

# STATISTICAL ANALYSIS OF SINGLE-SERVER LOSS QUEUEING SYSTEMS

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ABSTRACT. In this article statistical bounds for certain output characteristics of the  $M/GI/1/n$  and  $GI/M/1/n$  loss queueing systems are derived on the basis of large samples of an input characteristic of these systems, such as service time in the  $M/GI/1/n$  queueing system or interarrival time in the  $GI/M/1/n$  queueing system. The analysis of this article is based on application of Kolmogorov's statistics for empirical probability distribution functions.

## 1. INTRODUCTION

In theoretical problems of queueing theory, the input characteristics such as interarrival and service time distributions are assumed to be known. For example, if we speak about an  $M/GI/1$  queueing system, we assume that the arrival process is Poisson with rate  $\lambda$ , and service times are independent and identically distributed random variables with a given probability distribution function  $B(x)$ . In practice we, however, have only input data characterizing arrival and departure processes, and our conclusion about output characteristics depend on accuracy of approximation of aforementioned input characteristics.

In certain queueing systems some output characteristics can be insensitive to the type of probability distribution function of a service time. For example, in  $M/GI/m/0$  queueing systems the stationary state probabilities are defined by Erlang-Sevastyanov formulae, which are independent of the type of probability distribution function of a service time. In the  $M/G/1$  queueing system, in which the arrival is Poisson with rate  $\lambda$ , and the expected service time is  $b < \frac{1}{\lambda}$ , the expected length of a busy period is  $\frac{b}{1-\lambda b}$ , i.e. insensitive to the type of probability distribution  $B(x)$ . Therefore, the problem of estimation of the expected busy period and certain other output characteristics, such as, for example, the expected number of served customer during a busy period, reduces to those estimations of the parameters  $\lambda$  and  $b$  of the system.

The queueing systems, for which the output characteristics are insensitive to the types of probability distributions of input characteristics, are rather exceptions. In most cases the output characteristics depends on probability distribution functions, and this dependence can be very complicated. So, the problem of estimating the output characteristics of queueing systems is generally difficult problem.

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In the present paper we demonstrate the methods of statistical analysis of certain output characteristics of single-server loss queueing systems such as  $M/GI/1/n$  and  $GI/M/1/n$ , which are *not* insensitive, but expressed specifically via transforms of probability distribution function of interarrival or service times. We estimate the output characteristics of these systems, such as the expected busy period, expected numbers of lost and served customers during a busy period for the  $M/GI/1/n$  and the stationary loss probability for the  $GI/M/1/n$  queueing systems on the basis of the known information about input characteristics of these systems. The concrete problem formulations are given later.

Statistics of queueing systems is a distinguished area of queueing theory. The first publications in this area appeared long time ago (see the textbook of Ivchenko, Kashtanov and Kovalenko [13], review paper of Bhat and Rao [10] and the references in these sources). In the textbook [13] various traditional methods of statistical inference to queueing problems have been demonstrated. In [10], a review of different aspects of queueing systems, including identification of models, parameters estimation by the maximum likelihood method and the method of moments as well as estimates of mean value processes and auto-covariance functions, hypothesis testing and other topics of statistical analysis up to the publication date is done. For some recent publications in the area of statistical analysis of queueing systems see also [8], [9], [12], [17], [19]. Nevertheless, despite their importance, the papers on statistical inference of queueing systems appears much rarely compared to many of those that use the methods of stochastic analysis, optimization, control and asymptotic methods of Mathematical analysis. Moreover, the methods of statistical analysis of queueing systems known from the literature are traditional.

The statistical analysis of the present paper is based on application of Kolmogorov's statistics characterizing empirical probability distribution function of an interarrival or service time on the basis of large observations of these input characteristics.

To our knowledge, Kolmogorov's statistics are never used in statistical analysis of output characteristics of queueing systems. In recent paper [4], Kolmogorov's metric (which is associated with one of Kolmogorov's statistics) is used for establishing the bounds for the loss probability in certain queueing systems with large buffers. In other papers [5] and [6] Kolmogorov's metric is used for establishing conditions for the continuity in the  $M/M/1/n$  queueing system and, respectively, for the continuity of non-stationary state probabilities in large closed queueing networks with bottlenecks.

In the present paper we solve the following problems. Consider, for instance, the  $M/GI/1/n$  queueing system in which the arrival rate  $\lambda$  and the expected service time  $b$  are assumed to be known, however, the probability distribution function  $B(x)$  of a service time is unknown. On the basis of  $N$  observations of service times we build an empirical probability distribution function  $B_{\text{emp}}(x, N)$ . Let us denote

$$\delta_N = \sup_{x>0} |B(x) - B_{\text{emp}}(x, N)|.$$

Using this statistic, we estimate the expected busy period, and other significant characteristics such as the expected numbers of served and lost customers during that busy period. We find confidence intervals (ranges) for these characteristics based on a confidence probability  $P$ . The similar ranges will be obtained with the help of other Kolmogorov's statistics mentioned later.

Recall that according to Kolmogorov's theorem (see [16] or [21], p. 170) we have:

$$(1.1) \quad \lim_{N \rightarrow \infty} \Pr \left\{ \delta_N < \frac{z}{\sqrt{N}} \right\} = K(z) = \begin{cases} \sum_{j=-\infty}^{+\infty} (-1)^j e^{-2j^2 z^2}, & \text{for } z > 0, \\ 0, & \text{for } z \leq 0. \end{cases}$$

So, for the range

$$\delta_N < \epsilon,$$

based on a chosen confidence probability  $P$ , the value  $\epsilon = \frac{z}{\sqrt{N}}$  depending on the large parameter  $N$  can be chosen such that the equation  $K(z) = P$  is satisfied.

Along with  $\delta_N$ , the other relevant Kolmogorov's statistics are as follows:

$$(1.2) \quad \delta_N^- = \sup_{x>0} [B(x) - B_{\text{emp}}(x, N)],$$

and

$$(1.3) \quad \delta_N^+ = \sup_{x>0} [B_{\text{emp}}(x, N) - B(x)].$$

It is known (see e.g. [21], p. 170) that both (1.2) and (1.3) have the same limiting distribution:

$$(1.4) \quad \lim_{N \rightarrow \infty} \Pr \left\{ \delta_N^- \leq \frac{z}{\sqrt{N}} \right\} = \lim_{N \rightarrow \infty} \Pr \left\{ \delta_N^+ \leq \frac{z}{\sqrt{N}} \right\} = \begin{cases} 1 - e^{-2z^2}, & \text{for } z \geq 0, \\ 0, & \text{for } z < 0. \end{cases}$$

For fixed  $N$  the statistics  $\delta_N^-$  and  $\delta_N^+$  are dependent. However, as  $N$  increases to infinity, they become asymptotically independent. Explanation of this fact is discussed later in Section 4.2 (see Lemma 4.3 and its proof). This fact is essentially used in our further analysis.

For the  $GI/M/1/n$  queueing system, similar Kolmogorov's statistics are used for statistical analysis of stationary loss probabilities, where the probability distribution function of interarrival time is denoted by  $A(x)$ , empirical probability distribution of interarrival time based on  $N$  observations of interarrival times is denoted by  $A_{\text{emp}}(x, N)$ , and statistics  $\delta_N$ ,  $\delta_N^-$  and  $\delta_N^+$  are, correspondingly, as follows:

$$\delta_N = \sup_{x>0} |A(x) - A_{\text{emp}}(x, N)|,$$

$$\delta_N^- = \sup_{x>0} [A(x) - A_{\text{emp}}(x, N)],$$

and

$$\delta_N^+ = \sup_{x>0} [A_{\text{emp}}(x, N) - A(x)].$$

The aim of the present paper is twofold. Along with obtaining the ranges for aforementioned characteristics of  $M/GI/1/n$  and  $GI/M/1/n$  queueing systems for a given probability  $P$  (which is the first aim of the paper) we also answer to the question: *Which of these Kolmogorov's statistics is better?* That is, under which of these Kolmogorov's statistics the difference between an upper and lower bounds based on the chosen probability  $P$  is smaller?

The rest of the paper is organized as follows. In Section 2, we recall the known representations for the characteristics studied in the paper for  $M/GI/1/n$  and  $GI/M/1/n$  queueing systems. In Section 3, we prove the main lemmas, which are then used to find the estimators for required characteristics of the queueing systems.

The estimators themselves are derived in Section 4. In Section 5 we compare our statistical results and address the question formulated above. In Section 6 numerical results are presented. In Section 7 we conclude the paper.

## 2. THE RECURRENCE RELATION FOR MAIN CHARACTERISTICS OF THE $M/GI/1/n$ AND $GI/M/1/n$ QUEUEING SYSTEMS

In this section, we recall the known results for the main characteristics of the  $M/GI/1/n$  and  $GI/M/1/n$  queueing systems during their busy periods. For a more detailed information see [1] or [7].

**2.1. The  $M/GI/1/n$  queueing system.** Consider the  $M/GI/1/n$  queueing system, in which arrival flow is Poisson with parameter  $\lambda$ , and the probability distribution function of the service time is  $B(x)$  having the expectation  $b$ . Parameter  $n$  denotes the number of waiting places, i.e. the capacity for the customer in service is not taken into account. Let  $T_n$  denote the length of a busy period of this system, and let  $\nu_n$  and  $L_n$  denote the number of served and, respectively, lost customers during that busy period. The recurrence relation for  $ET_n$  has been obtained by Tomko [22]:

$$(2.1) \quad ET_n = \sum_{i=0}^n ET_{n-i+1} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x),$$

where  $ET_0 = b$ , and  $T_i$  denotes the length of a busy period in the  $M/GI/1/i$  queueing systems having the same arrival rate  $\lambda$  and the same probability distribution of the service time as the original queueing system  $M/GI/1/n$ . The expected number of served customers during the same busy period satisfies the recurrence relation similar to (2.1):

$$(2.2) \quad E\nu_n = \sum_{i=0}^n E\nu_{n-i+1} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x),$$

where  $\nu_i$  denotes the number of served customers during a busy period  $T_i$ . The main difference between recurrence relations (2.1) and (2.2) is that (2.2) starts from  $E\nu_0 = 1$ , while (2.1) starts from  $ET_0 = b$ .

For the expected number of losses during a busy period we correspondingly have the following recurrence relation (see e.g. [1], [2] or [7] for more details), which is similar to the previous two recurrence relations given by (2.1) and (2.2):

$$(2.3) \quad EL_n - 1 = \sum_{i=0}^n (EL_{n-i+1} - 1) \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x),$$

where  $EL_0 = \lambda b$ , and  $L_i$  denotes the number of losses during a busy period  $T_i$ .

All (2.1), (2.2) and (2.3) are the following convolution type recurrence relation:

$$(2.4) \quad Q_n = \sum_{i=0}^n Q_{n-i+1} r_i,$$

where  $r_i = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)$ .

**2.2. The  $GI/M/1/n$  queueing system.** Consider the  $GI/M/1/n - 1$  queueing system, where an interarrival time has the probability distribution function  $A(x)$ , and the parameter of an exponentially distributed service time is  $\mu$ . (The number of waiting places  $n - 1$  excludes the place for a customer in service.) Let  $\pi_n$  denote the stationary loss probability. It is shown in [3] that  $\pi_n$  satisfied the recurrence relation:

$$(2.5) \quad \frac{1}{\pi_n} = \sum_{i=0}^n \frac{1}{\pi_{n-i+1}} \int_0^\infty e^{-\mu x} \frac{(\mu x)^i}{i!} dA(x),$$

where  $\pi_0 = 1$ . Recurrence relation (2.5) is similar to that of (2.1), (2.2) and (2.3), and has general form (2.4). The only difference is that in representation (2.5)  $r_i = \int_0^\infty e^{-\mu x} \frac{(\mu x)^i}{i!} dA(x)$ , while in representations (2.1), (2.2) and (2.3)  $r_i = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)$ .

Another representation for  $\pi_n$  has been obtained by Miyazawa [18].

### 3. MAIN LEMMAS

In this section we prove main lemmas, which help us to obtain then the desired estimators for characteristics of queueing systems studied in this paper.

Let  $F_1(x)$  and  $F_2(x)$  denote arbitrary probability distribution functions of positive random variables. For a positive parameter  $\alpha$ , let us denote

$$r_i(F_1) = \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^i}{i!} dF_1(x),$$

and

$$r_i(F_2) = \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^i}{i!} dF_2(x).$$

**Lemma 3.1.** *Assume that  $\sup_{x>0} |F_1(x) - F_2(x)| < \epsilon$ . Then,*

$$(3.1) \quad |r_0(F_1) - r_0(F_2)| < \epsilon,$$

and for all  $i = 1, 2, \dots$ ,

$$(3.2) \quad |r_i(F_1) - r_i(F_2)| < 2\epsilon.$$

*Proof.* By partial integration we have

$$(3.3) \quad \int_0^\infty e^{-\alpha x} dF_k(x) = \alpha \int_0^\infty e^{-\alpha x} F_k(x) dx, \quad k = 1, 2.$$

Therefore,

$$\begin{aligned} \left| \int_0^\infty e^{-\alpha x} dF_1(x) - \int_0^\infty e^{-\alpha x} dF_2(x) \right| &= \alpha \left| \int_0^\infty e^{-\alpha x} F_1(x) dx - \int_0^\infty e^{-\alpha x} F_2(x) dx \right| \\ &\leq \underbrace{\alpha \int_0^\infty e^{-\alpha x} dx}_{=1} \underbrace{\left( \sup_{x>0} |F_1(x) - F_2(x)| \right)}_{< \epsilon \text{ by the assumption}} \\ &< \epsilon. \end{aligned}$$

Inequality (3.1) is proved.

Let us now prove inequalities (3.2). For  $i = 1, 2, \dots$  and  $k = 1, 2$  by partial integration we have:

$$(3.4) \quad \begin{aligned} & \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^i}{i!} dF_k(x) \\ &= \alpha \left( \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^i}{i!} F_k(x) dx - \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^{i-1}}{(i-1)!} F_k(x) dx \right) \end{aligned}$$

Therefore,

$$(3.5) \quad \begin{aligned} & \left| \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^i}{i!} dF_1(x) - \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^i}{i!} dF_2(x) \right| \\ &= \alpha \left| \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^i}{i!} F_1(x) dx - \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^i}{i!} F_2(x) dx \right. \\ & \quad \left. - \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^{i-1}}{(i-1)!} F_1(x) dx + \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^{i-1}}{(i-1)!} F_2(x) dx \right| \\ &\leq \frac{\alpha}{i!} \int_0^\infty e^{-\alpha x} (\alpha x)^i dx \underbrace{\left( \sup_{x>0} |F_1(x) - F_2(x)| \right)}_{< \epsilon \text{ by the assumption}} \\ & \quad + \frac{\alpha}{(i-1)!} \int_0^\infty e^{-\alpha x} (\alpha x)^{i-1} dx \underbrace{\left( \sup_{x>0} |F_1(x) - F_2(x)| \right)}_{< \epsilon \text{ by the assumption}}. \end{aligned}$$

Notice, that

$$\int_0^\infty e^{-\alpha x} (\alpha x)^i dx = \frac{1}{\alpha} \int_0^\infty e^{-y} y^i dy = \frac{1}{\alpha} \Gamma(i+1),$$

and

$$\int_0^\infty e^{-\alpha x} (\alpha x)^{i-1} dx = \frac{1}{\alpha} \int_0^\infty e^{-y} y^{i-1} dy = \frac{1}{\alpha} \Gamma(i).$$

where  $\Gamma(x)$  is Euler's Gamma function. Taking into account that  $\Gamma(i+1) = i!$ , from the last inequality of (3.5) we arrived at the desired result  $|r_i(F_1) - r_i(F_2)| < 2\epsilon$  for  $i = 1, 2, \dots$ . Inequalities (3.2) are proved, and the proof of the lemma is completed.  $\square$

**Lemma 3.2.** *Assume that  $\sup_{x>0} [F_1(x) - F_2(x)] < \epsilon_1$  and  $\sup_{x>0} [F_2(x) - F_1(x)] < \epsilon_2$ . Then,*

$$(3.6) \quad r_0(F_1) - r_0(F_2) < \epsilon_1,$$

and for all  $i = 1, 2, \dots$ ,

$$(3.7) \quad r_i(F_1) - r_i(F_2) < \epsilon_1 + \epsilon_2.$$

*Proof.* The proof of this lemma is similar to that of Lemma 3.1. From (3.3) we have:

$$\begin{aligned} \int_0^\infty e^{-\alpha x} dF_1(x) - \int_0^\infty e^{-\alpha x} dF_2(x) &= \alpha \int_0^\infty e^{-\alpha x} F_1(x) dx - \int_0^\infty e^{-\alpha x} F_2(x) dx \\ &\leq \underbrace{\alpha \int_0^\infty e^{-\alpha x} dx}_{=1} \underbrace{\left( \sup_{x>0} [F_1(x) - F_2(x)] \right)}_{< \epsilon_1 \text{ (assumption)}} \\ &< \epsilon_1. \end{aligned}$$

Inequality (3.6) is proved. Next, from (3.4) we have:

$$\begin{aligned} &\int_0^\infty e^{-\alpha x} \frac{(\alpha x)^i}{i!} dF_1(x) - \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^i}{i!} dF_2(x) \\ &= \alpha \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^i}{i!} F_1(x) dx - \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^i}{i!} F_2(x) dx \\ &\quad - \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^{i-1}}{(i-1)!} F_1(x) dx + \int_0^\infty e^{-\alpha x} \frac{(\alpha x)^{i-1}}{(i-1)!} F_2(x) dx \\ &\leq \underbrace{\frac{\alpha}{i!} \int_0^\infty e^{-\alpha x} (\alpha x)^i dx}_{=\frac{i!}{\alpha}} \underbrace{\left( \sup_{x>0} [F_1(x) - F_2(x)] \right)}_{< \epsilon_1 \text{ (assumption)}} \\ &\quad + \frac{\alpha}{(i-1)!} \underbrace{\int_0^\infty e^{-\alpha x} (\alpha x)^{i-1} dx}_{=\frac{(i-1)!}{\alpha}} \underbrace{\left( \sup_{x>0} [F_2(x) - F_1(x)] \right)}_{< \epsilon_2 \text{ (assumption)}}. \\ &< \epsilon_1 + \epsilon_2. \end{aligned}$$

Desired inequality (3.7) follows. The proof is complete.  $\square$

#### 4. EXPLICIT RECURSION AND ESTIMATES FOR CHARACTERISTICS OF QUEUEING SYSTEMS

The recurrence relation in form (2.4) is not convenient for estimating the aforementioned characteristics of the queueing systems. In the left-hand side of (2.4)  $Q_n$  is, while in the right-hand side of (2.4) the linear combination of  $Q_1, Q_2, \dots, Q_{n+1}$  is presented. In order to derive the appropriate estimates, we first should rewrite (2.4) in the form of explicit recursion. We have:

$$(4.1) \quad Q_1 = \frac{1}{r_0} Q_0,$$

$$(4.2) \quad Q_2 = \frac{1-r_1}{r_0} Q_1,$$

$$(4.3) \quad Q_n = \frac{1}{r_0} [(1-r_1)Q_{n-1} + r_2Q_{n-2} + \dots + r_{n-1}Q_1], \quad n \geq 3.$$

Explicit recurrence relations of (4.1)-(4.3) can be now used to establish necessary estimates for characteristics of the loss queueing systems  $M/GI/1/n$  and  $GI/M/1/n$ . To build the recursion for  $ET_n$ , we start from the  $M/GI/1/0$  queueing system. The busy period of this system contains only a single service time. In the sequel we assume that the parameters  $\lambda$  and  $b$  of the queueing systems are known. As the

expected service time  $b$  is known, we set  $\widehat{T}_0 = b$ , that is, the expected busy period of the  $M/GI/1/0$  queueing system is reckoned to be estimated exactly. In the following we use the notation:

$$r_i(B) = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x),$$

and

$$r_i(B_{\text{emp}}) = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB_{\text{emp}}(x, N),$$

where  $B_{\text{emp}}(x, N)$  is an empirical probability distribution function based on  $N$  observations. In the following the estimator for  $Q_n$  will be denoted  $\widehat{Q}_n$  and specifically that for the expected busy period will be denoted  $\widehat{T}_n$ . (Similarly, for  $GI/M/1/n$  queueing systems, the notation  $r_i(A)$  and  $r_i(A_{\text{emp}})$  can be used where the parameter  $\lambda$  should be replaced with the parameter  $\mu$ .)

**4.1. Estimators based on the statistic  $\delta_N$ .** Assume that the inequality

$$(4.4) \quad \delta_N < \epsilon$$

holds with probability  $P$ . Since  $N$  is assumed to be large enough, this probability can be chosen from limit relation (1.1).

Then, according to Lemma 3.1, with the probability not smaller than  $P$  we have

$$|r_0(B) - r_0(B_{\text{emp}})| < \epsilon,$$

and for  $i = 1, 2, \dots, n$

$$|r_i(B) - r_i(B_{\text{emp}})| < 2\epsilon.$$

For further simplifications, we will write  $r_i = r_i(B_{\text{emp}})$ ,  $i = 0, 1, 2, \dots$ , omitting the argument  $B_{\text{emp}}$  in the notation.

On the basis (4.1)-(4.3) and Lemma 3.1 we have the following.

**Theorem 4.1.** *The estimator  $\widehat{Q}_n$  for a required characteristic of a queueing system is recurrently defined as*

$$(4.5) \quad \widehat{Q}_1 = \frac{1}{r_0} Q_0,$$

$$(4.6) \quad \widehat{Q}_2 = \frac{1 - r_1}{r_0} \widehat{Q}_1,$$

$$(4.7) \quad \widehat{Q}_n = \frac{1}{r_0} \left[ (1 - r_1) \widehat{Q}_{n-1} + r_2 \widehat{Q}_{n-2} + \dots + r_{n-1} \widehat{Q}_1 \right], \quad n \geq 3.$$

*Then, the bounds of this estimator with a confidence probability non-smaller than  $P$  are recurrently defined as*

$$(4.8) \quad \widehat{Q}_1^{\text{lower}} = \frac{1}{r_0 + \epsilon} Q_0 < \widehat{Q}_1 < \frac{1}{r_0 - \epsilon} Q_0 = \widehat{Q}_1^{\text{upper}},$$

$$(4.9) \quad \widehat{Q}_2^{\text{lower}} = \frac{1 - r_1 - 2\epsilon}{[r_0 + \epsilon]^2} Q_0 < \widehat{Q}_2 < \frac{1 - r_1 + 2\epsilon}{[r_0 - \epsilon]^2} Q_0 = \widehat{Q}_2^{\text{upper}},$$

$$\begin{aligned}
 \widehat{Q}_n^{\text{lower}} &= \frac{1}{r_0 + \epsilon} \left[ (1 - r_1 - 2\epsilon)\widehat{Q}_{n-1}^{\text{lower}} \right. \\
 &\quad \left. + (r_2 - 2\epsilon)\widehat{Q}_{n-2}^{\text{lower}} + \dots + (r_{n-1} - 2\epsilon)\widehat{Q}_1^{\text{lower}} \right] \\
 (4.10) \quad &< \widehat{Q}_n < \frac{1}{r_0 - \epsilon} \left[ (1 - r_1 + 2\epsilon)\widehat{Q}_{n-1}^{\text{upper}} \right. \\
 &\quad \left. + (r_2 + 2\epsilon)\widehat{Q}_{n-2}^{\text{upper}} + \dots + (r_{n-1} + 2\epsilon)\widehat{Q}_1^{\text{upper}} \right] \\
 &= \widehat{Q}_n^{\text{upper}}, \quad n \geq 3.
 \end{aligned}$$

The value  $\epsilon$  is determined from the equation:  $K(\epsilon\sqrt{N}) = P$ ,  $K(z)$  is Kolmogorov's function defined in (1.1), and in the notation of (4.8)-(4.10)  $\widehat{Q}_i^{\text{lower}}$  and  $\widehat{Q}_i^{\text{upper}}$  are used for lower and, respectively, upper bound of the estimators.

Using relations (4.5)-(4.10) one can built estimators and the bounds for these estimators for all of the aforementioned characteristics of the queueing systems  $M/GI/1/n$  and  $GI/M/1/n$ .

*Remark 4.2.* For the same relations (4.8)-(4.10) the following conventions are made. If  $r_0 \leq \epsilon$ , then the right-hand sides of (4.8)-(4.10) are set to be equal to infinity. If  $1 - r_1 - 2\epsilon$  is negative, then the left-hand side of (4.9) is set to be equal to zero. In addition, if one or other term  $1 - r_i - 2\epsilon$ ,  $i = 1, 2, \dots, n$  of the left-hand side of (4.10) is negative, that term is set to zero. The value  $\epsilon$  should be chosen small enough, i.e. appropriate sample size  $N$  should be chosen large enough, such that all of the above inconsistencies can occur with negligibly small probability.

**4.2. Estimators based on the statistic  $\delta_N^-$  or  $\delta_N^+$ .** Since both of these statistics are similar, we only consider the first one,  $\delta_N^-$ . Assume that  $N$  is large enough and  $\delta_N^- < \epsilon$  occurs with probability  $P$ .

Then, according to Lemma 3.2 with the same probability  $P$

$$r_0(B) - r_0(B_{\text{emp}}) < \epsilon,$$

and with the same probability  $P$

$$r_0(B_{\text{emp}}) - r_0(B) < \epsilon.$$

Then, by the same Lemma 3.2 from (3.7) for  $i = 1, 2, \dots, n$  one can write:

$$r_i(B) - r_i(B_{\text{emp}}) < \gamma,$$

and

$$r_i(B_{\text{emp}}) - r_i(B) < \gamma,$$

where the value  $\gamma$  is the solution of the equation

$$\Pr\{\delta_N^+ + \delta_N^- < \gamma\} = P.$$

**Lemma 4.3.** *As  $N$  increases to infinity, the statistics  $\sqrt{N}\delta_N^+$  and  $\sqrt{N}\delta_N^-$  become asymptotically independent.*

*Proof.* This fact can be proved with the aid of construction used in [21], p.171-172 as follows. Assume that  $B(x)$  is a continuous probability distribution function, and  $B_{\text{emp}}(x, N)$  is the empirical probability distribution function based on the sample  $(\xi_1, \xi_2, \dots, \xi_N)$ , that is,  $B_{\text{emp}}(x, N)$  is defined as the fraction of random variables that less than or equal to  $x$ . Let us write the sample  $(\xi_1, \xi_2, \dots, \xi_N)$

in ascending order as  $(\xi_1^*, \xi_2^*, \dots, \xi_N^*)$ . Consider the deviations  $\delta_N^-(n) = B(\xi_n^*) - B_{\text{emp}}(\xi_n^*, N)$  and  $\delta_N^+(n) = B_{\text{emp}}(\xi_n^*, N) - B(\xi_n^*)$ ,  $n = 1, 2, \dots, N$ . Apparently, the random variables  $\delta_N^-(n)$  and  $\delta_N^+(n)$  all are continuous ( $n = 1, 2, \dots, N$ ), and all their joint distributions are independent of the choice of the probability distribution  $B(x)$ . That is, without loss of generality one can assume that  $B(x)$  is the uniform distribution in  $[0, 1]$ . Then,  $\delta_N^+(n) = \xi_n^* - \frac{n}{N}$  and  $\delta_N^-(n) = \frac{n}{N} - \xi_n^*$ . Let  $\kappa_n$  be the number of random variables amongst  $\xi_1, \xi_2, \dots, \xi_N$  falling into the interval  $(\frac{n-1}{N}, \frac{n}{N}]$  and  $\mathcal{N}_n = \kappa_1 + \kappa_2 + \dots + \kappa_n$ ,  $n = 1, 2, \dots, N$ . Then the number of points where  $\delta_N^-(n)$  are positive is  $\sum_{n=1}^N \mathbf{I}\{\mathcal{N}_n \geq n\}$ , and, respectively, the number of points where  $\delta_N^+(n)$  are positive is  $\sum_{n=1}^N \mathbf{I}\{\mathcal{N}_n < n\}$ . For  $N$  fixed, the random variables  $\delta_N^- = \max_{1 \leq n \leq N} \delta_N^-(n)$  and  $\delta_N^+ = \max_{1 \leq n \leq N} \delta_N^+(n)$  are generally dependent. As  $N \rightarrow \infty$ , the random variables  $\delta_N^+$  and  $\delta_N^-$  both vanish. The aim is to prove that the normalized random variables  $\sqrt{N}\delta_N^+$  and  $\sqrt{N}\delta_N^-$  become asymptotically independent as  $N \rightarrow \infty$ .

For this purpose we will use the known fact that the random variable  $\xi_n^*$  has the Beta distribution with parameters  $n$  and  $N - n + 1$  (see e.g. David and Nagaraja [11], p.14). In other words, this means that the random variable  $\xi_n^*$  has the same distribution as

$$\frac{\eta_1 + \eta_2 + \dots + \eta_n}{\eta_1 + \eta_2 + \dots + \eta_{N+1}},$$

where  $\eta_1, \eta_2, \dots, \eta_N, \dots$  is the sequence of independent exponentially distributed random variables with parameter 1 (see Karlin [14], p.242-244). According to the strong law of large numbers,

$$\Pr \left\{ \lim_{N \rightarrow \infty} \frac{N}{\eta_1 + \eta_2 + \dots + \eta_{N+1}} = 1 \right\} = 1.$$

Therefore, as  $N \rightarrow \infty$ , the random variable  $N\delta_N^+(n) = N\xi_n^* - n$  can be asymptotically represented as

$$(\eta_1 - 1) + (\eta_2 - 1) + \dots + (\eta_n - 1).$$

From this asymptotic representation, apparently that  $\lim_{N \rightarrow \infty} \mathbf{E}(N\xi_n^* - n) = 0$  for all  $n = 1, 2, \dots, N$ , and the expected number of positive and negative values of  $N\xi_n^* - n$  for  $n = 1, 2, \dots, N$  is approximately the same. That is the numbers of points where  $\delta_N^+(n)$  and  $\delta_N^-(n)$  are positive are asymptotically equal.

For large  $N$ , let  $j$  denote an index such that  $N\delta_N^+(j)$  is negative while  $N\delta_N^+(j+1)$  is positive. Then there is a random number  $v$  of consequent positive values of  $N\delta_N^+(j+i)$ ,  $i = 1, 2, \dots, v$ . This random number is called *length of positive period* or simply *positive period*. Notice that,

$$(4.11) \quad \Pr \left\{ \sum_{i=1}^{j+1} (\eta_i - 1) \leq x \mid \sum_{i=1}^j (\eta_i - 1) < 0, \sum_{i=1}^{j+1} (\eta_i - 1) > 0 \right\} = 1 - e^{-x}.$$

Indeed, let  $\tau_j = j - \sum_{i=1}^j \eta_i + 1$ . Then, the conditional probability of the left-hand side of (4.11) can be written

$$\begin{aligned} \Pr\{\eta_{j+1} \leq x + \tau_j | \eta_{j+1} > \tau_j > 1\} &= \int_0^\infty \Pr\{\eta_{j+1} \leq x + y | \eta_{j+1} > y\} d\Pr\{\tau_j \leq y\} \\ &= (1 - e^{-x}) \int_0^\infty d\Pr\{\tau_j \leq y\} \\ &= 1 - e^{-x}. \end{aligned}$$

(4.11) follows. Thus, the limiting distribution of  $N\delta_N^+(j+1)$  as  $N \rightarrow \infty$  is exponential, and  $N\delta_N^+(j+1)$  is asymptotically independent of the past values of  $N\delta_N^+(k)$ ,  $k = 1, 2, \dots, j$ . The following random variables  $N\delta_N^+(j+i)$ ,  $i = 2, 3, \dots, v$  are also asymptotically independent of the aforementioned past values  $N\delta_N^+(k)$ ,  $k = 1, 2, \dots, j$ . So, the positive period and associated random variables  $N\delta_N^+(j+i)$ ,  $i = 2, 3, \dots, v$  all are asymptotically independent of the past.

Consequently, let  $j_1, j_2, \dots$  be the sequence of values such that  $N\delta_N^+(j_k)$ ,  $k = 1, 2, \dots$  are negative while  $N\delta_N^+(j_k + 1)$  are positive. Then, according to the above statement, the positive periods are asymptotically mutually independent and identically distributed random variables.

The same result can be obtained for *negative* periods which are constructed similarly to those positive periods. Thus, as  $N \rightarrow \infty$ , positive and negative periods, alternatively changing one another, are asymptotically mutually independent and identically distributed random variables.

As  $N \rightarrow \infty$ , the sets of values where  $N\delta_N^+(n)$  and  $N\delta_N^-(n)$  are positive, are asymptotically independent, because the positive and negative periods are asymptotically independent. Note, that the set of indexes where the periods are positive is  $\{n \leq N : \mathcal{N}_n < n\}$  and that set where periods are negative is  $\{n \leq N : \mathcal{N}_n \geq n\}$ . Furthermore,

$$\Pr\left\{\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbb{I}\{\mathcal{N}_n < n\} = \frac{1}{2}\right\} = 1$$

On the other hand, as  $N \rightarrow \infty$ , the limiting distributions of  $\sqrt{N}\delta_N^-$  and  $\sqrt{N}\delta_N^+$  do exist, and they are defined by (1.4).

Therefore,

$$\begin{aligned} &\lim_{N \rightarrow \infty} \Pr\left\{\sqrt{N}\delta_N^+ \leq x, \sqrt{N}\delta_N^- \leq y\right\} \\ &= \lim_{N \rightarrow \infty} \Pr\left\{\frac{1}{\sqrt{N}} \max_{\{n \leq N : \mathcal{N}_n < n\}} N\delta_N^+(n) \leq x, \frac{1}{\sqrt{N}} \max_{\{n \leq N : \mathcal{N}_n \geq n\}} N\delta_N^-(n) \leq y\right\} \\ &= \lim_{N \rightarrow \infty} \Pr\left\{\frac{1}{\sqrt{N}} \max_{\{n \leq N : \mathcal{N}_n < n\}} N\delta_N^+(n) \leq x\right\} \\ &\quad \times \lim_{N \rightarrow \infty} \Pr\left\{\frac{1}{\sqrt{N}} \max_{\{n \leq N : \mathcal{N}_n \geq n\}} N\delta_N^-(n) \leq y\right\} \\ &= \lim_{N \rightarrow \infty} \Pr\left\{\sqrt{N}\delta_N^+ \leq x\right\} \lim_{N \rightarrow \infty} \Pr\left\{\sqrt{N}\delta_N^- \leq y\right\}, \end{aligned}$$

and the proof of the statement of the lemma is completed.  $\square$

From Lemma 4.3 and (1.4) we have:

$$(4.12) \quad \lim_{N \rightarrow \infty} \Pr \left\{ \delta_N^- + \delta_N^+ \leq \frac{z}{\sqrt{N}} \right\} = \begin{cases} (1 - e^{-2z^2}) * (1 - e^{-2z^2}), & \text{for } z \geq 0, \\ 0, & \text{for } z < 0. \end{cases}$$

where the asterisk denotes convolution.

For the convolution of the probability distribution  $F(z) = 1 - e^{-2z^2}$  with itself we have:

$$(4.13) \quad \int_0^z (1 - e^{-2(z-x)^2}) 4xe^{-2x^2} dx = 1 - e^{-z^2} - \sqrt{\pi}ze^{-z^2} [2\Phi(\sqrt{2}z) - 1],$$

where  $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{y^2}{2}} dy$ . So, the right-hand side of (4.12) is explicitly determined.

Now, we have the following theorem.

**Theorem 4.4.** *The estimator  $\widehat{Q}_n$  for a required characteristic of a queueing system is recurrently defined by (4.5)-(4.7). Then, the bounds of this estimator with a confidence probability non-smaller than  $P$  are recurrently defined as*

$$(4.14) \quad \widehat{Q}_1^{\text{lower}} = \frac{1}{r_0 + \epsilon} Q_0 < \widehat{Q}_1 < \frac{1}{r_0 - \epsilon} Q_0 = \widehat{Q}_1^{\text{upper}},$$

$$(4.15) \quad \widehat{Q}_2^{\text{lower}} = \frac{1 - r_1 - \gamma}{[r_0 + \epsilon]^2} Q_0 < \widehat{Q}_2 < \frac{1 - r_1 + \gamma}{[r_0 - \epsilon]^2} Q_0 = \widehat{Q}_2^{\text{upper}},$$

$$(4.16) \quad \begin{aligned} \widehat{Q}_n^{\text{lower}} &= \frac{1}{r_0 + \epsilon} \left[ (1 - r_1 - \gamma) \widehat{Q}_{n-1}^{\text{lower}} \right. \\ &\quad \left. + (r_2 - \gamma) \widehat{Q}_{n-2}^{\text{lower}} + \dots + (r_{n-1} - \gamma) \widehat{Q}_1^{\text{lower}} \right] \\ &< \widehat{Q}_n < \frac{1}{r_0 - \epsilon} \left[ (1 - r_1 + \gamma) \widehat{Q}_{n-1}^{\text{upper}} \right. \\ &\quad \left. + (r_2 + \gamma) \widehat{Q}_{n-2}^{\text{upper}} + \dots + (r_{n-1} + \gamma) \widehat{Q}_1^{\text{upper}} \right] \\ &= \widehat{Q}_n^{\text{upper}}, \quad n \geq 3, \end{aligned}$$

where the value  $\epsilon$  is determined from the equation:  $1 - \exp(-2N\epsilon^2) = P$ , and the value  $\gamma$  is determined from the equation:

$$1 - \exp(-N\gamma^2) - \sqrt{\pi N} \gamma \exp(-N\gamma^2) [2\Phi(\sqrt{2}N\gamma^2) - 1] = P,$$

where  $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dx$ .

## 5. WHICH OF THE STATISTICS IS BETTER?

In this section we address the question which of the statistics  $\delta_N$ ,  $\delta_N^+$  or  $\delta_N^-$  is better, or under which of them the difference between upper and lower bound is smaller? The statistics  $\delta_N^+$  and  $\delta_N^-$  are symmetric and they give the same bounds. Therefore the question of comparison should be addressed to the statistic  $\delta_N$  and one of the statistics  $\delta_N^+$  or  $\delta_N^-$ , say  $\delta_N^-$ .

Clearly, that

$$\begin{aligned}
 \delta_N &:= \sup_{x \geq 0} |B(x) - B_{\text{emp}}(x, N)| \\
 (5.1) \quad &\geq \sup_{x \geq 0} [B(x) - B_{\text{emp}}(x, N)] \\
 &:= \delta_N^-.
 \end{aligned}$$

Therefore, for the same probability  $P$  the  $z$ -value of the equation  $P = K(z)$  is not smaller than the  $z$ -value of the equation  $P = 1 - e^{-2z^2}$  for any given probability  $P$ . As a result,  $\epsilon(\delta_N) \geq \epsilon(\delta_N^-)$ , where  $\epsilon(\delta_N)$  is the value  $\epsilon$  obtained for the statistic  $\delta_N$ , and  $\epsilon(\delta_N^-)$  is the value  $\epsilon$  obtained for the statistic  $\delta_N^-$  both for the same value of probability  $P$ .

On the other hand, for any probability distribution function  $F(x)$  of a positive random variable we have as follows. Let  $F_1(x) = F\left(\frac{x}{2}\right)$  and  $F_2(x) = (F * F)(x)$ , where the asterisk denotes convolution. Apparently,  $\int_0^\infty x dF_1(x) = \int_0^\infty x dF_2(x)$ , and  $\int_0^\infty x^2 dF_1(x) \geq \int_0^\infty x^2 dF_2(x)$  (the integrals are assumed to converge). Then, there exists a point  $x_0$  where the probability distribution functions  $F_1(x)$  and  $F_2(x)$  cut one another (see Karlin and Novikoff [15] or Stoyan [20], p. 13).

In the given case  $F(x) = 1 - e^{-x^2}$ . Therefore,  $F_1(x) = 1 - e^{-\frac{1}{2}x^2}$ , while  $F_2(x) = 1 - e^{-x^2} - \sqrt{\pi}xe^{-x^2}[2\Phi(\sqrt{2}x) - 1]$ . Apparently, there exists a point  $x_0$  such that  $F_1(z) \leq F_2(z)$  for all  $z \geq x_0$ . Let  $z_1$  and  $z_2$  be two  $z$ -points associated with the solutions of the corresponding equations  $F_1(z) = P$  and  $F_2(z) = P$ , where  $P$  is some given level of probability. Clearly, that if both  $z_1 \geq x_0$  and  $z_2 \geq x_0$ , then, since  $F_1(z) \leq F_2(z)$  in this set, we also have  $z_1 \geq z_2$ .

This enables us to conclude, that correspondingly to these  $z$ -points of the aforementioned probability distribution functions we obtain  $2\epsilon(\delta_N^-) \geq \gamma(\delta_N^-)$ . The following notation is used here. As before, by  $\epsilon(\delta_N^-)$  we mean the value  $\epsilon$  associated with the solution  $1 - e^{-2N\epsilon^2} = P$ . By  $\gamma(\delta_N^-)$  we mean the value  $\gamma$  associated with the solution

$$1 - \exp(-N\gamma^2) - \sqrt{\pi N}\gamma \exp(-N\gamma^2) [2\Phi(\sqrt{2}N\gamma^2) - 1] = P.$$

Hence, from these results we conclude that for large enough  $z$ -values we have  $\epsilon(\delta_N^-) \leq \epsilon(\delta_N)$  and  $\gamma(\delta_N^-) \leq 2\epsilon(\delta_N)$ . (Recall that by  $\epsilon(\delta_N)$  we mean the value of  $\epsilon$  associated with the solution  $K(\epsilon\sqrt{N}) = P$ .)

Thus, the statistic  $\delta_N^-$  becomes better than the statistic  $\delta_N$  for large enough  $z$ -values, i.e. for large enough values of probability  $P$ . This conclusion is supported by the numerical examples in the next section.

## 6. NUMERICAL EXAMPLES

Numerical examples are provided for  $N = 10,000$ . For simplicity we take  $B(x) = 1 - e^{-\mu x}$ , and  $\lambda = \mu = 1$ . The value  $n$  is taken 4. In the numerical examples below we build estimators for the expected busy periods of the  $M/GI/1/n$  queueing systems.

**6.1. The statistic  $\delta_N$ .** From the equation  $K(z) = P = 0.95$  we obtain  $z \approx 1.3581$ . Therefore,

$$\epsilon \approx \frac{1.3581}{\sqrt{10,000}} = 0.013581.$$

Statistics	Expected (theoretical) value $r_i(B)$	The value obtained by simulating empirical distr. $r_i(B_{\text{emp}})$
$r_0$	.5	.501599
$r_1$	.25	.250059
$r_2$	.125	.126067
$r_3$	.0625	.064396
$r_4$	.03125	.03494

TABLE 1. The table for statistics  $r_i(B_{\text{emp}})$  obtained by simulating empirical probability distribution function

Statistics	Expected (theoretical) value $ET_i$	Estimated value $\hat{T}_i$	Lower bound $\hat{T}_i^{\text{lower}}$	Upper bound $\hat{T}_i^{\text{upper}}$
$T_0$	1	1	1	1
$T_1$	2	1.993624	1.941069	2.049105
$T_2$	3	2.980669	2.72325	3.262924
$T_3$	5	4.70773	4.193271	5.83915
$T_4$	8.5	8.043607	6.54611	10.707

TABLE 2. Estimators for expected busy periods in  $M/GI/1/n$  queueing systems based on the statistics  $\delta_N$

With the aid of simulation, we build empirical probability distribution, and on the basis of that empirical probability distribution we obtained  $r_i(B_{\text{emp}})$ ,  $i = 0, 1, 2, 3, 4$ , given in Table 1.

Then, on the basis of statistics  $r_i(B_{\text{emp}})$ ,  $i = 0, 1, 2, 3, 4$  the lower and upper bounds for estimators  $\hat{T}_i$ ,  $i = 1, 2, 3, 4$  and these estimators themselves are shown in Table 2.

**6.2. The statistic  $\delta_N^-$  or  $\delta_N^+$ .** From the equation  $1 - e^{-2z^2} = P = 0.95$  we obtain  $z = 1.224$ . Therefore  $\epsilon = 0.01224$ . From the other equation

$$1 - e^{-z^2} - \sqrt{\pi}ze^{-z^2} [2\Phi(\sqrt{2}z) - 1] = P = 0.95$$

(see (4.13)), we have  $z = 2.025$ . Therefore  $\gamma = 0.02025$ . In this case the lower and upper bounds for estimators  $\hat{T}_i$ ,  $i = 1, 2, 3, 4$  are shown in Table 3. (For convenience the estimators  $\hat{T}_i$ ,  $i = 1, 2, 3, 4$  themselves are duplicated from Table 2.)

## 7. CONCLUDING REMARKS

We provided statistical analysis of certain output characteristics of  $M/GI/1/n$  and  $GI/M/1/n$  queueing systems by using the known statistics associated with empirical distribution. It follows from our results and numerical analysis that in certain cases the use of the statistic  $\delta_N^-$  or  $\delta_N^+$  is more profitable than that of  $\delta_N$ . Namely, in these certain cases the differences between the upper and lower

Statistics	Expected (theoretical) value	Estimated value	Lower bound	Upper bound
	$ET_i$	$\hat{T}_i$	$\hat{T}_i^{\text{lower}}$	$\hat{T}_i^{\text{upper}}$
$T_0$	1	1	1	1
$T_1$	2	1.993624	1.946001	2.04479
$T_2$	3	2.980669	2.763262	3.216612
$T_3$	5	4.70773	4.325278	5.673235
$T_4$	8.5	8.043607	6.879717	10.25456

TABLE 3. Estimators for expected busy periods in  $M/GI/1/n$  queueing systems based on the statistics  $\delta_N^-$  or  $\delta_N^+$

bounds for the output characteristics are smaller when we use the statistic  $\delta_N^-$  or  $\delta_N^+$  compared to these differences between the bounds when we use the statistic  $\delta_N$ .

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