^R(f)$. Then $r \in \mathcal{A}^+(n, \varepsilon)$, so by the induction hypothesis, there exist $0 = u_0 < u_1 < \cdots < u_{n+1} \leq 1$ such that

$$(-1)^{i+1}\ (r(u_i) - r(u_{i-1})) \geq \varepsilon\ , \qquad 1 \leq i \leq n+1\,.$$

As r is nonnegative and continuous, we may assume without loss of generality that $r(t) > 0$ for all $u_{i-1} < t < u_i$ and, thus, $s(u_i) = s(u_{i-1})$ if $2 \le i \le n+1$ is odd. For all odd $2 \le i \le n+1$, observe that

$$(-1)^{i+2}\,(f(u_i) - f(u_{i-1})) \;=\; (-1)^{i+1}\,(r(u_i) - r(u_{i-1})) \;\ge\; \varepsilon.$$

Note that $s(u_i) \ge s(u_{i-1})$ for all $1 \le i \le n+1$; hence, for all even $2 \le i \le n+1$,

$$(-1)^{i+2}\,(f(u_i) - f(u_{i-1})) \;\ge\; (-1)^{i+1}\,(r(u_i) - r(u_{i-1})) \;\ge\; \varepsilon\,.$$

Pick $t_1 \in [0, u_1]$ with $f(t_1) = s(u_1)$ and recall that f is nonnegative. Then $\varepsilon \le r(u_1) = s(u_1) - f(u_1) \le s(u_1) = f(t_1)$ and $\varepsilon \le r(u_1) = s(u_1) - f(u_1) = -(f(u_1) - f(t_1))$. Consequently, $f \in \widetilde{\mathcal{A}}^+(n+1, \varepsilon)$, giving $\mathcal{A}^+(n+1, \varepsilon) \subseteq \widetilde{\mathcal{A}}^+(n+1, \varepsilon)$. Similarly, one gets the other inclusion, $\widetilde{\mathcal{A}}^+(n+1, \varepsilon) \subseteq \mathcal{A}^+(n+1, \varepsilon)$. □

Define $I : \mathcal{A}^+ \to [0, \infty]$ by $I(g) = \int_0^1 \dot{g}^2(u)\,du$. For $n \in \mathbb{N}_0$ and $\varepsilon > 0$, let $f(n, \varepsilon) \in \mathcal{A}^+(n, \varepsilon)$ be the function which is obtained by taking the linear interpolation between the points $(0, 0)$, $(1/(n+1), \varepsilon), \ldots, (n/(n+1)), \varepsilon)$, $(1, 0)$ if n is odd, or $(0, 0)$, $(1/(n+1), \varepsilon), \ldots, (n/(n+1), 0)$, $(1, \varepsilon)$, if n is even.

Lemma 5 *Let $\varepsilon > 0$ and $n \in \mathbb{N}_0$. Then $I : \mathcal{A}^+ \to [0, \infty]$ attains its unique minimum on $\mathcal{A}^+_{n,\varepsilon}$ at $f_{n,\varepsilon}$.*

Proof of Lemma 5 Fix $\varepsilon > 0$ and $n \in \mathbb{N}_0$. For a partition $\mathcal{Z} = \{0 = t_0 < t_1 < \cdots < t_{n+1} \le 1\}$ define $\mathcal{A}^+_{\mathcal{Z}}$ to be the set of functions $f \in \mathcal{A}^+$ satisfying (46) for this particular choice $0 = t_0 < t_1 < \cdots < t_{n+1} \le 1$.

Let $f_{\mathcal{Z}} \in \mathcal{A}^+_{\mathcal{Z}}$ be the function which is obtained by taking the linear interpolation between the points $(0, 0)$, $(t_1, \varepsilon), \ldots, (t_n, \varepsilon)$, $(t_{n+1}, 0)$, $(1, 0)$ if n is odd, or $(0, 0)$, $(t_1, \varepsilon), \ldots, (t_n, 0)$, (t_{n+1}, ε), $(1, \varepsilon)$, if n is even.

Let $f \in \mathcal{A}^+_{\mathcal{Z}}$ with $I(f) < \infty$. From the Cauchy-Schwarz inequality and (46) we get that

$$\begin{aligned} I(f) &\ge \sum_{i=1}^{n+1} \int_{t_{i-1}}^{t_i} \dot{f}^2(u)\,du \\ &\ge \sum_{i=1}^{n+1} \frac{\left(\int_{t_{i-1}}^{t_i} \dot{f}(u)\,du\right)^2}{t_i - t_{i-1}} = \sum_{i=1}^{n+1} \frac{\left(f(t_i) - f(t_{i-1})\right)^2}{t_i - t_{i-1}} \\ &\ge \sum_{i=1}^{n+1} \frac{\varepsilon^2}{t_i - t_{i-1}} = I(f_{\mathcal{Z}})\,. \end{aligned}$$

The chain of inequalities in the last display becomes a chain of identities if and only if $f = f_{\mathcal{Z}}$. Consequently, I attains its unique minimum on $\mathcal{A}^+_{\mathcal{Z}}$ at $f_{\mathcal{Z}}$.

For $\alpha < \beta$, observe that the function $\phi_{\alpha,\beta} : (\alpha, \beta) \to \mathbb{R}$, $\phi_{\alpha,\beta}(x) = 1/(x - \alpha) + 1/(\beta - x)$ attains its unique minimum at $x_0 = (\alpha + \beta)/2$. Using this device, it is straightforward to see that $I(f_{\mathcal{Z}})$ attains its unique minimum on $\mathcal{V}$ at $\mathcal{Z}_0 = \{0, 1/(n+1), \ldots, n/(n+1), 1\}$, where $\mathcal{V}$ is the set of all partitions $\mathcal{Z} = \{0 = t_0 < t_1 < \cdots < t_{n+1} \leq 1\}$. In view of Lemma 4, as $\mathcal{A}^+(n, \varepsilon) = \bigcup_{\mathcal{Z}\in\mathcal{V}} \mathcal{A}^+_{\mathcal{Z}}$, this completes the proof of the lemma. □

We return to the proof of the theorem. Fix $n \in \mathbb{N}$ and let Z be a.s. sequentially relatively compact in $D_0[0, 1]$ with $\mathcal{C}(Z) = \mathcal{K}$. Then $\pi_1 \circ \Psi_n^R : D_0[0, 1] \to \mathbb{R}$ and $\pi_1 \circ \Psi^S \circ \Psi_n^R : D_0[0, 1] \to \mathbb{R}$ are continuous mappings. It follows from the continuous mapping theorem and Theorem 4 that, for $n \geq 1$, $(R^{(n)}(t)/b(t))$ is sequentially relatively compact in $\mathbb{R}$ with

$$\mathcal{C}\left(\left(\frac{R^{(n)}(t)}{b(t)}\right)_{t>0}\right) = \pi_1 \circ \Psi_n^R \circ \mathcal{K} = \pi_1 \circ \Psi_{n-1}^R \circ \mathcal{K}^R .$$

Note that

$$\inf\{\pi_1 \circ \Psi_{n-1}^R(r) : r \in \mathcal{K}^R\} = 0 ,$$

where the infimum is attained at 0. As $\mathcal{K}^R$ is compact and $\pi_1 \circ \Psi_{n-1} : \mathcal{K}^R \to \mathbb{R}$ is continuous, we have

$$\rho_R^{(n)} := \sup\{\pi_1 \circ \Psi_{n-1}^R(r) : r \in \mathcal{K}^R\} < \infty$$

and

$$\mathcal{K}_n^R := \{r \in \mathcal{K}^R : \pi_1 \circ \Psi_{n-1}^R(r) = \rho_R^{(n)}\} \neq \emptyset .$$

Observe that

$$\pi_1 \circ \Psi_{n-1}^R(\alpha r) = \alpha \pi_1 \circ \Psi_{n-1}^R(r), \quad r \in D_0[0, 1], \quad \alpha \geq 0, \tag{47}$$

implying

$$\mathcal{C}\left(\left(R^{(n)}(t)/b(t)\right)_{t>0}\right) = [0, \rho_R^{(n)}] .$$

Similarly, we have

$$\pi_1 \circ \Psi^S \circ \Psi_{n-1}^R(\alpha r) = \alpha \pi_1 \circ \Psi^S \circ \Psi_{n-1}^R(r), \quad r \in D_0[0, 1], \quad \alpha \geq 0, \tag{48}$$

and, thus,

$$\mathcal{C}\left(\left(S^{(n)}(t)/b(t)\right)_{t>0}\right) = [0, \rho_S^{(n)}] ,$$

where

$$\rho_S^{(n)} := \sup\{\pi_1 \circ \Psi^S \circ \Psi_{n-1}^R(r) : r \in \mathcal{K}^R\} < \infty \tag{49}$$

and

$$\mathcal{K}_n^S := \{r \in \mathcal{K}^R \,:\, \pi_1 \circ \Psi^S \circ \Psi_{n-1}^R(r) = \rho_S^{(n)}\} \neq \emptyset .$$

It follows from (47) and (48) that

$$r \in \mathcal{K}_n^S \cup \mathcal{K}_n^R \quad \text{implies} \quad I(r) = \int_0^1 \dot{r}^2(u)\, du = 1. \tag{50}$$

Next note that

$$r \in \mathcal{K}_n^S \cup \mathcal{K}_n^R \quad \text{implies} \quad \pi_1 \circ \Psi^S \circ \Psi_{n-1}^R(r) = \pi_1 \circ \Psi_{n-1}^R(r) . \tag{51}$$

Indeed, suppose that $r \in \mathcal{K}_n^S$, but $\pi_1 \circ \Psi^S \circ \Psi_{n-1}^R(r) \neq \pi_1 \circ \Psi_{n-1}^R(r)$. Then there exists $\tau \in [0, 1)$ such that, for all $\tau < t \leq 1$,

$$\pi_\tau \circ \Psi_{n-1}^R(r) = \pi_1 \circ \Psi^S \circ \Psi_{n-1}^R(r) > \pi_t \circ \Psi_{n-1}^R(r) .$$

In particular, we must have $\int_\tau^1 \dot{r}^2(u)\, du > 0$. Define $x(t) = r(t)$ for $0 \leq t \leq \tau$ and $x(t) = r(\tau)$ for all $\tau < t \leq 1$. Observe that $x \in \mathcal{K}^R$, but $I(x) < 1$. For all $k \in \mathbb{N}_0$, we have $\pi_t \circ \Psi_k^R(x) = \pi_t \circ \Psi_k^R(r)$ for all $0 \leq t \leq \tau$ and $\pi_t \circ \Psi_k^R(x) = \pi_\tau \circ \Psi_k^R(r)$ for all $\tau \leq t \leq 1$. Consequently,

$$\pi_1 \circ \Psi^S \circ \Psi_{n-1}^R(x) \geq \pi_\tau \circ \Psi_{n-1}^R(r) = \pi_1 \circ \Psi^S \circ \Psi_{n-1}^R(r) ,$$

and, thus, $x \in \mathcal{K}_n^S$, contradicting (50); similarly, for the other case ($r \in \mathcal{K}_n^R$).

It follows from (51) that $\mathcal{K}_n^S = \mathcal{K}_n^R$. Furthermore, as $f(n-1, 1/n) \in \mathcal{K}^R$, we have $\rho_S^{(n)} \geq 1/n$; in particular, $\rho_S^{(n)}$ is strictly positive. Fix $r \in \mathcal{K}_n^S$ and observe that $r \in \mathcal{A}^+(n-1, \rho^{(n)})$. In fact, I attains its minimum on $\mathcal{A}^+(n-1, \rho^{(n)})$, at r. Suppose not, then there exists $y \in \mathcal{A}^+(n-1, \rho^{(n)})$ with $I(y) < I(r) = 1$. In particular, $I(y) > 0$ and, thus, $y/\sqrt{I(y)} \in \mathcal{K}^R$. Recall that $\rho_S^{(n)}$ is strictly positive. By (48), $\pi_1 \circ \Psi^S \circ \Psi_{n-1}^R(y/\sqrt{I(y)}) \geq \rho_S^{(n)}/\sqrt{I(y)} > \rho_S^{(n)}$, contradicting (49). In view of Lemma 5, $r = f(n-1, \rho_S^{(n)})$ and, by (50), $\rho_S^{(n)} = 1/n$. To summarize, we get

$$\mathcal{K}_n^R = \mathcal{K}_n^S = \{f_{n-1,1/n}\} , \quad \rho_R^{(n)} = \rho_S^{(n)} = \frac{1}{n} ,$$

completing the proof of the theorem.

6.3 Proof of Theorem 6

Let α be the Lipschitz constant of B. From (15)

$$\frac{L(tu) - X(tu)}{b(t)} = \frac{t}{b(t)} \int_0^u B(L(tv))\, dv\,, \qquad 0 \le u \le 1, \tag{52}$$

so

$$\|Z^L - Z^X\| \le \frac{t}{b(t)} \left(B(0) + \alpha \|L\|\right) .$$

As Z^X is a.s. sequentially relatively compact as $t \downarrow 0$, $\mathcal{K} = \mathcal{C}_0(Z^X)$ attracts Z^X a.s. Note that sample paths of L are a.s. càdlàg, thus, a.s. bounded on $[0, 1]$. As $t/b(t) \to 0$ as $t \downarrow 0$, the right-hand side of (52) tends to zero a.s. as $t \downarrow 0$. Consequently, $\mathcal{K}_2 := \{(f, f) \in D_0^2[0, 1] : f \in \mathcal{K}\}$ attracts (Z^X, Z^L) a.s. and $\mathcal{C}_0(Z^X, Z^L) = \mathcal{K}_2$. Clearly, $\mathcal{K}_2$ is compact and, thus, (Z^X, Z^L) is a.s. sequentially relatively compact as $t \downarrow 0$ in $D_0^2[0, 1]$ with $\mathcal{C}_0(Z^X, Z^L) = \mathcal{K}_2$.

Assume that B is $n+1$-times continuously differentiable in a neighbourhood of 0. As L is càdlàg with $L(0) = 0$, for each $\varepsilon > 0$ and ω from a set of probability one, we may find $T_0(\omega) > 0$ such that $|L(u, \omega)| \le \varepsilon$ for all $0 \le u \le T_0(\omega)$. For convenience, we may thus assume that B is $n+1$-times continuously differentiable on the whole of $\mathbb{R}$ with $B^{(n+1)}$ globally bounded by α. Applying the continuous mapping theorem shows that

$$\left(\int_0^u (L(tv)/b(t))^n\, dv \right)_{0 \le u \le 1}, \quad t > 0,$$

has as its a.s. cluster set (as $t \downarrow 0$) the set $\mathcal{K}^n$, where

$$\mathcal{K}^n = \left\{ u \mapsto \int_0^u (f(v))^n\, dv : \; f \in \mathcal{K} \right\}.$$

Exploring (52), Taylor's formula gives us the following identity:

$$\frac{L(tu) - X(tu)}{tb^n(t)} - \frac{B^{(n)}(0)}{n!} \int_0^u \left(\frac{L(tv)}{b(t)} \right)^n dv$$
$$= \frac{b(t)}{n!} \int_0^u \left(\frac{L(tv)}{b(t)} \right)^{n+1} \int_0^1 B^{(n+1)}\left(rL(tv)\right) (1-r)^n\, dr\, dv\,,$$

where we set $x^0 = 1$. Recall $b(t) \to 0$ for $t \downarrow 0$. The expression in the last display is uniformly bounded in $u \in [0, 1]$ by $\alpha\, b(t) \| Z^L \|^{n+1}$, which tends to zero almost surely as $t \downarrow 0$, giving the result.

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References

1. Baldi, P.: Large deviations and functional iterated logarithm laws for diffusion processes. Prob. Theor. Rel. Fields **71**, 435–453 (1986)
2. Bertoin, J.: Lévy Processes. Cambridge Tracts in Mathematics, vol 121. Cambridge University Press, Cambridge (1996)
3. Bertoin, J., Doney, R.A., Maller, R.A.: Passage of Lévy processes across power law boundaries at small times. Ann. Prob. (to appear 2007)
4. Billingsley, P.: Convergence of Probability Measures. Wiley, New York (1968)
5. Caramellino, L.: Strassen's law of the iterated logarithm for diffusion processes for small time. Stoch. Proc. Appl. **74**, 1–19 (1998)
6. Csáki, E.: A relation between Chung's and Strassen's laws of the iterated logarithm. Z. Wahrsch. Verw. Gebiete **54**, 287–301 (1980)
7. Csáki, E., Földes, A., Shi, Z.: Adjoint functional law for the Wiener process and principal value. Studia Sci. Math. Hungar. **40**, 213–241 (2003)
8. de Acosta, A.: Small deviations in the functional central limit theorem with applications to functional laws of the iterated logarithm. Ann. Prob. **11**, 78–101 (1983)
9. Deuschel, J.D., Stroock, D.W.: Large Deviations. Academic, Boston (1981)
10. Gantert, N.: An inversion of Strassen's law of the iterated logarithm for small time. Ann. Prob. **21**, 1045–1049 (1993)
11. Finkelstein, H.: The law of the iterated logarithm for empirical distributions. Ann. Math. Stat. **42**, 607–615 (1971)
12. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus. Springer, New York (1991)
13. Kesten, H.: The limit points of a normalized random walk. Ann. Math. Stat. **41**, 1173–1205 (1970)
14. Khintchine, A.Ya.: Sur la croissance locale des processes stochastiques homogènes à acroissements indépendants. Izv. Ahad. Nauk SSSR **3**, 487–508 (1939)
15. Sato, K.: Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, Cambridge (1999)
16. Strassen, V.: An invariance principle for the law of the iterated logarithm. Z. Wahrscheinlichkeitstheorie Und Verw. Gebiete **3**, 211–226 (1964)