THE OTHER LAW OF THE ITERATED LOGARITHM FOR LÉVY PROCESSES AT SMALL AND LARGE TIMES*

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Suppose $X(t)$ is a Lévy process with nonzero Gaussian component $\sigma B(t)$. Then we have the *small time* result:

$$\liminf_{t\downarrow 0} \frac{\sup_{0\le s\le t}|X(s)|}{\sqrt{t/\log|\log t|}} = \frac{\pi\sigma}{\sqrt{8}}, \qquad \text{a.s.}$$

Suppose $X(t)$ is a Lévy process with $0 < \widetilde{\sigma}^2 := EX^2(1) < \infty$ and $EX(1) = 0$. Then we have the *large time* result:

$$\liminf_{t\uparrow\infty} \frac{\sup_{0\le s\le t}|X(s)|}{\sqrt{t/\log\log t}} = \frac{\pi\widetilde{\sigma}}{\sqrt{8}}, \qquad \text{a.s.}$$

As a special case, both results hold when X is a Brownian motion with variance $\sigma^2 > 0$. In this case we have $\sigma^2 = \widetilde{\sigma}^2$.

These are analogues of the "other" law of the iterated logarithm for random walks.

1. Introduction. The "other law of the iterated logarithm", as a large-sample theorem for a random walk S_n, has a long and distinguished history. It was originally conceived as a counterpart to the "usual" law of the iterated logarithm by Chung [4], and proved in the form

$$\liminf_{n\uparrow\infty} \frac{\max_{1\le j\le n}|S_j|}{\sqrt{n/\log\log n}} = \frac{\pi\sqrt{ES_1^2}}{\sqrt{8}}, \qquad \text{a.s.,}$$

under the assumption that the i.i.d. increments of S_n have a finite absolute third moment and expectation 0. This was subsequently improved to a second moment assumption by Jain and Pruitt [8].

There have been many extensions and spin-off results related to this "other law"; among them we mention a converse by Csáki [5], a definitive

*This research was partially supported by ARC grant DP0664603.

AMS 2000 subject classifications: primary 60G51, 60F15, 60F17, 60F05; secondary 60J65, 60J75

Keywords and phrases: Lévy process, local behaviour, global behaviour, almost sure convergence, iterated logarithm laws, other law of the iterated logarithm

upper/lower class integral test by Einmahl [6], and very general formulations by Einmahl and Mason [7] and Kesten [10]. Our aim in this paper is to obtain similar laws for Lévy processes, not only as the time parameter becomes large, which gives an analogue to the Chung law, but also as the time parameter becomes small. This requires the development of some new methods, and also suggests some interesting new areas of research.

Notation is as follows. Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a Lévy process X with $X(0) = 0$ and canonical triplet (γ, σ^2, Π), where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, and Π is a nonnegative measure on $\mathbb{R}$ satisfying $\Pi(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (x^2 \wedge 1)\Pi(\mathrm{d}x) \; < \; \infty. \tag{1.1}$$

The finite dimensional distributions of X are determined by the distribution of $X(1)$ via the celebrated Lévy-Khintchine formula, which states that $Ee^{\mathrm{i}\theta X(t)} = e^{\mathrm{i}t\Psi(\theta)}$, where

$$\Psi(\theta) \; = \; \mathrm{i}\gamma\theta - \frac{\sigma^2\theta^2}{2} + \int_{\mathbb{R}} (e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x 1_{|x|\leq 1})\, \Pi(\mathrm{d}x)\,, \tag{1.2}$$

for $t > 0$ and $\theta \in \mathbb{R}$. Without loss of generality, we may assume throughout that, for all $\omega \in \Omega$, the sample paths $t \mapsto X(t, \omega)$ are càdlàg functions, i.e., right-continuous with left limits. See Bertoin [2] and Sato [12] for basic properties of Lévy processes.

Our "small time result" is stated in the next section, while Section 3 has the large time result. All proofs are in Section 4.

2. Small Time.

THEOREM 2.1. *Suppose $\sigma^2 > 0$ in (1.2). Then*

$$\liminf_{t\downarrow 0} \frac{\sup_{0\leq s\leq t} |X(s)|}{\sqrt{t/\log|\log t|}} \; = \; \frac{\pi\sigma}{\sqrt{8}}\,, \qquad \text{a.s.} \tag{2.1}$$

REMARK 2.1. (i) The form of (2.1) suggests that the liminf should equal 0 when $\sigma^2 = 0$. So far we have not proved this in general, though it holds in some special cases. Another interesting question is how to normalise $\sup_{0\leq s\leq t} |X(s)|$ so as to get a finite nonzero liminf, a.s., when $\sigma^2 = 0$. (See [3] for this kind of result for the limsup.)

(ii) Some interesting issues arise in connection with possible centering functions in (2.1). This topic is addressed by Kesten [10] in his very general random walk version. We expand on the issues in Remark 4.1 below. □

3. Large Time.

THEOREM 3.1. *Suppose* $0 < \widetilde{\sigma}^2 := EX^2(1) = \sigma^2 + \int_{\mathbb{R}} x^2 \Pi(\mathrm{d}x) < \infty$ *and* $EX(1) = 0$. *Then*

$$\liminf_{t \uparrow \infty} \frac{\sup_{0 \le s \le t} |X(s)|}{\sqrt{t / \log\log t}} = \frac{\pi \widetilde{\sigma}}{\sqrt{8}}, \qquad \text{a.s.} \tag{3.1}$$

REMARK 3.1. When $EX^2(1) = \infty$ the lefthand side of (3.1) is infinite, a.s. This follows easily from a corresponding random walk version of Csáki [5], which can be applied to show that, when $EX^2(1) = \infty$,

$$\lim_{n \uparrow \infty} \frac{\max_{1 \le j \le n} |X(j)|}{\sqrt{n / \log\log n}} = \infty, \qquad \text{a.s.}$$

□

4. Proofs.

4.1. *Preliminary Results.* We will make use of a "truncated" and "centered" decomposition of X. Take $b > 0$ and define Lévy measures $\Pi_i^{(b)}$ for $i = 1, 2$ by setting

$$\Pi_1^{(b)}(B) = \Pi([-b, b] \cap B) \quad \text{and} \quad \Pi_2^{(b)}(B) = \Pi(B \backslash [-b, b]),$$

for all Borel sets $B \subseteq \mathbb{R}$. Define a function $\nu : (0, \infty) \to \mathbb{R}$ as follows:

$$\nu(b) = \begin{cases} \gamma - \int_{b < |x| \le 1} x \Pi(\mathrm{d}x), & \text{if } 0 < b \le 1, \\ \gamma + \int_{1 < |x| \le b} x \Pi(\mathrm{d}x), & \text{if } b > 1. \end{cases} \tag{4.1}$$

This will play the role of a centering function. For any $b > 0$ we have the decomposition

$$X(t) = t\nu(b) + X^{(b)}(t) + W^{(b)}(t), \qquad t \ge 0, \tag{4.2}$$

where, for $t \ge 0$,

$$X^{(b)}(t) := \sigma B(t) + X^{(b,S)}(t), \tag{4.3}$$

with

$$X^{(b,S)}(t) = \text{a.s.} \lim_{\varepsilon \downarrow 0} \left(\sum_{0 < s \le t} \Delta X(s) \mathbf{1}_{\{\varepsilon < |\Delta X(s)| \le b\}} - t \int_{\varepsilon < |x| \le b} x \Pi(\mathrm{d}x) \right),$$

and

$$W^{(b)}(t) \;=\; \sum_{0<s\le t} \Delta X(s) \mathbf{1}_{\{|\Delta X(s)|>b\}}\,.$$

For each $b > 0$, $(X^{(b)})_{t\ge0}$ and $(W^{(b)})_{t\ge0}$ are independent Lévy processes. If $0 < b \le 1$ then it is easy to check that $X^{(b)}$ and $W^{(b)}$ have the triplets $(0, \sigma^2, \Pi_1^{(b)})$ and $(\int_{b<|x|\le1} x\Pi(\mathrm{d}x), 0, \Pi_2^{(b)})$, respectively, while if $b > 1$, then $X^{(b)}$ and $W^{(b)}$ have the triplets $(-\int_{1<|x|\le b} x\Pi(\mathrm{d}x), \sigma^2, \Pi_1^{(b)})$ and $(0, 0, \Pi_2^{(b)})$, respectively. In particular, jumps are smaller than or equal in magnitude to b for $X^{(b)}$ and bigger in magnitude than b for $W^{(b)}$.

Define functions $V(x)$ and $\rho(x)$ for $x > 0$ by

$$V(x) \;=\; \sigma^2 + \int_{0<|y|\le x} y^2\Pi(\mathrm{d}y) \quad \text{and} \quad \rho(x) \;=\; \int_{0<|y|\le x} |y|^3\Pi(\mathrm{d}y)\,. \tag{4.4}$$

Of course $V(x)$ and $\rho(x)$ are finite for each $x > 0$ by virtue of the integrability properties of Π.

For all $b > 0$ and $t \ge 0$, observe that

$$EX^{(b)}(t) \;=\; 0 \quad \text{and} \quad \mathrm{Var}(X^{(b)}(t)) \;=\; tV(b)\,.$$

The next lemma uses a result of Bertoin, Doney and Maller [3] to transfer an inequality of Sakhanenko [11] from discrete to continuous time.

LEMMA 4.1. *(No assumptions on $\sigma^2 \ge 0$.) Let $b > 0$ and let $X^{(b)}(t)$ be the Lévy process in (4.3). Suppose that the function V in (4.4) satisfies $V(b) > 0$.*

Suppose functions $g_i(t)$, $t \ge 0$, $i = 1, 2$, satisfy

$$g_2(t) \;<\; g_1(t)\,, \quad t \ge 0, \quad \text{and} \quad g_2(0) \;<\; 0 \;<\; g_1(0)\,,$$

and

$$|g_i(t+h) - g_i(t)| \;\le\; Lh, \qquad i = 1, 2\,,\; h > 0\,,\; t \ge 0\,, \tag{4.5}$$

for some constant $L > 0$. Then for any $t > 0$, we have the bound

$$\begin{aligned}\Bigg| P\left(g_2(s) \le \frac{X^{(b)}(s)}{\sqrt{V(b)}} \le g_1(s),\ 0 \le s \le t\right) \\ -\; P\big(g_2(s) \le B(s) \le g_1(s),\ 0 \le s \le t\big)\Bigg| \\ \le \frac{C(1+L\sqrt{t})\rho(b)}{\sqrt{t}(V(b))^{3/2}}\,,\end{aligned} \tag{4.6}$$

where $(B(t))_{t\ge0}$ is a standard Brownian motion, and C is an absolute constant.

PROOF OF LEMMA 4.1: Fix $b, t > 0$, and g_1, g_2, as specified, and assume (4.5). The functions g_1 and g_2 are then continuous, and as probability measures are continuous from above, we have

$$
\begin{aligned}
(4.7)\quad & P\left(g_2(s) \le \frac{X^{(b)}(s)}{\sqrt{V(b)}} \le g_1(s),\ 0 \le s \le t\right) \\
= \; & P\left(g_2(ts) \le \frac{X^{(b)}(ts)}{\sqrt{V(b)}} \le g_1(ts),\ 0 \le s \le 1\right) \\
= \; & \lim_{n\to\infty} P\left(g_2(tj2^{-n}) \le \frac{X^{(b)}(tj2^{-n})}{\sqrt{V(b)}} \le g_1(tj2^{-n}),\ 1 \le j \le 2^n\right) \\
= \; & \lim_{n\to\infty} P\left(\frac{g_2(tj2^{-n})}{\sqrt{t}} \le \frac{\sum_{m=1}^{j} \Delta_m^{(b,t)}(n)}{2^{n/2}} \le \frac{g_1(tj2^{-n})}{\sqrt{t}},\ 1 \le j \le 2^n\right),
\end{aligned}
$$

where, for $m \in \mathbb{N} = \{1, 2, \dots\}$, we set

$$
\Delta_m^{(b,t)}(n) \;=\; 2^{n/2}\,(tV(b))^{-1/2}\left[X^{(b)}(tm2^{-n}) - X^{(b)}(t(m-1)2^{-n})\right].
$$

For all $n \in \mathbb{N}$ observe that $(\Delta_m^{(b,t)}(n))_{m\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables having the same distribution as $2^{n/2}\,(tV(b))^{-1/2}X^{(b)}(t2^{-n})$, thus, having zero mean and variance one. By the inequality following Eq.(4) of Sakhanenko [11], we have for a Brownian motion $(B(t))_{t\ge 0}$ that

$$
\begin{aligned}
& \left| P\left(\frac{g_2(tj2^{-n})}{\sqrt{t}} \le \frac{\sum_{m=1}^{j} \Delta_m^{(b,t)}(n)}{2^{n/2}} \le \frac{g_1(tj2^{-n})}{\sqrt{t}},\ 1 \le j \le 2^n\right)\right. \\
& \qquad\qquad \left. - P\left(\frac{g_2(ts)}{\sqrt{t}} \le B(s) \le \frac{g_1(ts)}{\sqrt{t}},\ 0 \le s \le 1\right)\right| \\
& \le \; C' 2^{-n/2}(1 + L\sqrt{t})E|\Delta_1^{(b,t)}(n)|^3 \\
(4.8)\quad & = \; \frac{C'(1 + L\sqrt{t})}{\sqrt{t}(V(b))^{3/2}}\ \frac{E|X^{(b)}(t2^{-n})|^3}{t2^{-n}}, \quad n = 1, 2, \dots,
\end{aligned}
$$

where C' is an absolute constant. By (4.7), the lefthand side of (4.8) tends as $n \to \infty$ to the lefthand side of (4.6). For the righthand side, note that, for $u > 0$,

$$
E|X^{(b)}(u)|^3 \;=\; E|\sigma B(u) + X^{(b,S)}(u)|^3 \;\le\; 8\sigma^3 E|B(u)|^3 + 8E|X^{(b,S)}(u)|^3 .
$$

We have $u^{-1}E|B(u)|^3 = u^{1/2}E|B(1)|^3 \to 0$, and $u^{-1}E|X^{(b,S)}(u)|^3 \to \rho(b)$ as $u \to 0$, the latter by Lemma 4.3(i) of [3]. Thus letting $n \to \infty$ in the righthand side of (4.8) gives (4.6) for the choice $C = 8C'$. □

Throughout, the "tail sum" $\overline{\Pi}(x)$ is defined by

$$\overline{\Pi}(x) \;=\; \Pi(\mathbb{R}\backslash[-x,x])\,, \quad x > 0\,.$$

The next technical lemma proves some convergences that we will need. The proof is deferred to an appendix.

LEMMA 4.2. *(i) Suppose $(t_n)_{n=1,2,\ldots}$ is either the sequence r^n, where $0 < r < 1$, or the sequence n^{-n}. Then (with no further assumptions) we have*

$$\sum_{n\geq 1} t_n\overline{\Pi}(t_n^{1/2}) + \sum_{n\geq 1} t_n^{-1/2}\rho(t_n^{1/2}) + \sum_{n\geq 1} t_n^{1/2}|\nu(t_n^{1/2})| \;<\; \infty\,. \tag{4.9}$$

(ii) Let $(t_n)_{n=1,2,\ldots}$ be either the sequence λ^n, where $\lambda > 1$, or the sequence n^n. If $EX^2(1) < \infty$ and $EX(1) = 0$ then (4.9) holds as well, exactly as stated.

Let ν be as in (4.1) and define $K : (0,\infty) \to \mathbb{R}$ by setting

$$K(x) \;=\; \begin{cases} \nu(x)/\sqrt{V(x)}\,, & \text{if } V(x) > 0\,, \\ 0\,, & \text{if } V(x) = 0\,, \end{cases} \qquad x > 0\,. \tag{4.10}$$

Finally, we write

$$\|h\|_t \;:=\; \sup_{0\leq s\leq t} |h(s)|\,, \qquad t \geq 0\,,$$

for a locally bounded function $h : [0,\infty) \to \mathbb{R}$.

In the next lemma we provide a useful upper bound to control the uniform norm of a Brownian motion with drift.

LEMMA 4.3. *Let $0 < u_n, v_n < \infty$ for all $n \geq 1$. Let $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$ be the identity function and let K be the function in (4.10). Then*

$$\sum_{n\geq 1} \sup_{x\geq 0} \Big| P(\|B\|_1 \leq x) - P(\|B + u_n K(v_n)\mathrm{id}\|_1 \;\leq\; x)\Big| < \infty\,, \tag{4.11}$$

provided one of the following conditions is satisfied:
(i) We have $\sigma^2 > 0$ in (1.2). Furthermore, $u_n, v_n \downarrow 0$, as $n \to \infty$, are sequences such that

$$\sum_{n\geq 1} u_n \, |\nu(v_n)| \, < \, \infty\,. \tag{4.12}$$

(ii) We have $V(x_0) > 0$ for some $x_0 > 0$ in (4.4) and, furthermore, $u_n, v_n \uparrow \infty$, as $n \to \infty$, are sequences, satisfying (4.12), exactly as stated.

Proof of Lemma 4.3: Recall (4.4) and (4.10). In either case (i) or (ii) it follows from (4.12) that there exists $C \in (0, \infty)$ such that

$$\sup_{n\geq 1} u_n |K(v_n)| \leq \sum_{n\geq 1} u_n |K(v_n)| \, \leq \, C \, < \, \infty\,. \tag{4.13}$$

It follows from Girsanov's theorem (cf. Karatzas and Shreve [9], Theorem 3.5.1) that, for all $n \in \mathbb{N}$ and $x \geq 0$,

$$P(\|B + u_n K(v_n)\mathrm{id}\|_1 \leq x) = E\mathbf{1}_{\{\|B\|_1 \leq x\}} \exp\left(u_n K(v_n) B_1 - \frac{1}{2} u_n^2 K^2(v_n)\right)\,,$$

and, thus, by (4.13) and the elementary inequality $|e^x - 1| \leq |x| e^{|x|}$ $(x \in \mathbb{R})$,

$$\begin{aligned}
&\sum_{n\geq 1} \sup_{x\in\mathbb{R}} \left| P(\|B\|_1 \leq x) - P(\|B + u_n K(v_n)\mathrm{id}\|_1 \leq x) \right| \\
\leq\ & \sum_{n\geq 1} E \left| \exp\left(u_n K(v_n) B_1 - \frac{1}{2} u_n^2 K^2(v_n)\right) - 1 \right| \\
\leq\ & \sum_{n\geq 1} u_n |K(v_n)| \, E\left[\left(|B_1| + \frac{C}{2}\right) \exp\left(C|B_1| + \frac{C^2}{2}\right)\right] \\
\leq\ & C\, E\left[\left(|B_1| + \frac{C}{2}\right) \exp\left(C|B_1| + \frac{C^2}{2}\right)\right] \, < \, \infty\,,
\end{aligned}$$

which completes the proof of (4.11). □

4.2. *Proof of Theorem 2.1.* Now we are in the position to show the first part of Theorem 2.1.

LEMMA 4.4. *(No assumptions on X.) We have*

$$\liminf_{t\downarrow 0} \frac{\|X\|_t}{\sqrt{t/\log|\log t|}} \geq \frac{\pi\sigma}{\sqrt{8}}, \qquad \text{a.s.} \tag{4.14}$$

PROOF OF LEMMA 4.4: Without loss of generality we may take $\sigma^2 > 0$ in (4.4). Otherwise, (4.14) becomes trivial. Consequently, $V(x) \geq \sigma^2 > 0$, for all $x > 0$. Throughout the proof, we set

$$\ell_2(x) := \begin{cases} \log|\log(x)|, & 0 < x < e^2, \\ \log\log(e^2), & x \geq e^2. \end{cases} \tag{4.15}$$

Take $0 < r < 1$. We have the bounds

$$\begin{aligned} &\frac{4}{\pi}\left(\exp\left(-\frac{\pi^2}{8x^2}\right) - \frac{1}{3}\exp\left(-\frac{9\pi^2}{8x^2}\right)\right) \\ &\qquad \leq \; P(\|B\|_1 \leq x) \; \leq \; \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8x^2}\right), \end{aligned} \tag{4.16}$$

for all $x > 0$ [cf. Chung [4], p.221, or Jain and Pruitt [8]]. Hence

$$\sum_{n\geq 1} P\left(\|B\|_1 \leq \frac{a}{\sqrt{\ell_2(r^n)}}\right) \leq \frac{4}{\pi}\sum_{n\geq 1}\exp\left(-\frac{\pi^2\ell_2(r^n)}{8a^2}\right) < \infty,$$

for all $0 < a < \pi/\sqrt{8}$. It thus follows from (4.11) with $u_n = v_n = r^{n/2}$ (and using (4.9) to verify (4.12)), together with the self-similarity of Brownian motion, that

$$\begin{aligned} &\sum_{n\geq 1} P\left(\|B + K(r^{n/2})\mathrm{id}\|_{r^n} \leq \frac{a\sqrt{r^n}}{\sqrt{\ell_2(r^n)}}\right) \\ &= \sum_{n\geq 1} P\left(\|B + r^{n/2}K(r^{n/2})\mathrm{id}\|_1 \leq \frac{a}{\sqrt{\ell_2(r^n)}}\right) < \infty, \end{aligned} \tag{4.17}$$

for all $0 < a < \pi/\sqrt{8}$.

Now recall (4.1), (4.2) and (4.10). We have $V(x) \geq \sigma^2$, with $\sigma^2 > 0$ assumed. Applying Lemma 4.1 with

$$\begin{aligned} g_1(s) &= g_{1,n}^{(r,a)}(s) = -K(r^{n/2})s + a\sqrt{r^n/\ell_2(r^n)}, \\ g_2(s) &= g_{2,n}^{(r,a)}(s) = -K(r^{n/2})s - a\sqrt{r^n/\ell_2(r^n)}, \end{aligned}$$

($s \geq 0$, $a, r > 0$ and $n \geq 1$), which functions satisfy (4.5) with $L = |K(r^{n/2})|$, we get from (4.6) that

$$\sum_{n\geq 1}\left| P\left(\|X^{(\sqrt{r^n})} + \nu(r^{n/2})\mathrm{id}\|_{r^n} \leq a\sqrt{\frac{V(r^{n/2})r^n}{\ell_2(r^n)}}\right) - P\left(\|B + K(r^{n/2})\mathrm{id}\|_{r^n} \leq a\sqrt{\frac{r^n}{\ell_2(r^n)}}\right)\right|$$

$$\leq C\sigma^{-3}\sum_{n\geq 1} r^{-n/2}\rho(r^{n/2})(1 + r^{n/2}|K(r^{n/2})|)\,, \tag{4.18}$$

where C is the universal constant in Lemma 4.1. The right hand-side in the last display is finite by (4.9). Consequently, by (4.17) and (4.18),

$$\sum_{n\geq 1} P\left(\|X^{(r^{n/2})}) + \nu(r^{n/2})\mathrm{id}\|_{r^n} \leq a\sqrt{V(r^{n/2})r^n/\ell_2(r^n)}\right) < \infty\,, \tag{4.19}$$

for all $0 < a < \pi/\sqrt{8}$. Since we have $\lim_{t\downarrow 0} V(t) = \sigma^2$, it follows from (4.19) and the Borel-Cantelli lemma that

$$\liminf_{n\to\infty} \frac{\|X^{(r^{n/2})} + \nu(r^{n/2})\mathrm{id}\|_{r^n}}{\sqrt{r^n/\ell_2(r^n)}} \geq \frac{\pi\sigma}{\sqrt{8}}\,, \qquad \text{a.s.} \tag{4.20}$$

Next recall (4.2), and note that

$$P\left(\|W^{(r^{n/2})}\|_{r^n} > 0\right) \tag{4.21}$$

$$= P\left(\sup_{0<y\leq r^n}\left|\sum_{0<s\leq y}\Delta X(s)\mathbf{1}_{\{|\Delta X(s)|>r^{n/2}\}}\right| > 0\right)$$

$$\leq P(X \text{ has at least one jump up to time } r^n \text{ exceeding } r^{n/2} \text{ in magnitude})$$

$$= 1 - \exp(-r^n\overline{\Pi}(r^{n/2})) \leq r^n\overline{\Pi}(r^{n/2}), \qquad n \in \mathbb{N}\,.$$

This is summable over $n \in \mathbb{N}$ by (4.9). Thus

$$P\left(\|W^{(r^{n/2})}\|_{r^n} > 0 \text{ i.o. as } n\to\infty\right) = 0. \tag{4.22}$$

In view of (4.2) and (4.22), we can replace $X^{(r^{n/2})} + \nu(r^{n/2})\mathrm{id}$ with X in (4.20).

For the final step in the lemma, pick $0 < t < 1$ and choose $n = n(t) \geq 1$ so that $r^n \leq t < r^{n-1}$. Then

$$\frac{\|X\|_t}{\sqrt{t/\ell_2(t)}} \geq \frac{\|X\|_{r^n}}{\sqrt{r^n/\ell_2(r^n)}} \sqrt{r\frac{\ell_2(r^{n-1})}{\ell_2(r^n)}}\,, \tag{4.23}$$

in which we can let $t \downarrow 0$ (thus $n \uparrow \infty$) and then $r \uparrow 1$ to get (4.14). □

REMARK 4.1. (i) The crucial inequality (4.17) also follows directly from Anderson's inequality [[1], Corollary 5], which implies

$$P\left(\|B + r^{n/2}K(r^{n/2})\mathrm{id}\|_1 \leq \frac{a}{\sqrt{\ell_2(r^n)}}\right) \leq P\left(\|B\|_1 \leq \frac{a}{\sqrt{\ell_2(r^n)}}\right),$$

for all $0 < r < 1$ and $n \geq 1$. This offers an alternative approach to using Girsanov's theorem in the above proof.

(ii) It is trivial to give a slightly more general result than in (2.1), viz, for any locally Lipschitz function $a(\cdot)$ with $a(0) = 0$ we have that

$$\liminf_{t\downarrow 0} \frac{\sup_{0\leq s\leq t}|X(s) - a(s)|}{\sqrt{t/\log|\log t|}} = \frac{\pi\sigma}{\sqrt{8}}, \qquad \text{a.s.}$$

This follows because $\sup_{0\leq s\leq t}|a(s)| = o\left(\sqrt{t/\log|\log t|}\right)$ as $t \downarrow 0$ for such functions.

While this covers some cases of interest, others are not covered. The usual centering of interest will be of the form $t\nu(b(t))$, for a certain $b(t)$ (perhaps, $b(t) = \sqrt{t}$). For example, when X is of bounded variation, then ν is bounded near 0 and so $a(t) = t\nu(\sqrt{t})$ is locally Lipschitz. But there are Lévy processes X for which this argument does not apply.

We believe that the lower bound for the liminf in (4.14) should hold for arbitrary centering. The analogue of this is shown by Kesten [10] in his very general random walk version. But our method of proof of Theorem 2.1, via the generalised Sakhanenko bound in Lemma 4.1, requires the Lipschitz condition.

We further remark that we are a very long way from being able to give an analogue of the complete Kesten result for Lévy processes at 0. □

Next we show the other part of Theorem 2.1.

LEMMA 4.5. *Take $\sigma^2 > 0$. Then*

$$\liminf_{t\downarrow 0} \frac{\|X\|_t}{\sqrt{t/\log|\log t|}} \leq \frac{\pi\sigma}{\sqrt{8}}, \qquad \text{a.s.} \tag{4.24}$$

PROOF OF LEMMA 4.5: Let $\ell_2 : (0,\infty) \to \mathbb{R}$ be as defined in (4.15). Throughout the proof of the lemma, let $t_n := n^{-n}$, $n = 1, 2, \ldots$, and set

$$U_n := \sup_{t_{n+1} < s \le t_n} |X(s) - X(t_{n+1})| .$$

Suppose we can show that

$$\liminf_{n\to\infty} \frac{U_n}{\sqrt{t_n/\ell_2(t_n)}} \le \frac{\pi\sigma}{\sqrt{8}}, \qquad \text{a.s.} \tag{4.25}$$

We get from the iterated law of logarithm for a general Lévy process for small times [cf. Sato ([12], Proposition 47.11)],

$$\limsup_{n\to\infty} \frac{\|X\|_{t_{n+1}}}{\sqrt{2t_{n+1}\ell_2(t_{n+1})}} \le \limsup_{t\downarrow 0} \frac{|X(t)|}{\sqrt{2t\ell_2(t)}} = \sigma , \quad \text{a.s.},$$

and, thus,

$$\begin{aligned} \limsup_{n\to\infty} \frac{\|X\|_{t_{n+1}}}{\sqrt{t_n/\ell_2(t_n)}} & \\ \le & \lim_{n\to\infty} \sqrt{\frac{2t_{n+1}}{t_n}\,\ell_2(t_{n+1})\ell_2(t_n)} \times \limsup_{n\to\infty} \frac{\|X\|_{t_{n+1}}}{\sqrt{2t_{n+1}\ell_2(t_{n+1})}} \\ \le & \sqrt{2e^{-1}\sigma^2} \lim_{n\to\infty} \frac{\log n}{\sqrt{n}} = 0 , \quad \text{a.s.} \end{aligned} \tag{4.26}$$

On the other hand,

$$\begin{aligned} \|X\|_{t_n} &= \|X\|_{t_{n+1}} \vee \sup_{t_{n+1} < s \le t_n} |X(s)| \\ &\le \|X\|_{t_{n+1}} \vee (U_n + \|X\|_{t_{n+1}}) = U_n + \|X\|_{t_{n+1}} , \qquad \text{a.s.}, \end{aligned}$$

and, thus, by combining (4.25) and (4.26),

$$\liminf_{t\downarrow 0} \frac{\|X\|_t}{\sqrt{t/\ell_2(t)}} \le \liminf_{n\to\infty} \frac{\|X\|_{t_n}}{\sqrt{t_n/\ell_2(t_n)}} \le \frac{\pi\sigma}{\sqrt{8}}, \qquad \text{a.s.}, \tag{4.27}$$

which would give us (4.24).

To complete the proof of the lemma it thus remains only to show (4.25). Observe that, for all $n \ge 1$, we may write

$$U_n \le U_n^{(\sqrt{t_n})} + \widetilde{U}_n^{(\sqrt{t_n})} , \tag{4.28}$$

where, for $n \ge 1$, we set

$$U_n^{(\sqrt{t_n})} = \sup_{t_{n+1} < s \le t_n} \left| X^{(\sqrt{t_n})}(s) - X^{(\sqrt{t_n})}(t_{n+1}) + (s - t_{n+1})\nu(\sqrt{t_n}) \right| ,$$

and

$$\widetilde{U}_n^{(\sqrt{t_n})} = \sup_{t_{n+1}<s\le t_n} |W^{(\sqrt{t_n})}(s) - W^{(\sqrt{t_n})}(t_{n+1})| ,$$

with $\nu(\cdot)$, $X^{(\cdot)}(t)$ and $W^{(\cdot)}(t)$ as in (4.1) and (4.2). Note that

$$U_n^{(\sqrt{t_n})} \overset{\mathcal{D}}{=} \sup_{0<s\le t_n-t_{n+1}} \left|X^{(\sqrt{t_n})}(s) + s\nu(\sqrt{t_n})\right| . \tag{4.29}$$

From (4.16) we get (recall that $t_n = n^{-n}$ and $\lim_{x\downarrow 0} V(x) = \sigma^2 > 0$)

$$\sum_{n\ge 1} P\left(\|B\|_1 \le a\sqrt{\frac{t_n}{(t_n-t_{n+1})V(\sqrt{t_n})\ell_2(t_n)}}\right) = \infty , \tag{4.30}$$

for all $a > \pi\sigma/\sqrt{8}$. Note that (4.9) is satisfied with $t_n = n^{-n}$. Consequently,

$$\sum_{n\ge 1} |\nu(\sqrt{t_n})|\sqrt{t_n - t_{n+1}} \le \sum_{n\ge 1} |\nu(\sqrt{t_n})|\sqrt{t_n} < \infty .$$

So (4.12) holds with $u_n = \sqrt{t_n - t_{n+1}}$ and $v_n = \sqrt{t_n}$. Thus, from the self-similarity of Brownian motion, (4.11) and (4.30) we get that

$$\begin{aligned} &\sum_{n\ge 1} P\left(\|B + K(\sqrt{t_n})\mathrm{id}\|_{t_n-t_{n+1}} \le a\sqrt{\frac{t_n}{V(\sqrt{t_n})\ell_2(t_n)}}\right) \\ = &\sum_{n\ge 1} P\left(\|B + K(\sqrt{t_n})\sqrt{t_n-t_{n+1}}\ \mathrm{id}\|_1 \le a\sqrt{\frac{t_n}{(t_n-t_{n+1})V(\sqrt{t_n})\ell_2(t_n)}}\right) \\ = &\ \infty , \end{aligned} \tag{4.31}$$

for all $a > \pi\sigma/\sqrt{8}$. For $s \ge 0$ and $a > 0$ set

$$\begin{aligned} g_{1,n}^{(a)}(s) &= -K(\sqrt{t_n})s + a\sqrt{\frac{t_n}{V(\sqrt{t_n})\ell_2(t_n)}} , \\ g_{2,n}^{(a)}(s) &= -K(\sqrt{t_n})s - a\sqrt{\frac{t_n}{V(\sqrt{t_n})\ell_2(t_n)}} . \end{aligned}$$

Then Lemma 4.1 is applicable to $g_1 = g_{1,n}^{(a)}$ and $g_2 = g_{2,n}^{(a)}$, with $L = |K(\sqrt{t_n})|$. Recall that $V(x) \ge \sigma^2 > 0$ for all $x > 0$. Further, t_n satisfies both (4.9) and $\lim_{n\to\infty} t_n/(t_n - t_{n+1}) = 1$; consequently,

$$\sum_{n\ge 1} \frac{\left(1 + \sqrt{t_n - t_{n+1}}|K(\sqrt{t_n})|\right)\rho(\sqrt{t_n})}{\sqrt{(t_n - t_{n+1})(V(\sqrt{t_n}))^3}} < \infty .$$

We thus get from (4.6), (4.29) and (4.31) that

$$\sum_{n=1}^{\infty} P\left(U_n^{(\sqrt{t_n})} \le a\sqrt{t_n/\ell_2(t_n)}\right) = \sum_{n=1}^{\infty} P\left(\|X^{(\sqrt{t_n})} + \nu(\sqrt{t_n})\mathrm{id}\|_{t_n - t_{n+1}} \le a\sqrt{t_n/\ell_2(t_n)}\right) = \infty\,, \tag{4.32}$$

for all $a > \pi\sigma/\sqrt{8}$. As $(U_n^{(\sqrt{t_n})})_{n\ge1}$ is a sequence of independent random variables, it follows from the Borel-Cantelli lemma and (4.32) that

$$\liminf_{n\to\infty} \frac{U_n^{(\sqrt{t_n})}}{\sqrt{t_n/\ell_2(t_n)}} \le \frac{\pi\sigma}{\sqrt{8}}\,, \qquad \text{a.s.} \tag{4.33}$$

Since, by (4.9) (with $t_n = n^{-n}$),

$$\sum_{n\ge1}(t_n - t_{n+1})\overline{\Pi}(\sqrt{t_n}) \le \sum_{n\ge1} t_n\overline{\Pi}(\sqrt{t_n}) < \infty\,,$$

we have with the same argument as in (4.21) that

$$P\left(\widetilde{U}_n^{(\sqrt{t_n})} > 0 \text{ i.o. as } n\to\infty\right) = 0\,. \tag{4.34}$$

Combining (4.28), (4.33) and (4.34), this yields (4.25) and completes the proof of the lemma. □

4.3. *Proof of Theorem 3.1.* The proof is quite analogous to that of Theorem 2.1 in Subsection 4.2, so we only indicate the necessary changes. Therefore assume in the sequel that both

$$0 < EX^2(1) = \widetilde{\sigma}^2 < \infty \quad \text{and} \quad EX(1) = 0\,.$$

Note that Lemma 4.2(ii) is applicable to this situation. Further, we have

$$\lim_{x\to\infty} V(x) = \sigma^2 + \int y^2\Pi(\mathrm{d}y) = EX^2(1) = \widetilde{\sigma}^2\,. \tag{4.35}$$

In particular, there is some $x_0 > 0$ such that $V(x_0) > 0$ (and, thus, $V(x) > 0$ for all $x \ge x_0$).

Now replace in Lemmas 4.4 and 4.5 the function ℓ_2 with

$$\ell_2^\infty(x) := \begin{cases} \log\log(x)\,, & x > e^2\,, \\ \log\log(e^2)\,, & 0 \le x \le e^2\,. \end{cases}$$

Lemma 4.6. *Suppose that* $X^2(1) = \widetilde{\sigma}^2 < \infty$ *and* $EX(1) = 0$. *Then*

$$\liminf_{t\to\infty} \frac{\|X\|_t}{\sqrt{t/\log\log t}} \geq \frac{\pi\widetilde{\sigma}}{\sqrt{8}}, \qquad \text{a.s.} \tag{4.36}$$

Proof of Lemma 4.6: In view of Lemma 4.2(ii) and assumption (ii) of Lemma 4.3 we may follow the proof of Lemma 4.4 as far as (4.18), by replacing r^n with λ^n where $\lambda > 1$. Then we compare the supnorm $\|\cdot\|_{\lambda^n}$ of Brownian motion (with drift) with that of $X^{(\lambda^{n/2})}$, using Lemma 4.1. We can do this provided we replace the factor σ with $\sqrt{V(x_0)}$ in (4.18). Of course, this does not alter our arguments. It then follows from (4.35) that (4.20) can be replaced with

$$\liminf_{n\to\infty} \frac{\|X^{(\lambda^{n/2})} + \nu(\lambda^{n/2})\mathrm{id}\|_{\lambda^n}}{\sqrt{\lambda^n/\ell_2^\infty(\lambda^n)}} \geq \frac{\pi\widetilde{\sigma}}{\sqrt{8}}, \qquad \text{a.s.,}$$

for all $\lambda > 1$. Again, the large jumps in $W^{(\lambda^{n/2})}$ are asymptotically negligible, now by Lemma 4.2(ii). A similar argument as in (4.23) completes the proof of (4.36). □

Next we indicate the necessary changes in the proof of Lemma 4.5 to find the upper bound for large times.

Lemma 4.7. *Suppose that* $0 < X^2(1) = \widetilde{\sigma}^2 < \infty$ *and* $EX(1) = 0$. *Then*

$$\liminf_{t\to\infty} \frac{\|X\|_t}{\sqrt{t/\ell_2^\infty(t)}} \leq \frac{\pi\widetilde{\sigma}}{\sqrt{8}}, \qquad \text{a.s.}$$

Proof of Lemma 4.7: In the proof of Lemma 4.5 replace $t_n = n^{-n}$ and σ^2 with $t_n = n^n$ and $\widetilde{\sigma}^2$, respectively. Also replace ℓ_2 with ℓ_2^∞. With the convention $t_0 = 1$, define

$$U_{\infty,n} := \sup_{t_{n-1} < s \leq t_n} |X(s) - X(t_{n-1})|.$$

Then we get from the iterated law of logarithm for a general Lévy process for large times [cf. [12], Proposition 48.9], that

$$\limsup_{n\to\infty} \frac{\|X\|_{t_{n-1}}}{\sqrt{2t_{n-1}\ell_2(t_{n-1})}} \leq \limsup_{t\to 0} \frac{|X(t)|}{\sqrt{2t\ell_2(t)}} = \widetilde{\sigma}, \quad \text{a.s.}$$

This implies (4.27) (but with $t \to \infty$) by the same arguments, provided (4.25) is in place, where we replace U_n and σ with $U_{\infty,n}$ and $\widetilde{\sigma}$, respectively. Then set

$$U_{\infty,n}^{(\sqrt{t_n})} = \sup_{t_{n-1}<s\le t_n} \left|\nu(t_n^{1/2})(s-t_{n-1}) + X^{(\sqrt{t_n})}(s) - X^{(\sqrt{t_n})}(t_{n-1})\right|,$$

and

$$\widetilde{U}_{\infty,n}^{(\sqrt{t_n})} = \sup_{t_{n-1}<s\le t_n} |W^{(\sqrt{t_n})}(s) - W^{(\sqrt{t_n})}(t_{n-1})|,$$

where $\nu(\cdot)$, $X^{(\cdot)}(t)$ and $W^{(\cdot)}(t)$ are as in (4.1) and (4.2).

Replacing $U_n^{(\sqrt{t_n})}$ and $\widetilde{U}_n^{(\sqrt{t_n})}$ with analogous functions $U_{\infty,n}^{(\sqrt{t_n})}$ and $\widetilde{U}_{\infty,n}^{(\sqrt{t_n})}$, we end up with an analogous inequality as in (4.28) for large times. By the same argument as in (4.34), using Lemma 4.2(ii), we can disregard the $\widetilde{U}_{\infty,n}^{(\sqrt{t_n})}$ component, so it remains only to show (4.33), where we replace $U_n^{(\sqrt{t_n})}$ and σ with $U_{\infty,n}^{(\sqrt{t_n})}$ and $\widetilde{\sigma}$, respectively. The rest of the proof runs through by replacing t_{n+1} by t_{n-1} and reference to $V(x) \to \widetilde{\sigma} \in (0,\infty)$, as $x \to \infty$. □

APPENDIX

PROOF OF LEMMA 4.2: (i) First suppose $t_n = r^n$, $0<r<1$. Then the first two parts of (i) are shown in [3]; they are consequences of the fact that $\int_{|x|\le 1} x^2 \Pi(\mathrm{d}x) < \infty$ (always, for any Lévy canonical measure). So is the third part of (i), as we demonstrate. Using (4.1), write

$$\begin{aligned}
\sum_{n\ge1} \sqrt{r^n}|\nu(\sqrt{r^n})| &\le |\gamma| \sum_{n\ge1}\sqrt{r^n} + \sum_{n\ge1}\sqrt{r^n}\int_{(\sqrt{r^n},1]} x|\mathrm{d}\overline{\Pi}(x)| \\
&\le \frac{|\gamma|}{1-\sqrt{r}} + \sum_{n\ge1}\sqrt{r^n}\sum_{j=1}^{n}\int_{(\sqrt{r^j},\sqrt{r^{j-1}}]} x|\mathrm{d}\overline{\Pi}(x)| \\
&= \frac{|\gamma|}{1-\sqrt{r}} + \frac{1}{1-\sqrt{r}}\sum_{j\ge1}\sqrt{r^j}\int_{(\sqrt{r^j},\sqrt{r^{j-1}}]} x|\mathrm{d}\overline{\Pi}(x)| \\
&\le \frac{|\gamma|}{1-\sqrt{r}} + \frac{1}{1-\sqrt{r}}\int_{(0,1]} x^2|\mathrm{d}\overline{\Pi}(x)| < \infty\,.
\end{aligned}$$

Next suppose $t_n = n^{-n}$. Then for some $c \in (0,\infty)$,

$$\sum_{n\ge1} t_n\overline{\Pi}(\sqrt{t_n}) \le \sum_{n\ge1}\frac{t_n}{t_n - t_{n+1}}\int_{t_{n+1}}^{t_n}\overline{\Pi}(\sqrt{x})\mathrm{d}x \le c\int_0^1 \overline{\Pi}(\sqrt{x})\mathrm{d}x\,,$$

and this is finite by (1.1).

For the second part of (i), in the case $t_n = n^{-n}$, we write

$$\begin{aligned}\sum_{n\ge1}\frac{\rho(\sqrt{t_n})}{\sqrt{t_n}} &= \sum_{n\ge1}\frac{1}{\sqrt{t_n}}\int_{(0,\sqrt{t_n}]} x^3|\mathrm{d}\overline{\Pi}(x)| \\ &= \sum_{n\ge1}\frac{1}{\sqrt{t_n}}\sum_{j\ge n}\int_{(\sqrt{t_{j+1}},\sqrt{t_j}]} x^3|\mathrm{d}\overline{\Pi}(x)| \\ &= \sum_{j\ge1}\int_{(\sqrt{t_{j+1}},\sqrt{t_j}]} x^3|\mathrm{d}\overline{\Pi}(x)|\sum_{n=1}^{j} n^{n/2}\,.\end{aligned}$$

For $j \ge 3$ we get

$$\begin{aligned}\sum_{n=1}^{j-1} n^{n/2} &\le \sum_{n=1}^{j-1}\int_n^{n+1} x^{x/2}\mathrm{d}x \le \int_0^j e^{\frac{x}{2}\log x}\mathrm{d}x \le j\int_0^1 e^{\frac{jx}{2}(\log j+\log x)}\mathrm{d}x \\ &= j\int_0^1 e^{\frac{jx}{2}\log j}\mathrm{d}x \le \frac{2j^{j/2}}{\log j} \le 2j^{j/2}\,.\end{aligned}$$

So for $j \ge 3$, we have $\sum_{n=1}^{j} n^{n/2} \le 3j^{j/2} = 3/\sqrt{t_j}$. Consequently, for some $c \in (0,\infty)$,

$$\begin{aligned}\sum_{n\ge1}\frac{\rho(\sqrt{t_n})}{\sqrt{t_n}} &\le c + 3\sum_{j\ge3}\frac{1}{\sqrt{t_j}}\int_{(\sqrt{t_{j+1}},\sqrt{t_j}]} x^3|\mathrm{d}\Pi(x)| \\ &\le c + 3\int_{(0,1]} x^2|\mathrm{d}\Pi(x)| < \infty\,.\end{aligned}$$

Next, still with $t_n = n^{-n}$ and some $c' \in (0,\infty)$,

$$\begin{aligned}\sum_{n\ge1}\sqrt{t_n}|\nu(\sqrt{t_n})| &\le c' + \sum_{n\ge2}\sqrt{t_n}\sum_{j=1}^{n-1}\int_{(\sqrt{t_{j+1}},\sqrt{t_j}]} x|\mathrm{d}\overline{\Pi}(x)| \\ &= c' + \sum_{j\ge2}\int_{(\sqrt{t_{j+1}},\sqrt{t_j}]} x|\mathrm{d}\overline{\Pi}(x)|\sum_{n\ge j+1} n^{-n/2}\,.\end{aligned}$$

Now for $j \ge 3$

$$\begin{aligned}\sum_{n\ge j+1} n^{-n/2} &\le \sum_{n\ge j+1}\int_{n-1}^{n} x^{-x/2}\mathrm{d}x = \int_j^\infty e^{-\frac{x}{2}\log x}\mathrm{d}x \\ &\le j\int_1^\infty e^{-\frac{jx}{2}\log j}\mathrm{d}x \le \frac{2j^{-j/2}}{\log j} = 2\sqrt{t_j}\,.\end{aligned}$$

Hence also

$$\sum_{n\ge j+1} n^{-n/2} \le \sqrt{t_{j+1}} + 2\sqrt{t_{j+1}} = 3\sqrt{t_{j+1}}\,, \qquad j \ge 2\,,$$

so we get

$$\sum_{n\ge1}\sqrt{t_n}|\nu(\sqrt{t_n})| \;\le\; c' + 3\sum_{j\ge2}\sqrt{t_{j+1}}\int_{(\sqrt{t_{j+1}},\sqrt{t_j}]} x|\mathrm{d}\overline{\Pi}(x)|$$
$$\le\; c' + 3\sum_{j\ge2}\int_{(\sqrt{t_{j+1}},\sqrt{t_j}]} x^2|\mathrm{d}\overline{\Pi}(x)| \;\le\; c' + 3\int_{(0,1]} x^2|\mathrm{d}\Pi(x)| \;<\; \infty\,.$$

(ii) Fix $\lambda > 1$. Recall that we have $EX_1^2 < \infty$ iff $\int x^2\Pi(\mathrm{d}x) < \infty$. For either choice $t_n = n^n$ or $t_n = \lambda^n$ (and $t_0 = 0$) we can argue

$$\sum_{n\ge1} t_n\overline{\Pi}(\sqrt{t_n}) \;\le\; \sum_{n\ge1}\frac{t_n}{t_n - t_{n-1}}\int_{t_{n-1}}^{t_n}\overline{\Pi}(\sqrt{x})\mathrm{d}x \;\le\; C\int_0^\infty\overline{\Pi}(\sqrt{x})\mathrm{d}x \;<\; \infty\,.$$

Next, using the bounds derived in Part (i), we can find a constant $C', D \in (0,\infty)$ such that

$$\sum_{n\ge1} t_n^{-1/2}\rho(\sqrt{t_n}) = \sum_{n\ge1}\sum_{j=1}^{n} t_n^{-1/2}\int_{(\sqrt{t_{j-1}},\sqrt{t_j}]} x^3|\mathrm{d}\Pi(x)|$$
$$= \;\; C' + D\sum_{j\ge1} t_j^{-1/2}\int_{(\sqrt{t_{j-1}},\sqrt{t_j}]} x^3|\mathrm{d}\Pi(x)| \le C' + D\int_{\mathbb{R}} x^2\Pi(dx) < \infty\,,$$

for either choice $t_n = n^n$ and $t_n = \lambda^n$.

Finally recall that we assumed $EX_1 = 0$ and, thus, [cf. Sato [12], Example 25.12]

$$\nu(b) = \nu(b) - EX_1 = -\int_{b<|x|<\infty} x\Pi(dx)\,, \qquad b > 1\,.$$

Use the bound $\sum_{n=1}^{j} n^{n/2} \le 3j^{j/2}$, for $j \ge 3$, derived as in Part (i), to find $C'', D' \in (0,\infty)$ such that

$$\sum_{n\ge1}\sqrt{t_n}|\nu(\sqrt{t_n})| \;\le\; \sum_{n\ge1}\sum_{j\ge n} t_n^{1/2}\int_{(\sqrt{t_j},\sqrt{t_{j+1}}]} x|\mathrm{d}\Pi(x)|$$
$$\le\;\; C'' + D'\sum_{j\ge1} t_j^{1/2}\int_{(\sqrt{t_j},\sqrt{t_{j+1}}]} x|\mathrm{d}\Pi(x)| \;\le\; C'' + D'\int_{\mathbb{R}} x^2\Pi(dx) < \infty\,,$$

for either choice $t_n = n^n$ or $t_n = \lambda^n$. □

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