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A CONTINUOUS TIME APPROXIMATION OF AN EVOLUTIONARY STOCK MARKET MODEL

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We derive a continuous time approximation of the evolutionary market selection model of Blume and Easley (1992). Conditions on the payoff structure of the assets are identified that guarantee convergence. We show that the continuous time approximation equals the solution of an integral equation in a random environment. For constant asset returns, the integral equation reduces to an autonomous ordinary differential equation. We analyze its long-run asymptotic behavior using techniques related to Lyapunov functions, and compare our results to the benchmark of profit-maximizing investors.

Keywords: Portfolio theory; evolutionary finance; continuous time Euler approximation; stochastic processes in random environments; Lyapunov function.

1. Introduction

The axiom of profit maximization is a cornerstone of neoclassical economics. Often it is justified by the market selection hypothesis, which argues that maximization describes the long-run market behavior induced by an evolutionary selection process, cf. [11] and [10]. While intuitively appealing, this argument clearly needs a rigorous analysis.

An explicit model for the market selection mechanism has been proposed in a seminal paper by [4]. In an asset model with endogenous prices in discrete time, agents follow simple trading strategies. They keep the proportion of wealth invested in each asset fixed over time and reinvest their payoffs. The market process induces a redistribution of wealth among traders. [4] investigate the long-run dynamics of the selection process. Under strong conditions on the underlying random variables and the payoff structure of the assets they identify the unique survivor of the market selection process.

This result has recently been generalized. [9] extend the model to a more complex payoff structure for the case that uncertainty is modeled by a sequence of independent random variables taking values in a finite state space. They identify the unique surviving strategy. For general ergodic states, [14] derive local stability results.

In the current paper, we provide a continuous time approximation for the model of [4] for general random payoffs of the assets. Here, we assume that trading takes place at a higher frequency and that in each trading interval agents reinvest only a fraction of their wealth. If the payoffs of the assets converge nicely (as the time between two successive trading dates approaches zero), then also the wealth process of the agents converges by a functional limit theorem which is closely related to the well known Euler scheme. The continuous time limit of the wealth process equals the solution of a nonlinear integral equation in a random environment.

The continuous time approximation of the wealth process relies on a proper convergence of the payoffs of the assets, as the length of the trading intervals tends to zero. We suggest an economically meaningful model for the dividend processes and their convergence. Dividend payments are modeled as increments of stochastic firm value processes. Conditions on these processes are identified, which ensure the applicability of the functional limit theorem. For this purpose, the notion of locally finite kernels turns out to be useful.

In a further step, we analyze the long-run asymptotic behavior of the continuous time approximation in the simplest special case. Namely, we assume that the dividend process of the assets is deterministic and constant. The Markovian case will be investigated in [7]. For constant dividend payments, the deterministic dynamics of the wealth process in continuous time is described by a nonlinear, autonomous ordinary differential equation. We characterize its long-run asymptotic behavior. Here, we employ the technique of Lyapunov functions. In particular, we prove that there exists a unique strategy that asymptotically gathers all wealth in any market without redundant trading strategies. The strategy consists in dividing income proportionally to relative payoffs of the assets. Our analysis provides a characterization of the long-wealth distribution of investors who follow conservative strategies and invest only a small fraction of wealth in risky assets. The strategy ”betting your beliefs” plays a special role in markets without any redundant strategies.

Finally, we compare these results to a Walrasian equilibrium of myopic agents who are price takers. In continuous time, the investors' objectives coincides with the growth optimality of their strategies. The equilibrium solutions are closely connected to the asymptotic behavior of the evolutionary model.

Evolutionary models of portfolio selection are related to the literature on growth optimal portfolios, see e.g. [1], [3], [6], [8], [12], [13], [15] and [16]. As common in mathematical finance and in contrast to the evolutionary approach, these models usually assume an exogenous price process. Equilibrium consequences are neglected in these models. The current model makes a connection between an evolutionary approach and continuous time processes which are commonly used in mathematical finance. This has two implication. Techniques from stochastic analysis can be used for the investigation of the proposed model. At the same time, equilibrium effects are treated endogenously.

The balance of this paper is organized as follows. In Section 2 we present the discrete time model of dynamic asset allocation of [4]. In Section 3 we provide a continuous time approximation of the wealth process and suggest an economically meaningful model for the dividend processes. In Section 4 we study the long-run asymptotic behavior of the continuous time approximation of the wealth process in the deterministic case and examine a rational benchmark. Section 5 concludes. All proofs are found in Section 6.

2. Modeling Dynamic Asset Allocation

2.1. *The economy*

In this section we provide a model of dynamic portfolio allocation and the evolution of wealth of investors in a financial market. By $i \in I = \{1, 2, \ldots, I\}$ we denote a finite set of investors who can invest into assets $k \in K = \{1, 2, \ldots, K\}$ at discrete points in time $t \in \mathbb{N}$.

At time t, investor $i \in I$ is endowed with wealth $w_i^t \in \mathbb{R}_+$. For the vector of agents' wealth we will write $w^t = (w_i^t)_{i \in I}$. At each point in time t each investor i acquires a portfolio $a_i^t = (a_{i,1}^t, a_{i,2}^t, \ldots, a_{i,K}^t)$; here $a_{i,k}^t$ denotes the number of shares of asset k in the portfolio. For simplicity, we assume that assets live only for one period and are re-born at every period. Denoting the price of one share of asset k by ρ_k^t, the I budget constraints of the investors $i \in I$ can be written in the following form:

$$w_i^t = \sum_{k=1}^{K} \rho_k^t \cdot a_{i,k}^t \,. \tag{2.1}$$

The prices are determined in a Walrasian market by the K equilibrium equations

$$\bar{a}_k^t = \sum_{i=1}^{I} a_{i,k}^t \,, \tag{2.2}$$

where $\bar{a}_k^t > 0$ is the total supply of asset k in period t. For simplicity, we suppose that the supply of each asset does not depend on time and is nonrandom. By an appropriate renormalization of the payoffs of the assets we may and will assume that $\bar{a}_k^t \equiv 1$ for all $k \in K$. The *budget shares* of the assets in the portfolio of the investors are given by

$$\lambda_{i,k}^t = \frac{\rho_k^t \cdot a_{i,k}^t}{w_i^t} \,. \tag{2.3}$$

The sequence of budget shares $\lambda_i = (\lambda_i^t)_{t \in \mathbb{N}} = (\lambda_{i,1}^t, \lambda_{i,2}^t, \ldots, \lambda_{i,K}^t)_{t \in \mathbb{N}}$ will be called the *trading strategy* of investor i.

Rewriting (2.2), we obtain the following equation for the market-clearing price:

$$\rho_k^t = \sum_{i=1}^{I} \lambda_{i,k}^t \cdot w_i^t \,. \tag{2.4}$$

The shares bought at time t pay a dividend at time t+1 which we will assume to be random. We let $(\Omega, \mathcal{F}, P)$ be a probability space. By $\tilde{d}_k^{t+1} : \Omega \to \mathbb{R}_+$ we denote the dividend payment of asset k at time $t+1$. We will assume that all random quantities under consideration, i.e., $\tilde{d}_k^t$, $a_{i,k}^t$, and w_i^t ($i \in I$, $k \in K$, $t \in \mathbb{N}$), are measurable.

Total dividend payments received by agents i at time $t+1$ can be calculated as

$$D_i^{t+1} = \sum_{k=1}^{K} a_{i,k}^t \cdot \tilde{d}_k^{t+1} \,. \tag{2.5}$$

The quantities we considered so far were given by their nominal value. In relative terms economic quantities are given by choosing total market wealth as numeraire. Relative wealth of investor i at time t is given by

$$r_i^t = \frac{w_i^t}{\sum_{j=1}^{I} w_j^t} \,. \tag{2.6}$$

Normalizing the prices of the assets by the market wealth we obtain the relative prices of asset k at date t:

$$q_k^t = \frac{\rho_k^t}{\sum_{i=1}^{I} w_i^t} = \sum_{i=1}^{I} \lambda_{i,k}^t r_i^t \,. \tag{2.7}$$

The relative payoff of asset k at time $t+1$ can be calculated as

$$R_k^{t+1} = \frac{\tilde{d}_k^{t+1}}{\sum_{l=1}^{K} \tilde{d}_l^{t+1}} \,. \tag{2.8}$$

2.2. *The wealth dynamics in discrete time*

Apart from the choice of the investments and the market structure, we have to describe how the wealth of the investors is determined in period t+1. We investigate the case of investors who never consume, but reinvest their investment earnings completely. For simplicity, we assume that investors do not receive income from labor. Hence, we suppose that $w_i^{t+1} = D_i^{t+1}$ for all times t and agents i. We may rewrite the evolution of relative wealth as

$$\begin{aligned} r_i^{t+1} &= \frac{D_i^{t+1}}{\sum_{j=1}^{I} D_j^{t+1}} \overset{(2.5)}{=} \sum_{k=1}^{K} a_{i,k}^t R_k^{t+1} \overset{(2.3)}{=} \sum_{k=1}^{K} \frac{\lambda_{i,k}^t \cdot w_i^t}{\rho_k^t} R_k^{t+1} = \sum_{k=1}^{K} \frac{\lambda_{i,k}^t r_i^t}{q_k^t} R_k^{t+1} \\ &= r_i^t \sum_{k=1}^{K} \frac{\lambda_{i,k}^t}{\sum_{j=1}^{I} \lambda_{j,k}^t r_j^t} R_k^{t+1} = r_i^t + r_i^t \left(\sum_{k=1}^{K} R_k^{t+1} \frac{\lambda_{i,k}^t}{\sum_{j=1}^{I} \lambda_{j,k}^t r_j^t} - 1 \right). \end{aligned} \tag{2.9}$$

The economic significance of this crucial equation is easily understood. Agents i holds $a^t_{i,k} = \frac{r^t_i \lambda^t_{i,k}}{\sum_{j=1}^I \lambda^t_{j,k} r^t_j}$ assets of type k from time t to $t+1$. This is a consequence of the Walrasian equilibrium. By assumption agents are fully invested in the asset market and do not receive any income from labor. Total relative wealth at time $t+1$ is therefore described by (2.9).

We will study the case in which the trading strategies $\lambda^t_{i,k} = \lambda_{i,k}$ do not depend on time. Hence, we will drop the index t. In this case, the wealth dynamic is only triggered by the random payments. We will always stick to the following technical assumption which will simplify our analysis.

Assumption 2.1. *All agents invest a strictly positive amount into any asset, i.e. the values* $\lambda_{i,k}$ *are strictly positive. In economic terms, all agents are completely diversified.*

3. The wealth dynamics in continuous time

3.1. *A continuous time approximation*

In the current section we describe how a continuous time approximation of the evolutionary model can be constructed. Assuming that dividends are paid at a higher frequency, we state precise conditions for the convergence of the discrete time wealth process to a continuous time limit. It turns out that the limiting process can be characterized as the solution of an integral equation in a random environment.

The functional limit theorem and technical conditions under which we obtain convergence are described in the current section. The next section provides an economic foundation. Our approximation results bridge the gap between the evolutionary approach and the theory of continuous time processes which are commonly used in mathematical finance.

Let us now turn to the construction of the continuous-time approximation. Given $n \in \mathbb{N}$, we let a new time grid be given by the time points $\left\{ l \cdot n^{-1} : l \in \mathbb{N}_0 \right\}$. Dividends are paid at these dates, and the corresponding dividend process is a discrete time stochastic process denoted by $(\tilde{d}^{(n),s/n})_{s \in \mathbb{N}_0}$. By convention, we fix $\tilde{d}^{(n),0} = a_0 \in \mathbb{R}^K_+$.

Assumption 3.1. *For all* $n \in \mathbb{N}$ *and* $s \in \mathbb{N}_0$*, we suppose that with probability one* $\sum_{k=1}^K \tilde{d}^{(n),s/n}_k > 0$.

Analogous to (2.8), the relative returns of the assets are given by the expressions

$$R^{(n),s/n}_k = \frac{\tilde{d}^{(n),s/n}_k}{\sum_{l=1}^K \tilde{d}^{(n),s/n}_l} . \tag{3.1}$$

As before we suppose that trading takes place immediately after dividends have been received, but we will no longer assume that total wealth is invested. At times $0, \frac{1}{n}, \frac{2}{n}, \ldots$ agents invest only a fraction $\alpha^n \in (0,1]$ of their wealth in the market.

Remark 3.1. *The economic interpretation of this assumption can be described as follows. We investigate an economy with agents who invest only a small fraction of* α^n *in the financial market and characterize their long-run wealth. These agents keep most of their wealth in a risk-free account and invest only a very small portion in risky assets. Nevertheless, in the long run evolutionary forces will drive some of the agents out of business while others accumulate wealth and survive. The purpose of this paper is to identify the successful strategies relative to different business environments, i.e., populations of strategies. Our characterization focuses on conservative investors. Since we are looking at the long-run behavior of wealth, the scaling of time is merely a convenient technical trick which allows us to use continuous-time processes for our analysis of the long-run.*

Our assumptions modify the dynamics described by equation (2.9). For fixed n, the wealth dynamics is given by the following recursive scheme

$$r_i^{(n)}(t_{n,l+1}) = (1-\alpha^n)\cdot r_i^{(n)}(t_{n,l}) + \alpha^n \cdot r_i^{(n)}(t_{n,l}) \sum_{k=1}^{K} \frac{\lambda_{i,k} R_k^{(n),t_{n,l+1}}}{\sum_{j=1}^{I} r_j^{(n)}(t_{n,l})\lambda_{j,k}}\,, \tag{3.2}$$

where $t_{n,l} = \frac{l}{n}$ and $r_0^{(n)} = r_0 \in \Delta_I$. Here, Δ_I denotes the simplex in $\mathbb{R}^I$.

We are interested in a continuous time approximation for $n \to \infty$ where we choose $\alpha^n = \frac{1}{n}$. For this purpose, it is convenient to extend all discrete time processes to continuous time. The continuous time extension of relative returns $R^{(n)}$ is defined by the piecewise constant process

$$R^{(n)} := R^{(n),0} \cdot \mathbf{1}_{\{0\}} + \sum_{s=0}^{\infty} R^{(n),(s+1)/n} \cdot \mathbf{1}_{\left(\frac{s}{n},\frac{s+1}{n}\right]}. \tag{3.3}$$

The wealth process $r^{(n)}$ is extended to continuous time by linear interpolation. For $t_{n,l} \le s \le t_{n,l+1}$ and $i = 1, 2, \ldots, I$, we let

$$\begin{aligned} r_i^{(n)}(s) &:= r_i^{(n)}(t_{n,l}) + n(s-t_{n,l})\left(r_i^{(n)}(t_{n,l+1}) - r_i^{(n)}(t_{n,l})\right) \qquad (3.4)\\ &= r_i^{(n)}(t_{n,l}) + \int_{t_{n,l}}^{s} r_i^{(n)}(t_{n,l}) \left(\sum_{k=1}^{K} \frac{\lambda_{i,k} R_k^{(n),t_{n,l+1}}}{\sum_{j=1}^{I} r_j^{(n)}(t_{n,l})\lambda_{j,k}} - 1\right) du\,. \end{aligned}$$

We will provide precise conditions under which the wealth processes $r^{(n)}$ converge to a continuous time limit r as $n \to \infty$. The limiting process r is characterized as the pathwise solution of an integral equation in a random environment. In the next proposition, we investigate the relevant family of integral equations. Under weak conditions, these possess a unique continuous solution.

Proposition 3.1. *Let* Δ_I *and* Δ_K *denote the simplices in* $\mathbb{R}^I$ *and* $\mathbb{R}^K$*, respectively. Let* $T : \mathbb{R}_+ \to \Delta_K$ *be measurable. Assume that* $r_0 \in \Delta_I$*. Then the coupled integral equations*

$$r_i(s) = r_{i,0} + \int_0^s r_i(s') \left(\sum_{k=1}^{K} \frac{\lambda_{i,k} T_k^{s'}}{\sum_{j=1}^{I} r_j(s')\lambda_{j,k}} - 1\right) ds', \tag{3.5}$$

with $i = 1, 2, \dots, I$, possess a unique continuous solution $r : \mathbb{R}_+ \to \Delta_I$.

The existence of a continuous-time limit of the evolutionary stock market model relies on appropriate conditions on the relative return processes. Key to the analysis is the following technical theorem which provides bounds on the pathwise approximation error. Economic conditions on the dividend processes guarantee that these errors are asymptotically zero (cf. Section 3.2).

Theorem 3.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. For each $n \in \mathbb{N}$, we let $(R^{(n),(s-1)/n})_{s\in\mathbb{N}}$ be a sequence of random variables on Ω with values in Δ_K. $R^{(n)}$ is extended to a continuous time process by (3.3). Assume that $r^{(n)}$ is defined according to (3.2) and (3.4) with $r^{(n)}(0) = r_0 \in \Delta_I$.*

Let $(T^s)_{s\in\mathbb{R}_+}$ be a stochastic process on Ω with values in Δ_K that is jointly measurable in $\omega \in \Omega$ and $s \in \mathbb{R}_+$. Suppose that r is the pathwise unique continuous solution of (3.5).

Then there exists for every $t \geq 0$ a nonrandom constant D such that for all $n \in \mathbb{N}$ the following inequality holds:

$$\sup_{0\leq s\leq t} \|r(s) - r^{(n)}(s)\|_{\mathbb{R}^I} \leq D \cdot \left(\frac{1}{n} + \int_0^{t+\frac{1}{n}} \|T^u - R^{(n),u}\|_{\mathbb{R}^K} \, du \right), \tag{3.6}$$

where $\|\cdot\|_{\mathbb{R}^I}$ and $\|\cdot\|_{\mathbb{R}^K}$ are given norms on $\mathbb{R}^I$ and $\mathbb{R}^K$, respectively.

As a consequence of the last theorem we obtain convergence of the discrete-time wealth processes to a continuous-time wealth processes, if the right hand side of (3.6) converges to zero as $n \to \infty$. For different modes of convergence, this fact is stated rigorously in the next corollary.

Corollary 3.1. *Consider the same setting as in Theorem 3.1. Let*

$$Y_t^n := \int_0^t \|T^u - R^{(n),u}\|_{\mathbb{R}^K} \, du \, . \tag{3.7}$$

Then the following implications hold:

(i) If Y_t^n converges for all $t \in \mathbb{R}_+$ to 0 almost surely, then $r^{(n)}$ converges to r uniformly on compacts with probability 1.

(ii) If Y_t^n converges for all $t \in \mathbb{R}_+$ to 0 in probability, then $r^{(n)}$ converges to r uniformly on compacts in probability and in L^p for $p \in [1, \infty)$.

(iii) If Y_t^n converges for all $t \in \mathbb{R}_+$ to 0 in L^∞, then $r^{(n)}$ converges to r uniformly on compacts in probability and in L^p for $p \in [1, \infty]$.

3.2. *Dividend processes in continuous time*

Assuming that dividends are paid at a higher frequencies, we derived in the last section precise conditions for the convergence of the discrete time wealth processes to a continuous limit. Our main result, Corollary 3.1, states that the wealth processes converge to such a limit, if the dividend processes converge in an appropriate sense.

These mathematical results are rather abstract and lack an economic foundation: for each $n \in \mathbb{N}$, we supposed that dividends $(\tilde{d}^{(n),s/n})_{s\in\mathbb{N}_0}$ are paid at the time points $\{l \cdot n^{-1} : \ l \in \mathbb{N}_0\}$, but what is the economic significance of these dividend processes at different frequencies $1/n$? In the current section we provide an economic rational.

We suppose that each asset $k = 1, 2, \dots, K$ corresponds to a firm. We assume that the value generated by each firm can be described by an increasing process S^t_k which is the basis for its dividends. For a fixed value process, dividends can be paid at different frequencies and correspond to the increments of the process.

In the current section we investigate under which conditions on the value processes S^t_k the dividend processes satisfy the conditions of Corollary 3.1 which guarantee the convergence of the wealth processes to a continuous time limit.

We will show that the firm value processes can be represented in terms of locally finite kernels. This concept turns out to be very useful. On the basis of a representation in terms of these kernels we can formulate precise conditions for a continuous time approximation of the wealth process. Moreover, the limiting wealth process can be characterized.

3.2.1. *The firm value processes*

Let us fix a probability space $(\Omega, \mathcal{F}, P)$ and assume that all random variables and processes are defined on Ω. We suppose that the dividend payments $(\tilde{d}^{(n),s/n})_{s\in\mathbb{N}_0}$ $(n \in \mathbb{N})$ are driven by value processes earned by firms. More specifically, let $S^t \in \mathbb{R}^K_+$ be the stochastic process of the excess value generated by K firms corresponding to the assets $k = 1, 2, \dots, K$, i.e., the process of cumulated dividends. We make the following assumption:

Assumption 3.2. *The value process S^t is cadlag and strictly increasing in the following sense: for given $t \in \mathbb{R}_+$, $\omega \in \Omega$ and $\epsilon > 0$ it holds that*

$$\forall k \quad S^{t+\epsilon}_k(\omega) - S^t_k(\omega) \geq 0 \qquad \textit{and} \qquad \exists k \quad S^{t+\epsilon}_k(\omega) - S^t_k(\omega) > 0\,.$$

We will assume that the value process S^t is related to the dividend payments in the following way:

$$\tilde{d}^0 = S^0\,, \qquad \tilde{d}^t = S^t - S^{t-1}\,, \quad t \in \mathbb{N}\,.$$

In other words, at time t the firm pays the complete incremental value generated between times $t-1$ and t as dividends to the investors. The relative payoff of asset k at time $t+1$ can therefore be calculated as

$$R^{t+1}_k = \frac{\tilde{d}^{t+1}_k}{\sum_{l=1}^K \tilde{d}^{t+1}_l} = \frac{S^{t+1}_k - S^t_k}{\sum_{l=1}^K (S^{t+1}_l - S^t_l)}\,. \tag{3.8}$$

In the continuous time approximation, dividends are paid at a higher frequency. Given $n \in \mathbb{N}$, we define on the new time grid $\frac{1}{n}\mathbb{N}$

$$\tilde{d}^{(n),0} = S^0\,, \qquad \tilde{d}^{(n),(s+1)/n} = S^{(s+1)/n} - S^{s/n}\,, \quad s \in \mathbb{N}_0\,.$$

In terms of the value process, relative returns are thus given by

$$R_k^{(n),(s+1)/n} = \frac{\tilde{d}_k^{(n),(s+1)/n}}{\sum_{l=1}^K \tilde{d}_l^{(n),(s+1)/n}} = \frac{S_k^{(s+1)/n} - S_k^{s/n}}{\sum_{l=1}^K (S_l^{(s+1)/n} - S_l^{s/n})}. \tag{3.9}$$

At time 0, we obtain returns not depending on n:

$$R_k^{(n),0} = \frac{\tilde{d}_k^{(n),0}}{\sum_{l=1}^K \tilde{d}_l^{(n),0}} = \frac{S_k^0}{\sum_{l=1}^K S_l^0}. \tag{3.10}$$

We extend again $R^{(n)}$ to a continuous time process $(R^{(n)}(\omega, u))_{u \geq 0}$ by formula (3.3). If the relative dividends $R^{(n)}$ converge in an appropriate sense to a limiting process T, we can apply Theorem 3.1 and Corollary 3.1 to obtain a continuous time approximation of the wealth process. We are thus interested in the question when the stochastic processes $R^{(n)}$ converge to a limiting process T and how this process is related to the firms' value process S. For this purpose, it is helpful to establish a representation of S in terms of locally finite kernels.

3.2.2. *A representation of the firm value process*

By Assumption 3.2, for $\omega \in \Omega$ the components $S_k(\omega)$ $(k = 1, \ldots, K)$ of the firm value process are cumulative distribution functions of a positive locally finite Borel measure $\mu_k(\omega)$ on $\mathbb{R}_+$. More precisely, μ_k $(k = 1, 2, \ldots, K)$ is a locally finite kernel from Ω to $\mathbb{R}_+$. Here, a mapping $\mu : \Omega \times \mathcal{B}(\mathbb{R}_+) \to \bar{\mathbb{R}}_+$ is called a locally finite kernel, if $\mu(\cdot, B) : \Omega \mapsto \bar{\mathbb{R}}_+$ is measurable and $\mu(\omega, \cdot)$ is a locally finite measure on $\mathbb{R}_+$ for all $\omega \in \Omega$, where $\mathcal{B}(\mathbb{R}_+)$ denotes the Borel-σ-algebra on $\mathbb{R}_+$.

Given a probability measure P on $(\Omega, \mathcal{F})$, every locally finite kernel μ from Ω to $\mathbb{R}_+$ induces a unique σ-finite measure $P\mu$ on $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$ (cf. Lemma 6.1). $P\mu$ is uniquely defined by setting

$$P\mu(A \times B) := \int_A \mu(\omega, B)\; P(d\omega)\,, \qquad A \in \mathcal{F},\; B \in \mathcal{B}(\mathbb{R}_+)\,. \tag{3.11}$$

The following theorem provides a canonical representation of the firm value process S which is useful when investigating the convergence to a continuous time dividend process. The notion of *exhausting sequence* is given in Definition 6.1.

Theorem 3.2. *Suppose that Assumption 3.2 holds. Then there exists a canonical representation of* S^t *in terms of a locally finite kernel* μ *from* Ω *to* $\mathbb{R}^+$ *and measurable functions* $f_k : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that the sum of the functions* $(\omega, u) \mapsto f_k(\omega, u)$, $k = 1, \ldots, K$, *is* $P\mu$*-almost everywhere positive. Namely, for every* $1 \leq k \leq K$ *and for every* $t \geq 0$ *the firm value process* $S : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}^K$ *satisfies for* P*-almost all* $\omega \in \Omega$,

$$S_k^t(\omega) = \int \mathbf{1}_{[0,t]}(u)\; f_k(\omega, u)\; \mu(\omega, du). \tag{3.12}$$

For every exhausting sequence (C_N) *for* P *and* μ, *the functions* $f_k 1_{C_N}$ *are* $P\mu$*-integrable .*

Convergence to a continuous time dividend process. Assumption 3.2 implies that the discrete-time relative return processes converge to a continuous-time limit. The proof is based on a martingale argument which can be found in the appendix.

Proposition 3.2. *Suppose that Assumption 3.2 holds. We suppose that S^t is represented according to (3.12). Then, for $P\mu$-almost all (ω, u), the limit of $R^{(n)}(\omega, u)$ exists for $n \to \infty$ and equals*

$$\lim_{n\to\infty} R_k^{(n)}(\omega, u) = \frac{f_k(\omega, u)}{\sum_{l=1}^K f_l(\omega, u)}. \tag{3.13}$$

3.2.3. *Dividend convergence and Euler approximation*

In this paragraph we provide sufficient conditions on the firms' value process S^t which ensure the convergence of the discrete time wealth processes $r^{(n)}$ to a continuous time process r. In terms of the family of random variables Y_t^n $(t \in \mathbb{R}_+,\ n \in \mathbb{N})$ conditions have been derived in Section 3.1, see in particular Corollary 3.1. We will now combine these results with representation (3.12) of Theorem 3.2.

Our two main results can be summarized as follows. The discrete time wealth processes converge to a continuous time limit, if the representing kernel or the representing densities are sufficiently regular. To be more precise, if one of the two following conditions is satisfied:

- The representing measure $\mu(\omega, \cdot)$ dominates the Lebesgue measure for P-almost all $\omega \in \Omega$.
- The representing functions $f_k(\omega, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ are Lebesgue-almost everywhere continuous for $1 \le k \le K$ with probability one.

Corollary 3.2. *Suppose that the Assumption 3.2 holds, and let a representation of the value process S be given according to Theorem 3.2. Suppose that P-almost surely μ dominates the Lebesgue measure. For $k = 1, \ldots, K$, we set $g_k = f_k$ if $\sum_{l=1}^K f_l > 0$, and $g_k = 1$ else. Define the process $T = g_k \cdot \left(\sum_{l=1}^K g_l\right)^{-1}$. Then $r^{(n)}$ converges to r defined in (3.5) uniformly on compacts with probability* 1.

Corollary 3.2 provides a sufficient condition on the firms' value process S^t in terms of the representing kernel μ in (3.12) which ensures the convergence of the discrete time wealth processes $r^{(n)}$ to a continuous time process r. The next proposition and Corollary 3.3 give a condition in terms of the representing densities f_k $(k = 1, \ldots, K)$. If these functions are sufficiently regular, then the continuous time approximation of the wealth process is valid – irrespectively of the properties of the representing kernel μ.

Proposition 3.3. *Suppose that Assumption 3.2 holds. Assume that there exists a canonical representation according to Theorem 3.2 such that the mappings $f_k(\omega, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ are Lebesgue-almost everywhere continuous for $1 \le k \le K$ with probability one. Then for P-almost all $\omega \in \Omega$ the sum of the functions*

$(f_k(\omega))_{k=1,\dots,K}$ *is Lebesgue-almost everywhere positive and the limit of* $R^{(n)}(\omega)$ *exists Lebesgue-almost everywhere and equals*

$$\lim_{n\to\infty} R_k^{(n)}(\omega) = \frac{f_k(\omega)}{\sum_{l=1}^K f_l(\omega)}. \tag{3.14}$$

Corollary 3.3. *Suppose that the assumptions of Proposition 3.3 are satisfied. For* $k = 1,\dots,K$*, we set* $g_k = f_k$ *if* $\sum_{l=1}^K f_l > 0$*, and* $g_k = 1$ *else. Define the process* $T = g_k \cdot \left(\sum_{l=1}^K g_l\right)^{-1}$*. Then* $r^{(n)}$ *converges to* r *defined in (3.5) uniformly on compacts with probability* 1*.*

We conclude this section with two examples.

Example 3.1. *(i) Observe that the condition of Corollary 3.2 is not always satisfied. Given a value process* S*, we can in general not expect to find a representation (3.12) such that* μ *dominates the Lebesgue measure. Therefore, let* $K = 1$*, and define* $\nu := \sum_{l\in\mathbb{N}} 2^{-l}\delta_{q_l}$*, where* $q: \mathbb{N} \to \mathbb{Q}$*,* $l \mapsto q_l$ *is a bijection. Assume that* $S: \mathbb{R}_+ \to \mathbb{R}_+$ *is given by* $S^t = \nu([0,t])$*. It is not possible to find a measure* μ *dominating the Lebesgue measure and a density* f *which is* μ*-almost surely positive such that* S *can be represented by* $S^t = \int \mathbf{1}_{[0,t]} f d\mu$ *.*

(ii) Let $M = (M^t(\omega))_{t\geq 0}$ *be a process with* $M^{t+\epsilon}(\omega) - M^t(\omega) > 0$ *for all* $\epsilon > 0$ *and for* P*-almost all* ω *(for instance,* M *is a Lévy subordinator with infinite activity). Now suppose that the firm value processes* S_k^t *are given in terms of deterministic multiples of* M*. More precisely, let* $f_1,\dots,f_K > 0$ *and suppose that* $S_k^t(\omega) = f_k M^t(\omega)$*. In view of* (3.8) *and Proposition 3.3, the kth component of* T_t *equals* $f_k/\sum_{l=1}^K f_l$*. This example reflects a situation where the limiting dynamics are deterministic and the corresponding relative dividends are constant although the original market dynamics are random.*

4. Deterministic dynamics

4.1. *The wealth dynamics and its semiflow*

The continuous time wealth dynamics (3.5) is driven by the relative dividend process T. In this section we will focus on those cases, where T is deterministic and constant which corresponds to no dividend risk. Although the analysis of a purely deterministic dynamics might be considered as rather limited, it prepares the analysis of cases where relative dividends remain random (cf. [7]). In addition, if T is random, but not time-dependent, our results hold path by path.

While fundamentals are by now fixed, prices and wealth still vary due to market interaction. The wealth dynamics in the absence of fundamental risk is described by a *nonlinear* autonomous differential equation. We are mainly interested in global asymptotics in order to characterize strategies which survive or die out (cf. Examples 4.1 and 4.2). Therefore we will employ the technique of Lyapunov functions.

In sequel we suppose $T \equiv \pi$ for fixed $\pi \in \Delta_K$. For the whole Chapter 4 we make the following assumption.

Assumption 4.1. *The relative dividends π are strictly positive, i.e., $\pi_k > 0$ for $1 \leq k \leq K$.*

We will now state the ordinary differential equation which describes the wealth dynamics in continuous time. To this end, we define a mapping $N : \Delta_I \to \mathbb{R}^I$ by

$$N_i(r) = \sum_{k=1}^{K} \frac{\pi_k \lambda_{i,k}}{\sum_{j=1}^{I} r_j \lambda_{j,k}} - 1 \,. \tag{4.1}$$

Letting the vector field $\psi : \Delta_I \to \mathbb{R}^I$ be given by

$$\psi_i(r) = r_i \cdot N_i(r)\,, \tag{4.2}$$

the integral equation (3.5) reduces to an autonomous differential equation,

$$\dot{r}(t) = \psi(r)\,, \qquad r(0) = r_0 \,. \tag{4.3}$$

This ordinary differential equation describes the wealth dynamics in continuous time.

It is implied by standard arguments ([2], Corollary 16.10) that Δ_I is an invariant set. Since Δ_I is also compact, the solution of the differential equation (4.3) exists for all times $t \in \mathbb{R}$ and initial values $r_0 \in \Delta_I$ ([2], Remark (17.3)). We associate a flow

$$\phi : \mathbb{R} \times \Delta_I \to \Delta_I\,, \quad (t, r_0) \mapsto \phi_t(r_0) \tag{4.4}$$

with the ordinary differential equation (4.3), where $\phi_t(r_0)$ is the value of the solution of (4.3) at time t when the initial value is $r_0 \in \Delta_I$.

4.2. *Invariant sets, equilibria and a Lyapunov function*

Besides the simplex Δ_I also the relative boundary[a] $\partial^*(\Delta_I)$ and the relative interior ri (Δ_I) are invariant. Moreover, the vertices e_i of the simplex Δ_I are fixed points of the flow where e_i denotes the ith unit vector in $\mathbb{R}^I$. Finally, we define for $J \subseteq I$ the subsimplices

$$\Delta_J := \left\{ \sum_{i \in J} r_i e_i : \; r \in \mathbb{R}_+, \quad \sum_{j \in J} r_j = 1 \right\}.$$

For $J \subseteq I$, $\Delta_J \subseteq \Delta_I$ is invariant. In economic terms, the restriction to a simplex Δ_J, $J \subseteq I$, $J \neq I$ corresponds to a smaller economy where only agents from set J are present. If the initial value is an element of the boundary, i.e., $r \in \partial^*(\Delta_I)$, the wealth dynamics is effectively of lower dimension and, thus, we will mainly be interested in initial values in ri (Δ_I).

---

[a] The *relative boundary* of a convex set C is defined by $\partial^*(C) := \bar{C} \backslash \text{ri }(C)$ where the *relative interior* of a convex set C is defined by ri $(C) = \{c \in C : \; \exists \epsilon > 0 \; \forall y \in C \quad \forall |\delta| < \epsilon \;\; c + \delta(y - c) \in C\}$.

The set of equilibria (fixed points) of the ordinary differential equation (4.3) in the simplex Δ_I is given by

$$\mathcal{A} := \left\{ r \in \Delta_I : \sum_{i=1}^{I} r_i N_i^2(r) = 0 \right\} . \tag{4.5}$$

In particular, $\big\{ r \in \Delta_I : \sum_{i=1}^{I} r_i \lambda_{i,k} = \pi_k \quad \text{for all } 1 \le k \le K \big\} \subseteq \mathcal{A}$ is a set of equilibria. A Lyapunov function Φ for the flow that describes the wealth dynamics is given in the following lemma.

Lemma 4.1. *Suppose that Assumptions 2.1 and 4.1 are satisfied. The function* $\Phi : \Delta_I \to \mathbb{R}$, *defined as*

$$\Phi(r) := -\sum_{k=1}^{K} \pi_k \log(\sum_{j=1}^{I} \lambda_{j,k} r_j) + \sum_{k=1}^{K} \sum_{j=1}^{I} \lambda_{j,k} r_j \, , \tag{4.6}$$

is a Lyapunov function for the flow ϕ *on* Δ_I. Φ *and* N *can be continuously differentiable extended to an open neighborhood of* Δ_I *where they satisfy the equation* $N = -\nabla_r \Phi$. *The Lyapunov function* Φ *is convex on* Δ_I.

4.3. *The minima of the Lyapunov function*

We denote the set of global minima of the Lyapunov function Φ on Δ_I by $\mathcal{A}_{\min}$. Lead by our intuition, we expect that $\operatorname{dist}\{\phi_t(r_0), \mathcal{A}_{\min}\}$ should tend to zero as $t \to \infty$, $r_0 \in \Delta_I$, as Φ is a Lyapunov function. Although this is not true in generally, there is still a close connection between $\mathcal{A}_{\min}$ and the global attractor (cf. Subsection 4.4). It is, thus, useful to have a good characterization of $\mathcal{A}_{\min}$.

As Φ is a convex function, global and local minima coincide and form a convex set. We define a function

$$\tilde{\Phi} : \left(\mathbb{R}^+ \setminus \{0\}\right)^K \to \mathbb{R} \, , \qquad \tilde{\Phi}(x) = -\sum_{k=1}^{K} \pi_k \log(x_k) + \sum_{k=1}^{K} x_k \, . \tag{4.7}$$

The Lyapunov function Φ can be recovered from $\tilde{\Phi}$ by

$$\Phi(r) = \tilde{\Phi}\left(\left(\sum_{i=1}^{I} r_i \lambda_{i,k} \right)_k \right) . \tag{4.8}$$

Recall that by (2.7) the argument of $\tilde{\Phi}$ equals the relative price vector $(q_k)_{k=1,2,\dots,K}$ of the assets. The minimization of the Lyapunov function Φ on the space of wealth distributions consists thus of the two steps: minimize firstly the associated Lyapunov function $\tilde{\Phi}$ on the price space, and find secondly the wealth distributions that support this price vector given the fixed strategy profile.

Lemma 4.2. *Suppose that Assumptions 2.1 and 4.1 are satisfied.*

(i) π *is the unconstraint absolute minimizer of* $\tilde{\Phi}$.

(ii) We denote by $\Lambda \subseteq \Delta_K$ the convex hull of the trading strategies $\lambda_1, \ldots, \lambda_I$. Then there exists a unique x^* such that

$$\tilde{\Phi}(x^*) = \inf_{x \in \Lambda} \tilde{\Phi}(x). \tag{4.9}$$

The minimizer x^* depends on both the relative dividends π and the polyhedral set Λ. $x^* = \pi$, if and only if $\pi \in \Lambda$.

Given the minimizer x^* of $\tilde{\Phi}$ on Λ, the set of minima $\mathcal{A}_{\min}$ of the Lyapunov function Φ on the simplex Δ_I is essentially determined by the solution of a linear equation. $\mathcal{A}_{\min}$ is a polyhedral set, that is, the convex hull of finitely many points.

Lemma 4.3. *$\mathcal{A}_{\min}$ is a nonempty polyhedral set and can be represented by*

$$\mathcal{A}_{\min} = \left\{ r \in \Delta_I : \sum_{i=1}^{I} r_i \lambda_{i,k} = x_k^* \quad \textit{for all } 1 \leq k \leq K \right\}. \tag{4.10}$$

Certain situations are easy to analyze and include those cases in which the minimizer x^* equals a trading strategy λ_i for some $i \in I$. Those are closely related to the existence of single survivors in the market (cf. Example 4.1). The following proposition formulates necessary and sufficient conditions.

Proposition 4.1. *Let $i \in I$. Then the following conditions are equivalent.*

(i) $\lambda_i = x^*$.

(ii) $\nabla \tilde{\Phi}(\lambda_i) \cdot (\lambda_j - \lambda_i) \geq 0$ *for all* $j \in I \setminus \{i\}$.

(iii) $\sum_{k=1}^{K} \pi_k \frac{\lambda_{j,k}}{\lambda_{i,k}} \leq 1$ *for all* $j \in I \setminus \{i\}$.

4.4. *The global attractor*

We return to the analysis of the long run behavior of the wealth dynamics. The asymptotic limit is characterized in terms of the minimal attractor, the smallest closed set which contains the ω-limit sets of all starting points (for the definition of a ω-limit set, see [2], Section 17).

We characterize first the minimal attractor of Δ_I which describes the long-run wealth distribution in the economy, if initially no more than I agents are present. A more detailed analysis allows us to determine the minimal attractor of ri (Δ_I). This second attractor captures the long-run wealth distribution in the economy, if initially the wealth of *all* I investors is positive, i.e., if initially (and thus for every finite time) exactly I agents are present.

Theorem 4.1. *If Assumptions 2.1 and 4.1 are satisfied then the minimal attractor of Δ_I for the flow ϕ is given by the set $\mathcal{A}$ of fix points in* (4.5) *and – in particular– for all $r \in \Delta_I$ the ω-limit set $\omega(r)$ is included in $\mathcal{A}$.*

Theorem 4.1 describes the long run wealth distribution in the economy, if initially no more than I agents are present. This result can be refined, if we assume that initially the wealth of *all* I investors is positive. The asymptotics is then described by the minimal attractor $\mathcal{B}$ of the relative interior ri (Δ_I), where $\mathcal{B}$ equals the closure of $\bigcup_{r\in \mathrm{ri}\,(\Delta_I)} \omega(r)$. The next theorem gives conditions such that the set of minima $\mathcal{A}_{\min}$ of the Lyapunov function describe the long-run wealth distributions $\mathcal{B}$ (cf. Lemma 6.3 for refinements).

Theorem 4.2. *Suppose that Assumptions 2.1 and 4.1 and one of the following two conditions is satisfied:*

(i) Φ *is strictly convex on the boundary* $\partial^*(\Delta_I)$*, that is* $\Phi : \Delta_I \to \mathbb{R}$ *is strictly convex for all convex subsets of the boundary* $\partial^*(\Delta_I)$*.*
(ii) $\Phi(e_i) = \min_{g\in\partial^*(\Delta_I)} \Phi(g)$ *for some* $i \in I$*.*

Then $\mathcal{B} \subseteq \mathcal{A}_{\min}$*. If additionally* $\mathcal{A}_{\min}$ *contains points of the relative interior of* Δ_I*, then* $\mathcal{B} = \mathcal{A}_{\min}$*.*

Remark 4.1. Φ *is strictly convex on the boundary if the dimension of the hyperspace generated by the trading strategies is large compared to the number of investors (cf. [17] for a detailed analysis).*

Next let us discuss the long run behavior of the model when the minimizer x^* equals a trading strategy λ_i for some $i \in I$ (cf. Proposition 4.1).

Corollary 4.1. *Assume that one and thus all of the equivalent conditions in Proposition 4.1 hold. Then*

$$\mathcal{B} \subseteq \mathcal{A}_{\min} = \left\{ r \in \Delta_I : \sum_{j=1}^{I} r_j \lambda_j = \lambda_i \right\}. \tag{4.11}$$

If additionally $\mathcal{A}_{\min}$ *contains a point in the relative interior ri* (Δ_I)*, then* $\mathcal{B} = \mathcal{A}_{\min}$ *in (4.11).*

Corollary 4.2. *Assume that one and thus all of the equivalent conditions in Proposition 4.1 hold. If* λ_i *is an extremal point of* Λ*, then* $\mathcal{B} = \mathcal{A}_{\min} = \{e_i\}$*.*

Example 4.1. *Let us consider a special case. Assume that* $\lambda_i = \pi$*,* $\lambda_j \neq \pi$ *for* $j \neq i$*. The strategy* π *is closely related to "betting your beliefs" as introduced by [5]. If* λ_i *is extremal in* Λ*, then agent* i *will asymptotically own total wealth, while all other agents loose everything. In particular, the implication holds, if all trading strategies are extremal points in* Λ*, i.e., if there are* no redundant strategies *present in the market. This parallels the results of [9] where an analogous strategy is characterized as the global attractor in a discrete-time model.*

In the preceding corollary it was assumed that some trading strategy is equal to the minimizer x^*. This hypothesis is, of course, not always satisfied. In general, the minimizer x^* of $\tilde{\Phi}$ is a convex combination of the trading strategies. This convex combination involves some subset of the trading strategies, but possibly not all of them. The next proposition characterizes trading strategies which will never contribute to x^*. The consequences of the long run of the wealth process are discussed afterwards.

Proposition 4.2. *Assume that for some $i \in I$ the following inequality is satisfied:*

$$\sum_{k=1}^{K} \pi_k \frac{\lambda_{i,k}}{x_k^*} \neq 1\,. \tag{4.12}$$

If $\sum_{j=1}^{I} r_j \lambda_j = x^*$ or, equivalently, $r \in \mathcal{A}_{\min}$ for some $r \in \Delta_I$, then $r_i = 0$.

Example 4.2. *Suppose now that r_0 is an initial value of the wealth distribution among investors with asymptotics $\omega(r_0) \subseteq \mathcal{A}_{\min}$. If the condition (4.12) of the preceding proposition is satisfied, then strategy λ_i dies out in the long run, that is, $r_i = 0$ for $r \in \omega(r_0)$. Condition (4.12) depends on Λ: whether a trading strategy dies out or not for initial value r_0 with $\omega(r_0) \subseteq \mathcal{A}_{\min}$, is determined by its business environment of competing trading strategies.*

4.5. *A rational benchmark*

The vector π is the unconstraint minimizer of the function $\tilde{\Phi}$. This implies that whenever $\pi \in \Lambda$, then

$$\mathcal{A}_{\min} = \left\{ r \in \Delta_I : \sum_{j=1}^{I} r_j \lambda_{j,k} = \pi_k \text{ for all } 1 \leq k \leq K \right\}.$$

By (2.7) the vector π equals the price vector $(q_k)_{k=1,2,\ldots,K}$ for any wealth distribution $r \in \mathcal{A}_{\min}$ and the given profile of trading strategies. Under conditions which we already discussed in previous sections the long-run wealth distributions $\mathcal{B}$ are characterized by $\mathcal{A}_{\min}$.

In this section we will compare our results to a rational benchmark of maximizing investors who are price takers in the Walrasian market. In contrast to the evolutionary perspective agents can now choose their trading strategies. It turns out that also in this context the vector π plays a special role.

We consider myopic agents who are price takers in a continuous time Walrasian market. The aim of the agents is to maximize the instantaneous gain or growth of their portfolio. By (4.3) and (4.1) the objective function of the agents $i = 1, 2, \ldots, I$ is thus equal to

$$V_i^q : \Delta_K \to \bar{\mathbb{R}}_+\,, \qquad V_i^q(\lambda_i) = \sum_{k=1}^{K} \frac{\pi_k}{q_k} \lambda_{i,k} - 1.$$

Here, $(q_k)_{k=1,2,\ldots,K}$ equals by (2.7) the relative price vector of the assets. In terms of the price, the market clearing conditions can be rewritten as

$$q_k = \sum_{i=1}^{I} \lambda_{i,k} \cdot r_i, \qquad k = 1, 2, \ldots, K. \tag{4.13}$$

Under these conditions we obtain the following result.

Proposition 4.3. *The set of Walrasian equlibria in the economy of price taking myopic investors is given by*

$$\mathcal{E} = \left\{(\lambda_1, \lambda_2, \ldots, \lambda_I) \in (\Delta_K)^I : \sum_{i=1}^{I} \lambda_{i,k} r_i = \pi_k \text{ for all } 1 \le k \le K\right\}.$$

In equilibrium the price vector q equals π. Moreover, in equilibrium the wealth vector $(r_i)_{i=1,2,\ldots,I}$ of the investors is constant.

Finally, observe that the set of Walrasian equilibria for given wealth vector $(r_i)_{i=1,2,\ldots,I}$,

$$\mathcal{E} = \left\{(\lambda_1, \lambda_2, \ldots, \lambda_I) \in (\Delta_K)^I : \sum_{j=1}^{I} \lambda_{j,k} r_j = \pi_k \text{ for all } 1 \le k \le K\right\},$$

and the set of minima of the Lyapunov function Φ,

$$\mathcal{A}_{\min} = \left\{r \in \Delta_I : \sum_{j=1}^{I} r_j \lambda_{j,k} = \pi_k \text{ for all } 1 \le k \le K\right\},$$

for given strategy profile $(\lambda_i)_{i=1,2,\ldots,I}$ with $\pi \in \Lambda$, are dual with respect to each other.

Remark 4.2. *Instead of price taking investors who maximize their objective functions V_i^q for given price vector q we could investigate an oligopolistic market game. In this situation the objective function of investors $i = 1, 2, \ldots, I$ equals*

$$U_i(\lambda_1, \lambda_2, \ldots, \lambda_I) = \sum_{k=1}^{K} \frac{\pi_k \lambda_{i,k}}{\sum_{j=1}^{I} r_j \lambda_{j,k}}.$$

In the Nash equilibrium of the strategic game, each investor $i \in I$ chooses her optimal λ_i given the trading strategies of the others. It can be shown that in this strategic situation the unique Nash equilibrium is the strategy profile $(\lambda_1, \lambda_2, \ldots \lambda_I) = (\pi, \pi, \ldots, \pi)$ (we omit the proof).

5. Conclusion

In this article, we discuss a continuous time approximation for the evolutionary stock market model of [4]. We proved convergence of the Euler scheme to a nonlinear integral equation in a random environment, provided that the underlying firm value processes relate nicely to the Lebesgue measure (cf. Section 3.2). For constant relative dividends, we investigate the long-run asymptotics of the continuous time wealth process. The analysis is quite involved as the underlying system is nonlinear. Thanks to the existence of a Lyapunov function, we were able to provide a detailed characterization the global attractor in terms of the set of minimizers of this function. As a consequence we obtain criteria for strategies which survive or die out. In particular, the strategy "betting your beliefs" is unique surviver in a market with no redundant strategies. For comparison, we have investigated a rational benchmark. The strategy "betting your beliefs" plays also a special role in a market of rational investors. The analysis of a purely deterministic case should be consider as a first step which provides the ground for further investigations focusing on stochastic relative dividends (cf. [7]).

The present analysis of the long-run behavior of the wealth dynamics rests on a continuous-time approximation. Economically, agents are assumed to follow conservative investment strategies which invest only a small fraction of wealth into risk securities. An extension to aggressive growth strategies requires other mathematical techniques than the ones proposed in this article. Other interesting, but more demanding problems arise when agents are updating their portfolios. In this case, it would be interesting to explore connections to control theory.

6. Proofs

6.1. *Proofs of Section 3.1*

For the proofs of Proposition 3.1, Theorem 3.1, and Corollary 3.1 the reader is referred to [17].

6.2. *Proofs of Section 3.2*

For technical reasons, we need the following concept of an *exhausting sequence.*

Definition 6.1. *Let P be a probability measure and μ be a locally finite kernel from Ω to $\mathbb{R}_+$. A sequence $(C_N)_{N\in\mathbb{N}} \subseteq \mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)$ is called* exhausting *for P and μ, if the following properties are satisfied:*

(i) $C_N \in \{F\times[0,\beta): \ F\in\mathcal{F}, \ \beta>0\}$ *for all* N.
(ii) $P\mu(C_N) < \infty$.
(iii) $\bigcup_N C_N = \Omega\times\mathbb{R}_+$.

Lemma 6.1. *Let $(\Omega,\mathcal{F},P)$ be a probability space. Let μ be a locally finite transition kernel from Ω to $\mathbb{R}^+$. Then there exists an exhausting sequence $(C_N)_{N\in\mathbb{N}}$ for P and μ. Thus, Definition (3.11) defines a unique σ-finite measure $P\mu$ on the whole σ-algebra $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)$.*

Proof. See [17]. □

Proof of Theorem 3.2. Let μ_k be the measure associated with the cumulative distribution function S_k. Define $\mu := \sum_{k=1}^K \mu_k$. By Assumption 3.2 $\mu_k(\omega,\cdot)$ is a locally finite measure on $\mathbb{R}_+$. The mapping $\mu_k(\cdot,B):\Omega\mapsto\bar{\mathbb{R}}_+$ is measurable. The same is true for μ.

By Lemma 6.1 both $P\mu$ and $P\mu_k$ are σ-finite measures. As $P\mu$ dominates $P\mu_k$ it follows from the Radon-Nikodym theorem that there exist densities $f_k:\Omega\times\mathbb{R}_+\to$

$\mathbb{R}_+$ such that $d(P\mu_k) = f_k d(P\mu)$ $(k = 1.\dots, K)$. For any $F \in \mathcal{F}$ we obtain

$$\int_F S_k^t(\omega)\, P(d\omega) = P\mu_k(F \times [0,t]) = \int_F \int 1_{[0,t]}(u) f(\omega,u)\, d\mu(\omega,u)\, P(d\omega)\,.$$

Since the equality holds for all $F \in \mathcal{F}$, we obtain (3.12).

Observe that $dP\mu = \sum_{k=1}^K dP\mu_k = (\sum_{k=1}^K f_k)dP\mu$. Hence, we can conclude that $\sum_{k=1}^K f_k = 1$ $P\mu$-almost everywhere. Finally, let (C_N) be an exhausting sequence for P and μ. This implies that $\int 1_{C_N} f_k\, dP\mu = P\mu_k(C_N) \le P\mu(C_N) < \infty$. Thus, the functions $1_{C_N} f_k$ are $P\mu$-integrable. ☐

Proof of Proposition 3.2. By Lemma 6.1 we can find an exhausting sequence $(C_N)_N$ for P and μ. It clearly suffices to verify the claim for $P\mu$-almost every $(\omega, s) \in C_N$ and any $N \in \mathbb{N}$. By definition, $C_N = F_N \times [0, \alpha_N)$ for some $F_N \in \mathcal{F}$ and $\alpha_N > 0$. W.l.o.g. suppose that $P\mu(C_N) > 0$. Since $P\mu(C_N)$ is finite, we may normalize $P\mu$. Thus, we assume w.l.o.g. that $P\mu$ is a probability measure on C_N. By $\mathcal{F}_n$ we denote the σ-algebra on $\mathbb{R}_+$ generated by the partition

$$\{\{0\}\} \cup \left\{ \left(\frac{l-1}{n}, \frac{l}{n} \right],\ l \in \mathbb{N} \right\}.$$

$\mathcal{F}_n$ induces a σ-algebra $\mathcal{G}_n^N$ on C_N, namely

$$\mathcal{G}_n^N = \Big(F_N \cap \mathcal{F} \Big) \otimes \Big([0, \alpha_N) \cap \mathcal{F}_n \Big),$$

where $F_N \cap \mathcal{F} = \{F_N \cap F \,:\, F \in \mathcal{F}\}$, $[0,\alpha_N) \cap \mathcal{F}_n = \{[0,\alpha_N) \cap E \,:\, E \in \mathcal{F}_n\}$, respectively.

Let $g : C_N \to \mathbb{R}$ be measurable and integrable with respect to $P\mu$. Doob's martingale convergence theorem for directed index sets implies that $E_{P\mu}(g|\mathcal{G}_n^N)$ converges $P\mu$-almost surely to g as $n \to \infty$. This result can be applied to $1_{C_N} f_k$, since $E_{P\mu}(1_{C_N} f_k) < \infty$ by Theorem 3.2.

For $(\omega, s) \in C_N$ there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ the number $(\lfloor sn \rfloor + 1)/n$ is strictly smaller than α_N. We obtain therefore for $n \ge n_0$

$$\begin{aligned}
R_k^{(n)}(\omega, s) &= \frac{S_k^{(\lfloor sn \rfloor)+1/n} - S_k^{\lfloor sn \rfloor/n}}{\sum_{l=1}^K \left(S_l^{(\lfloor sn \rfloor+1)/n} - S_l^{\lfloor sn \rfloor/n} \right)} \\
&= \left(\int \mathbf{1}_{(\lfloor sn \rfloor/n, (\lfloor sn \rfloor+1)/n]} f_k d\mu \right) \cdot \left(\sum_{l=1}^K \int \mathbf{1}_{(\lfloor sn \rfloor/n, (\lfloor sn \rfloor+1)/n]} f_l d\mu \right)^{-1} \\
&= \big(E_{P\mu}(f_k|\mathcal{G}_n^N)(\omega,s) \big) \cdot \left(\sum_{l=1}^K E_{P\mu}(f_l|\mathcal{G}_n^N)(\omega,s) \right)^{-1}.
\end{aligned}$$

The last term converges $P\mu$-almost everywhere to $f_k(\omega,s)/\sum_{l=1}^K f_l(\omega,s)$. ☐

Proof of Corollary 3.2. By Proposition 3.2 we obtain that $\lim_{n\to\infty} R^{(n)}(\omega,s) = T(\omega,s)$ $P\mu$-almost everywhere. Set $L \in \mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)$ be the set of all (ω,s) such that $\lim_{n\to\infty} R^{(n)}(\omega,s)$ exists and equals $T(\omega,s)$. Denote by L^c the complement of L, and let $B_\omega = \{s : \ (\omega,s)\notin L\}$. Then,

$$0 = \int L^c \, dP\mu = \int \mu(\omega, B_\omega)\, P(dw)\,.$$

Hence, $\mu(\omega,B_\omega) = 0$ for P-almost all $\omega\in\Omega$. Since $\mu(\omega,\cdot)$ dominates the Lebesgue measure for P-almost all $\omega\in\Omega$, we obtain that $\lambda(B_\omega)=0$ for P-almost all $\omega\in\Omega$ where λ is the Lebesgue measure. Thus, we conclude that P-almost surely for $t\in\mathbb{R}_+$,

$$\lim_{n\to\infty} Y_t^n = \lim_{n\to\infty}\int \|T^u - R^{(n),u}\| du = \int \lim_{n\to\infty} \|T^u - R^{(n),u}\| du = 0\,.$$

Interchanging limit and integral is justified by the dominated convergence theorem, since P-almost surely T and $R^{(n)}$ $(n\in\mathbb{N})$ are bounded in Δ_K. The result follows from Theorem 3.1. □

Proof of Proposition 3.3. Observe that

$$0 = P\mu\left(\sum_{k=1}^K f_k(\omega,u) = 0\right) = \int \mu\left(\omega, \left\{s : \sum_{k=1}^K f_k(\omega,s) = 0\right\}\right) P(d\omega).$$

Thus, the sum of the functions $(f_k(\omega))_{k=1,\dots,K}$ is $\mu(\omega,\cdot)$-almost surely positive for P-almost all $\omega\in\Omega$.

Assumption 3.2 implies that the complement of any $\mu(\omega,\cdot)$-nullset lies densely in $\mathbb{R}_+$. The regularity of f implies then the positivity of the sum of the components Lebesgue-almost everywhere with probability one.

Let $B_n(s) := \left[\frac{\lfloor sn\rfloor - 1}{n}, \frac{\lfloor sn\rfloor + 1}{n}\right]$. Note that $\mu(t,t+\epsilon) > 0$ for $t\in\mathbb{R}_+$ and $\epsilon>0$. Thus, by definition of $R^{(n)}$ it holds that

$$\frac{\inf_{u\in B_n(s)} f_k(u)}{\sum_{l=1}^K \sup_{u\in B_n(s)} f_l(u)} \le R_k^{(n)}(s) \le \frac{\sup_{u\in B_n(s)} f_k(u)}{\sum_{l=1}^K \inf_{u\in B_n(s)} f_l(u)}\,.$$

Since f is Lebesgue-almost everywhere continuous, the claim follows. □

Proof of Corollary 3.3. This is analogous to the proof of Corollary 3.2. □

6.3. *Proofs of Subsection 4.1*

6.3.1. *Equlibria and a Lyapunov function*

For the characterization of the equilibria we refer to the proof of Theorem 4.1.

Proof of Lemma 4.1. Recall N and Φ as defined in (4.1) and (4.6), respectively. In view of Assumption 2.1, recall that the zeros of the linear mapping $r\mapsto \sum_{j=1}^I r_j\lambda_{j,k}$

are not contained in Δ_I. Thus, N and Φ extend to continuously differentiable functions on an open neighborhood of Δ_I. The equation $N = -\nabla\Phi$ is easily verified. Let $r_0 \in \Delta_I$ and set $r(t) = \phi_t(r_0)$, $t \in \mathbb{R}$, $r_0 \in \Delta_I$, where ϕ is the flow of (4.3) on the simplex Δ_I. In particular, for all $t \in \mathbb{R}$, observe

$$\frac{d}{dt}\,\Phi(\phi_t(r)) \;=\; (\nabla_r\Phi)[\phi_t(r)]\;\psi(\phi_t(r)) \;=\; -\sum_{i=1}^{I}\phi_{i,t}(r)N_i^2(\phi_t(r)) \;\leq\; 0\,. \tag{6.1}$$

Consequently, Φ is a Lyapunov function for the flow ϕ on Δ_I, cf. [2], Section 18. The convexity of Φ follows from the concavity of the logarithm. □

6.3.2. *The minima of the Lyapunov function*

Proof of Lemma 4.2. Part (i) is implied by the strict convexity of $\tilde{\Phi}$ and the first order conditions. We only need to show that (ii) holds. The set $\Lambda \subseteq \Delta_K$ is a compact set included in ri (Δ_K). Assumption (4.1) ensures that $\tilde{\Phi}$ is strictly convex and continuous. Moreover, Λ is convex. Since $\tilde{\Phi}$ restricted to Λ is also continuous, there exists a unique global minimum attained at some point $x^* \in \Lambda$. This implies the uniqueness of x^*. Finally, if $\pi \in \Lambda$, then clearly $x^* = \pi$. Conversely, if $\pi \notin \Lambda$, then $\pi \neq x^* \in \Lambda$. □

Proof of Lemma 4.3. $\mathcal{A}_{\min}$ is nonempty as $\Phi : \Delta_I \to \mathbb{R}$ is continuous and Δ_I is compact. Representation (4.10) is an immediate consequence of (4.8) and (4.6), as the following calculation shows:

$$\min_{r\in\Delta_I}\Phi(r) \;=\; \min_{r\in\Delta_I}\tilde{\Phi}\Bigg(\bigg(\sum_{j=1}^{I} r_j\lambda_{j,k}\bigg)_{k=1,\ldots,K}\Bigg) \;=\; \min_{x\in\Lambda}\tilde{\Phi}(x) \;=\; \tilde{\Phi}(x^*)\,. \tag{6.2}$$

The solution of the linear system in $\mathbb{R}^I$, given by $\sum_{i=1}^{I} r_i\lambda_i = x^*$ with r unknown, is an affine subspace of $\mathbb{R}^I$. This implies that $\mathcal{A}_{\min}$ is the intersection of a simplex and an affine subspace, hence polyhedral. □

Proof of Proposition 4.1. (i) ⇒ (ii): Suppose not. Then there exists $j \in I \setminus \{i\}$ such that $\nabla\tilde{\Phi}(\lambda_i)\cdot(\lambda_j - \lambda_i) < 0$. Define for $\alpha \in [0,1]$ the vector $x(\alpha) := \alpha\lambda_j + (1-\alpha)\lambda_i \in \Lambda$. Then

$$\frac{d}{d\alpha}\,\tilde{\Phi}(x(\alpha))_{|\alpha=0} \;=\; \nabla\tilde{\Phi}(\lambda_i)\cdot(\lambda_j-\lambda_i) \;<\; 0\,.$$

Since $\alpha \mapsto \frac{d}{d\alpha}\tilde{\phi}(x(\alpha))$ is continuous, we can find $1 \geq \epsilon > 0$ such that $\alpha \mapsto \tilde{\Phi}(x(\alpha))$ is strictly decreasing on $[0,\epsilon]$. Thus, $\tilde{\Phi}(x(\epsilon)) < \tilde{\Phi}(\lambda_i)$. This implies $\lambda_i \neq x^*$, a contradiction.

(ii) ⇒ (i): Let $x = \sum_{j=1}^{I} r_j\lambda_j$ with $\sum_{j=1}^{I} r_j = 1$ and $r_j \geq 0$ for all $j \in I$. Then

$$\nabla\tilde{\Phi}(\lambda_i)\cdot(x-\lambda_i) \;=\; \sum_{j=1}^{I} r_j\nabla\tilde{\Phi}(\lambda_i)\cdot(\lambda_j-\lambda_i) \;\geq\; 0.$$

Thus, by the subgradient inequality for convex functions

$$\tilde{\Phi}(x) \,\geq\, \tilde{\Phi}(\lambda_i) + \nabla\tilde{\Phi}(\lambda_i)\cdot(x-\lambda_i) \,\geq\, \tilde{\Phi}(\lambda_i).$$

Since the minimum of $\tilde{\Phi}$ is unique, we obtain $\lambda_i = x^*$.
(ii) $\Leftrightarrow$ (iii): The equivalence of (ii) and (iii) follows from

$$\nabla\tilde{\Phi}(\lambda_i)\cdot(\lambda_j-\lambda_i) \,=\, \sum_{k=1}^{K}\left(-\frac{\pi_k}{\lambda_{i,k}}+1\right)\cdot(\lambda_{j,k}-\lambda_{i,k}) \,=\, 1 \,-\, \sum_{k=1}^{K}\pi_k\frac{\lambda_{j,k}}{\lambda_{i,k}}\,.$$

□

6.3.3. *The global attractor*

Technical lemmas needed here are found in Subsubsection 6.3.4.

Proof of Theorem 4.1. Recall (4.5) and denote the minimal attractor of Δ_I by $\tilde{\mathcal{B}}$. In view of Lemma 4.1 and (6.1), the inclusion $\tilde{\mathcal{B}} \subseteq \mathcal{A}$ is implied by LaSalle's invariance principle ([2], Theorem 18.3). Conversely, the condition $\sum r_i N_i^2(r) = 0$ implies that $r_i = 0$ or $N_i(r) = 0$ for all $1 \leq i \leq I$. Thus, $\psi(r) = 0$ for $r \in \mathcal{A}$. $\mathcal{A}$ is therefore a set of fix points for the flow ϕ, hence $\mathcal{A} \subseteq \tilde{\mathcal{B}}$. □

Proof of Theorem 4.2. First suppose that (i) holds and $\mathcal{B}\backslash\mathcal{A}_{\min} \neq \emptyset$. By Lemma 6.4 below there exists a connected set $\mathcal{C} \subseteq \partial^*(\Delta_I)$ satisfying the properties (1)–(3). It follows from (2) and (3) that $\mathcal{C}$ contains at least two points. Define $M_n = \bigcup_{\substack{J\subseteq I\\ |J|\leq n}} \Delta_J$. Note that $\mathcal{C} \subseteq \partial^*(\Delta_I) = M_{I-1}$. Take the minimal n such that $\mathcal{C} \subseteq M_n$. By minimality of n we find $J \subseteq I$ such that $|J| = n$ and $\mathcal{C} \cap \mathrm{ri}\,(\Delta_J) \neq \emptyset$. As all points $\mathcal{C} \cap \mathrm{ri}\,(\Delta_J)$ are minima of Φ on Δ_J and Φ is strictly convex on Δ_J, we obtain $|\mathcal{C} \cap \mathrm{ri}\,(\Delta_J)| = 1$, a contradiction, since $|\mathcal{C}| \geq 2$ and $\mathcal{C}$ connected.

If (ii) holds, then it follows by the subgradient inequality that for all $c \in \mathcal{A}\backslash\mathcal{A}_{\min}$ and for all $i \in I$,

$$N_i(c) \,=\, N_i(c) - \sum_{j=1}^{I} c_j N_j(c) \,=\, \nabla\Phi(c)(e_i - c) \,\geq\, \Phi(c) - \Phi(e_i) \,>\, 0\,.$$

In this case, a set $\mathcal{C}$ as stated in Lemma 6.4 satisfying (2) and (3) simultanously cannot exist. The last claim is immediate by Lemma 6.3(iii). □

Proof of Corollary 4.1. $\mathcal{A}_{\min} = \left\{r \in \Delta_I : \sum_{j=1}^{I} r_j\lambda_j = \lambda_i\right\}$ by (4.10). Then $e_i \in \mathcal{A}_{\min}$, hence $\mathcal{B} \subseteq \mathcal{A}_{\min}$ by Theorem 4.2(ii). The last claim follows from Lemma 6.3(iii). □

Proof of Corollary 4.2. If λ_i is an extremal point of the polyhedron Λ, then $\mathcal{A}_{\min} = \{e_i\}$. Since $\emptyset \neq \mathcal{B} \subseteq \mathcal{A}_{\min}$, we obtain $\mathcal{B} = \mathcal{A}_{\min}$. □

Proof of Proposition 4.2. Since $\nabla\tilde{\Phi}(x^*)\cdot(\lambda_i - x^*) = 1 - \sum_{k=1}^{K}\pi_k\cdot\frac{\lambda_{i,k}}{x_k^*}$, we obtain $\nabla\tilde{\Phi}(x^*)\cdot(\lambda_i - x^*) \neq 0$. Let now $y \in \Lambda$. Assume that $\nabla\tilde{\Phi}(x^*)\cdot(y - x^*) < 0$. For

$\alpha \in [0,1]$ define the vector $x(\alpha) := \alpha y + (1-\alpha)x^* \in \Lambda$. The same arguments as in the part (i)⇒(ii) of the proof of Proposition 4.1 show that there exists $0 < \epsilon \le 1$ such that $\tilde\Phi(x(\epsilon)) < \tilde\Phi(x^*)$. This implies that $x^* \neq \mathrm{argmin}_{x\in\Lambda}\tilde\Phi(x)$, a contradiction. Hence, for $y \in \Lambda$,

$$\text{(i)} \quad \nabla\tilde\Phi(x^*)\cdot(y-x^*) \ge 0 \qquad \text{and} \qquad \text{(ii)} \quad \nabla\tilde\Phi(x^*)\cdot(\lambda_i - x^*) > 0\,.$$

Now, let $r \in \Delta_I$ such that $x^* = \sum_{j=1}^I r_j\lambda_j$. Then,

$$0 = \nabla\tilde\Phi(x^*)\cdot(x^*-x^*) = \sum_{j=1}^{I} r_j \nabla\tilde\Phi(x^*)\cdot(\lambda_j - x^*).$$

Since each summand is nonnegative by (i), we obtain that $r_i\nabla\tilde\Phi(x^*)\cdot(\lambda_i - x^*) = 0$. Finally, (ii) implies that $r_i = 0$. □

6.3.4. *Auxiliary results on the global attractor*

Lemma 6.2. *Any $r \in \mathcal{A} \cap \partial^*(\Delta_I)$ is contained in the relative interior $ri\,(\Delta_J)$ for some $J \subseteq I$, $J \neq I$. r minimizes the Lyapunov function Φ on Δ_J.*

Proof. This is immediate from the subgradient inequality for convex functions and Theorem 4.1 (cf. (4.5)). □

Lemma 6.3. *If Assumptions 2.1 and 4.1 are satisfied then $\overline{ri\,(\Delta_I) \cap \mathcal{A}} \subseteq \mathcal{B} \subseteq \mathcal{A}$. Moreover, $\mathcal{A}_{\min} \subseteq \mathcal{A}$ is a nonempty, closed, convex set of fixed points for ϕ, and the following holds:*

(i) $\overline{ri\,(\Delta_I) \cap \mathcal{A}} \subseteq \mathcal{A}_{\min}$.

(ii) The converse inclusion $\mathcal{A}_{\min} \subseteq \overline{ri\,(\Delta_I) \cap \mathcal{A}}$ holds, if and only if the set $\mathcal{A}_{\min} \cap ri\,(\Delta_I)$ is nonempty. In this case, $\mathcal{A}_{\min} = \overline{ri\,(\Delta_I) \cap \mathcal{A}} = \overline{ri\,(\Delta_I) \cap \mathcal{A}_{\min}}$.

(iii) If $\mathcal{A}_{\min} \cap ri\,(\Delta_I)$ is nonempty, then $\mathcal{A}_{\min} \subseteq \mathcal{B}$.

(iv) $\mathcal{B} \setminus \mathcal{A}_{\min}$ is a subset of the boundary $\partial^*(\Delta_I)$. Any $r^* \in \mathcal{B} \setminus \mathcal{A}_{\min}$ is contained in the relative interior $ri\,(\Delta_J)$ of some subsimplex for some $J \subseteq I$, $J \neq I$. r^* minimizes the Lyapunov function Φ on $ri\,(\Delta_J)$.

Proof. See [17]. □

Lemma 6.4. *If Assumptions 2.1 and 4.1 are satisfied and $\mathcal{B} \setminus \mathcal{A}_{\min} \neq \emptyset$ then there exist $g \in \mathcal{B}\setminus\mathcal{A}_{\min}$ and $r \in ri(\Delta_I)$ with $g \in \omega(r)$. $\mathcal{C} := \omega(r) \subseteq \mathcal{B}\setminus\mathcal{A}_{\min}\cap\partial^*(\Delta_I)$ is a nonempty, connected set with the following properties:*

(1) $\forall c \in \mathcal{C}\ \forall J \subseteq I\ \big(c \in ri\,(\Delta_J) \ \Rightarrow\ \Phi(c) = \min_{d\in\Delta_J}\ \Phi(d)\big)$.

(2) $\forall c \in \mathcal{C}\ \exists i \in I\ N_i(c) > 0$.

(3) $\forall c \in \mathcal{C}\ \forall i \in I\ \big(N_i(c) > 0 \ \Rightarrow\ \exists d \in \mathcal{C}\ N_i(d) = 0\big)$.

Proof. See [17]. □

6.4. *Proofs of Section 4.5*

Proof of Proposition 4.3. Clearly, $q_k \neq 0$ for all $k = 1, 2, \ldots, K$. Namely, if $q_k = 0$, the demand for asset k is strictly positive (even infinite). Thus, $q_k \neq 0$ by (4.13), a contradiction. Assume that $\frac{\pi_k}{q_k} > \frac{\pi_l}{q_l}$ for some $l, k = 1, 2, \ldots, K$. Then clearly $\lambda_{j,l} = 0$ for $j = 1, 2, \ldots, I$, since agents are maximizers. Thus, $q_k = 0$ by (4.13), a contradiction. Hence, $\frac{\pi_k}{q_k} = \frac{\pi_l}{q_l}$ $(l, k = 1, 2, \ldots, K)$. Since $\sum_{k=1}^K \pi_k = \sum_{k=1}^K q_k$, this implies that $\pi_k = q_k$ $(k = 1, 2, \ldots, K)$. By (4.13) we obtain that

$$\mathcal{E} \subseteq \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I) \in (\Delta_K)^I : \sum_{i=1}^I \lambda_{i,k} r_i = \pi_k \text{ for all } 1 \leq k \leq K \right\} =: \mathcal{E}'.$$

Clearly, for $(\lambda_1, \lambda_2, \ldots, \lambda_I) \in \mathcal{E}'$ the price vector q equals π. For $q = \pi$ agents are indifferent between all strategies, thus no profitable deviation exists for any agent. Hence, $\mathcal{E}' \subseteq \mathcal{E}$.

For $(\lambda_1, \lambda_2, \ldots, \lambda_I) \in \mathcal{E}$ we obtain $q = \pi$, thus $V_i^q(\lambda_i) = 0$ for all $i = 1, 2, \ldots, I$. This implies that the wealth vector $(r_i)_{i=1,2,\ldots,I}$ of the investors is constant. ☐

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