

# CONTINUITY OF LARGE CLOSED QUEUEING NETWORKS WITH BOTTLENECKS

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ABSTRACT. This paper studies a closed queueing network containing a hub (a state dependent queueing system with service depending on the number of units residing here) and  $k$  satellite stations, which are  $GI/M/1$  queueing systems. The number of units in the system,  $N$ , is assumed to be large. After service completion in the hub, a unit visits a satellite station  $j$ ,  $1 \leq j \leq k$ , with probability  $p_j$ , and, after the service completion there, returns to the hub. The parameters of service times in the satellite stations and in the hub are proportional to  $\frac{1}{N}$ . One of the satellite stations is assumed to be a bottleneck station, while others are non-bottleneck. The paper establishes the continuity of the queue-length processes in non-bottleneck satellite stations of the network when the service times in the hub are close in certain sense (exactly defined in the paper) to the exponential distribution.

## 1. INTRODUCTION

In the present paper we study the continuity of large closed queueing networks with bottlenecks. The continuity of queueing systems is a known area in queueing theory, and the goal of this theory is to answer the question *how small change of input characteristics of the system affects its output characteristics*. This question is of especial significance for queueing networks modeling computer networks, which have a much more complicated structure than queueing systems.

In the present paper the continuity results are established for closed queueing networks describing client/server communication networks. The queueing networks considered in the paper are of the following configuration. There is a large server (hub), which is a specific state dependent queueing system (the details are given later) and  $k$  satellite stations, which are single server  $GI/M/1/\infty$  queueing systems. Being served in the hub, a unit is addressed to the  $j$ th satellite station,  $1 \leq j \leq k$ , with probability  $p_j > 0$   $\left( \sum_{j=1}^k p_j = 1 \right)$ , and then, being served there, it returns to the hub.

There are many papers in the literature that study similar type of queueing systems. In most of them the hub is a multiserver (or infinite server) queueing system. One of the earliest considerations goes back to a paper of Whitt [26]. A Markovian queueing network with one bottleneck satellite station has been studied by Kogan and Liptser [18]. Extensions of the results [18] for different non-Markovian

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models have been in a series of the papers of Abramov [1], [2], [3], [5]. Non-Markovian models of networks with a hub and one satellite station have also been considered by Krichagina, Liptser and Puhalskii [19] and Krichagina and Puhalskii [20]. For other studies of queueing networks with bottleneck see Berger, Bregman and Kogan [10], Brown and Pollett [11] and Pollett [21] and other papers.

Aforementioned papers [1], [2], [3], [5], [18], study non-stationary queue-length distributions in non-bottleneck satellite stations. Specifically, in [1], [3] and [18] these distributions have been studied in the presence of one bottleneck satellite station, and in [2] the only cases of bottleneck leading to diffusion approximations have been investigated. One of the main assumption leading to substantial technical simplification in [1], [2], [3] and [18] was that at the initial time moment all of  $N$  units are located in the hub, where the time instant  $t = 0$  is the moment of service starts for all of these  $N$  units. In the case of exponentially distributed service times in the hub, because of the property of the lack of memory of an exponential distribution, the above assumption is not the loss of generality. However, in the case where the service times in the hub are generally distributed, this assumption is an essential restriction, because its change can lead to a serious change of the limiting non-stationary queue-length distributions. The model considered in [2] is flexible to this change, since it models hub as a state dependent single-server queueing system in which the behavior of the queue-length processes is stable with respect to a slight change of the initial condition. However, there has been the quite general assumption on that state dependence in that paper, and except the cases where diffusion approximations are available no real bottleneck analysis in the model of [2] has been provided, and the most interesting cases of the problem remain unsolved. Although the models with state dependent single server system are quite general, it is very hard to derive the results for more particular models, for instance, for such models where the hub is a multiserver (or infinite-server) queueing system with homogeneous servers.

In the present paper we study a more particular model than that in [2], i.e. we study the model with the same state-dependent service mechanism in the hub as in [2], but specific additional assumptions on probability distribution functions of service times in the hub that will be discussed later. One of these assumptions, saying that service times in the hub are “close” (in the sense given in the paper) to the exponential distribution, enables us to easily extend the results to other schemes including realistic cases of multiserver (or infinite-server) hub stations.

The results of this study or similar results cannot be derive from other results obtained for known models such as considered in [18] or [1]. For example, it seems to be impossible to derive the results similar to those obtained in the present paper by the means of [18]. The methods of this and the following paper [1] fail if we assume that the probability distribution function of a service time in the hub differs from exponential distribution, since the proofs in these papers were based on martingale characterization of Poisson processes.

Let us recall the assumptions made in [2] in the description of the service mechanism in the hub. It is assumed in [2] that

(i) if immediately before a service start of a unit the queue-length in the hub is equal to  $K \leq N$ , then the probability distribution function is  $G_K(Kx)$ , and there is expectation  $\frac{1}{\lambda_K} = \int_0^\infty x dG_K(x) < \infty$ .

Hence the expectation of a service time in this case is

$$\int_0^\infty x dG_K(Kx) = \frac{1}{K} \int_0^\infty x dG_K(x) := \frac{1}{K\lambda_K}.$$

(ii) As  $N \rightarrow \infty$ , the sequence of probability distribution functions  $G_N(x)$  is assumed to converge weakly to  $G(x)$  with the finite expectation  $\frac{1}{\lambda} = \int_0^\infty x dG(x)$ , and  $G(0+) = 0$ . In addition, it is assumed in [2] that  $G_K(Kx) \leq G_{K+1}(Kx+x)$  for all  $x \geq 0$ .

In the present paper in addition to (i) we assume that

$$(1.1) \quad G_1(x) = G_2(x) = \dots = G_N(x) := G(x),$$

and there exists the second moment

$$\int_0^\infty x^2 dG(x) = r < \infty$$

satisfying the condition:

$$(1.2) \quad r > \frac{2}{\lambda^2}.$$

The meaning of Condition (1.2) will be clear later (see proof of Theorem 4.1).

In this case for the random variable  $\zeta_K$  having the probability distribution function  $F_K(x) = G(Kx)$  we have

$$(1.3) \quad E\zeta_K = \int_0^\infty x dF_K(x) = \int_0^\infty x dG(Kx) = \frac{1}{K\lambda}.$$

Under the above addition assumptions, earlier assumption (ii) of [2] becomes irrelevant here.

Following (1.1) and (1.3) one can “guess” that departures of units from the hub when  $K$  increases indefinitely should be asymptotically insensitive to the type of the distribution  $G(x)$ , i.e. the number of departures per unit of time should be the same as in the case when  $G(x)$  would be the exponential distribution. This guess however cannot be proved by the means of the earlier papers [18] and [1] and should involve more delicate methods (which are provided in the present paper).

In addition to the assumptions made, we consider two classes of probability distribution function  $G(x) := P\{\zeta \leq x\}$ . The first one satisfies the condition

$$(1.4) \quad \sup_{x,y \geq 0} |G(x) - G_y(x)| < \epsilon,$$

where  $\epsilon$  is a small positive number, and  $G_y(x) = P\{\zeta \leq x+y | \zeta > y\}$ .

For the second class of distributions along with (1.4) it is assumed that  $G(x)$  belongs to one of the classes NBU (New Better than Used) or NWU (New Worse than Used). Recall that a probability distribution function  $\Xi(x)$  of a nonnegative random variable is said to belong to the class NBU if for all  $x \geq 0$  and  $y \geq 0$  we have  $\Xi(x+y) \leq \Xi(x)\Xi(y)$ , where  $\bar{\Xi}(x) = 1 - \Xi(x)$ . If the opposite inequality holds, i.e.  $\bar{\Xi}(x+y) \geq \bar{\Xi}(x)\bar{\Xi}(y)$ , then  $\Xi(x)$  is said to belong to the class NWU.

Assumption (i) assumes that the probability distribution function is not changed when a new arrival occurs in the hub from one of the satellite stations. It is only changed at the moments when a service in the hub starts. However, the later assumption (1.4) enables us to consider more general schemes that take also into account the situation where the change of a probability distribution function can

occur at the moment of arrival of a unit in the hub from a satellite station. Indeed, from (1.4) by the triangle inequality for any  $y_2 \geq 0$  we have:

$$\begin{aligned} & \sup_{x, y_1 \geq 0} |G_{y_1}(x) - G_{y_2}(x)| \\ & \leq \sup_{x, y_1 \geq 0} |G(x) - G_{y_1}(x)| + \sup_{x \geq 0} |G(x) - G_{y_2}(x)| \\ & \leq \sup_{x, y_1 \geq 0} |G(x) - G_{y_1}(x)| + \sup_{x, y_2 \geq 0} |G(x) - G_{y_2}(x)| \\ & < 2\epsilon. \end{aligned}$$

Therefore

$$(1.5) \quad \sup_{x, y_1, y_2 \geq 0} |G_{y_1}(x) - G_{y_2}(x)| < 2\epsilon$$

as well. In addition, according to a characterization theorem of Azlarov and Volodin [9], [6] we also have:

$$(1.6) \quad \sup_{x \geq 0} |G(x) - (1 - e^{-\lambda x})| < 2\epsilon.$$

If, in addition, the probability distribution function  $G(x)$  belongs either to the class NBU or to the class NWU, then instead of (1.5) for any  $y_2 \geq 0$  we, respectively, have:

$$\begin{aligned} \sup_{x, y_1 \geq 0} |G_{y_1}(x) - G_{y_2}(x)| & \leq \sup_{x, y_1 \geq 0} \max_{i=1,2} |G(x) - G_{y_i}(x)| \\ & \leq \sup_{x, y_1 \geq 0} |G(x) - G_{y_1}(x)| \\ & < \epsilon. \end{aligned}$$

Hence,

$$(1.7) \quad \sup_{x, y_1, y_2 \geq 0} |G_{y_1}(x) - G_{y_2}(x)| < \epsilon.$$

In addition instead of (1.6) (see [6]) we have

$$(1.8) \quad \sup_{x \geq 0} |G(x) - (1 - e^{-\lambda x})| < \epsilon.$$

As in [2], it is assumed in the paper that the service times in the satellite station  $j$ ,  $j = 1, 2, \dots, k$ , are exponentially distributed with parameter  $N\mu_j$ , and  $\mu_j > \lambda p_j$  for  $j = 1, 2, \dots, k-1$  and  $\mu_k < \lambda p_k$ . This means that the  $k$ th satellite station is assumed to be a bottleneck station, while the first  $k-1$  satellite stations are non-bottleneck.

The problem of continuity of queues and networks is an old problem. It goes back to the papers of Kennedy [17] and Whitt [25] and then was developed in many papers. To mention only a few of them we refer Kalashnikov [15], Kalashnikov and Rachev [16], Rachev [22], Zolotarev [27], [28], Gordienko and Ruiz de Chávez [13], [14].

The continuity results of this paper are based on the estimates for the least positive root of the functional equation  $z = \widehat{G}(\mu - \mu z)$ , where  $\widehat{G}(s)$ ,  $s \geq 0$ , is the Laplace-Stieltjes transform of the probability distribution function  $G(x)$  of positive random variables having first two moments, and  $\mu \int_0^\infty x dG(x) > 1$ . It is well-known (see Takács [24] or Gnedenko and Kovalenko [12]) that the least positive root of the aforementioned functional equation belongs to the interval  $(0,1)$ . Within interval  $(0,1)$  Rolski [23] established the lower and upper bounds for this least positive root,

when there exist the first two moments of the probability distribution function  $G(x)$ . For the smaller class of probability distribution functions, when in addition the distance in Kolmogorov's metric between any pair of probability distribution functions is less than a small value  $\epsilon > 0$ , the new bounds for the aforementioned least positive root have been established in [7]. These bounds are used to establish the desired continuity results.

The rest of the paper is organized as follows. In Section 2 the known results for Markovian models containing only one satellite station, which is a bottleneck station, are recalled, and then in the same section the same result is obtained under our more general assumption for this simple model that the service times in the hub are state dependent and satisfy (1.1). In Section 3, the general model containing a bottleneck station and several non-bottleneck stations is studied. In Section 4, the main results of this paper related to continuity of queueing network are obtained.

## 2. A LARGE CLOSED NETWORK CONTAINING ONLY ONE SATELLITE STATION, WHICH IS BOTTLENECK

In this section we consider examples of simplest queueing networks in order to demonstrate that in the case of the model considered in the paper, the asymptotic behaviour of the queue-length process in the hub is the same as that in the case of the exponentially distributed service times in the hub, which was originally studied in [18].

**2.1. The case  $G(x) = 1 - e^{-\lambda x}$ .** In the case where  $G(x) = 1 - e^{-\lambda x}$  we deal with the model considered in the paper of Kogan and Liptser [18].

Consider the Markovian network (see Kogan and Liptser [18]) containing only one satellite station, which is assumed to be bottleneck. Let  $A(t)$  and  $D(t)$  denote arrival process to this satellite station and, respectively, departure process from this satellite station. Denoting the queue-length process in this satellite station by  $Q(t)$  one can write:

$$(2.1) \quad Q(t) = A(t) - D(t).$$

The departure process  $D(t)$  is defined via the Poisson process as follows. Service times in the satellite station are assumed to be exponentially distributed with parameter  $\mu N$ . Therefore, denoting the Poisson process with parameter  $\mu N$  by  $S(t)$ , for the departure process  $D(t)$  we have the equation:

$$(2.2) \quad D(t) = \int_0^t \mathbf{1}\{Q(s-) > 0\} dS(s),$$

where  $\mathbf{1}\{\mathcal{A}\}$  denotes the indicator of the event  $\mathcal{A}$ .

To define the arrival process, consider the collection of independent Poisson processes  $\pi_i(t)$ ,  $i = 1, 2, \dots, N$ . Then,

$$(2.3) \quad A(t) = \int_0^t \sum_{i=1}^N \mathbf{1}\{N - Q(s-) \geq i\} d\pi_i(s).$$

For the normalized queue-length process  $q_N(t) = \frac{1}{N}Q(t)$ , from (2.1) we have:

$$(2.4) \quad q_N(t) = \frac{1}{N}A(t) - \frac{1}{N}D(t).$$

As  $N \rightarrow \infty$ , both  $\frac{1}{N}A(t)$  and  $\frac{1}{N}D(t)$  converge a.s. to the limits, and these limits are the same as the correspondent limits of  $\frac{1}{N}\widehat{A}(t)$  and  $\frac{1}{N}\widehat{D}(t)$  (see [18] or [1] for further details), where  $\widehat{A}(t)$  and  $\widehat{D}(t)$  are the compensators in the Doob-Meyer semimartingale decompositions  $A(t) = M_A(t) + \widehat{A}(t)$  and, correspondingly,  $D(t) = M_D(t) + \widehat{D}(t)$  ( $M_A(t)$  and  $M_D(t)$  denote local square integrable martingales corresponding the processes  $A(t)$  and  $D(t)$ .) Therefore,

$$(2.5) \quad q_N(t) = \frac{1}{N}\widehat{A}(t) - \frac{1}{N}\widehat{D}(t).$$

The compensators  $\widehat{A}(t)$  and  $\widehat{D}(t)$  have the representations (for details see [18]):

$$(2.6) \quad \widehat{A}(t) = \lambda \int_0^t [N - Q(s)] ds,$$

and

$$(2.7) \quad \widehat{D}(t) = \mu N t - \mu N \int_0^t \mathbf{1}\{Q(s) = 0\} ds,$$

Therefore, from (2.5), (2.6) and (2.7)

$$(2.8) \quad \begin{aligned} q(t) : &= \text{P-} \lim_{N \rightarrow \infty} q_N(t) \\ &= \lambda \int_0^t [1 - q(s)] ds - \mu t \\ &\quad + \mu \int_0^t \mathbf{1}\{Q(s) = 0\} ds. \end{aligned}$$

In the case  $\mu < \lambda$ , using the Skorokhod reflection principle and techniques of the theory of martingales one can show that  $\mathbf{1}\{Q(s) = 0\}$  vanishes in probability for all  $s > 0$  as  $N$  increases to infinity (for details of this see [18] or [1]), and from (2.8) we arrive at

$$(2.9) \quad q(t) = \lambda \int_0^t [1 - q(s)] ds - \mu t.$$

Taking into account that  $q(0) = 1$ , from (2.9) we obtain:

$$(2.10) \quad q(t) = \left(1 - \frac{\mu}{\lambda}\right) (1 - e^{-\lambda t}).$$

**2.2. The case of arbitrary distribution  $G(x)$ .** We will show below that the equation (2.10) holds also in the case, when service times in the hub are generally distributed and state dependent (for further details see Section 1) but under the assumption that there is only one satellite station, which is assumed to be bottleneck.

Let  $\delta$  be a small enough positive fixed real number so the inequalities appearing later in the form  $\mu < \lambda(1 - c\delta)$  for specified constants  $c$  are assumed to be satisfied; the large parameter  $N$  is assumed to increase to infinity. Let  $Q_N(t)$  denote the queue-length in the satellite station (which is a bottleneck station), and let  $\overline{Q}_N(t)$  denote the queue-length in the hub ( $\overline{Q}_N(t) = N - Q_N(t)$ ). In a small interval  $[0, \delta)$  let us denote

$$\overline{Q}_N^+[0, \delta) = \sup_{0 \leq t < \delta} \overline{Q}_N(t)$$

and

$$\overline{Q}_N^-[0, \delta) = \inf_{0 \leq t < \delta} \overline{Q}_N(t).$$

Clearly, that  $\overline{Q}_N^+[0, \delta) = N$ , and, as  $N \rightarrow \infty$ , the a.s. limit of  $\overline{q}_N(t) = \frac{1}{N} \overline{Q}_N(t)$  as  $N \rightarrow \infty$ , denoted by  $\overline{q}(t)$ , must be between 1 and  $1 - (\lambda - \mu)\delta$  for all  $t \in [0, \delta)$ .

The last result is explained as follows. Let  $a_N[0, \delta)$  be the maximum arrival rate (that is maximum instantaneous arrival rate) in the interval  $[0, \delta)$ . Since  $\overline{Q}_N^+[0, \delta) = N$ , this instantaneous rate  $a_N[0, \delta) = \lambda N$ . This means that, as  $N \rightarrow \infty$ , then with probability approaching 1 the number of arrivals during the interval  $[0, \delta)$  is not greater than  $\lambda N \delta$ , i.e.

$$(2.11) \quad \Pr \left\{ \limsup_{N \rightarrow \infty} \frac{\mathcal{A}_N[0, \delta)}{N} \leq \lambda \delta \right\} = 1,$$

where  $\mathcal{A}_N[0, \delta)$  denotes the number of unit arrivals from the hub to satellite station.

Let  $\mathcal{S}_N[0, \delta)$  denote the number of service completions in the satellite station during the interval  $[0, \delta)$ . Similarly to (2.11), as  $N \rightarrow \infty$ , we obtain

$$(2.12) \quad \Pr \left\{ \limsup_{N \rightarrow \infty} \frac{\mathcal{S}_N[0, \delta)}{N} = \mu \delta \right\} = 1,$$

which means that, as  $N \rightarrow \infty$ , then with the probability approaching 1 the number of service completions during  $[0, \delta)$  becomes close to  $\mu N \delta$ .

The fact given in (2.12) can be easily proved as follows. Take  $\epsilon = \frac{\delta}{M}$ , where  $M$  is a sufficiently large number. As  $\mu < \lambda$ , the probability that a busy period of the  $GI/M/1/\infty$  queueing system is finite is  $\frac{\mu}{\lambda}$  (i.e. it is less than 1). Therefore the value  $N$  can be chosen such that during the time interval  $[0, \epsilon)$  there is only a finite number of busy periods with probability 1, and during the time interval  $[\epsilon, \delta)$  there is none finished busy period, and units are served continuously without delay with the rate  $\mu N$  per time unit. Therefore, as  $N \rightarrow \infty$ , with probability approaching 1 the number of service completions during the interval  $[0, \delta)$ ,  $\mathcal{S}_N[0, \delta)$ , is at least  $\mu N(\delta - \epsilon) = \mu N \frac{M-1}{M} \delta$ . As  $N \rightarrow \infty$ , the sequence of values  $M = M(N)$  can be taken increasing to infinity and associated sequence  $\epsilon = \epsilon(N)$  vanishing. Therefore, as  $N \rightarrow \infty$ , the value  $\mathcal{S}_N[0, \delta)$  becomes asymptotically equivalent to  $\mu N \delta$ . So, we arrive at (2.12).

From (2.11) and (2.12) we obtain

$$(2.13) \quad \begin{aligned} \Pr \left\{ 1 - \limsup_{N \rightarrow \infty} \left( \frac{\mathcal{A}_N[0, \delta)}{N} - \frac{\mathcal{S}_N[0, \delta)}{N} \right) \leq \lim_{N \rightarrow \infty} \overline{q}_N(t) \leq 1 \right\} \\ = \Pr \left\{ 1 - (\lambda - \mu)\delta \leq \lim_{N \rightarrow \infty} \overline{q}_N(t) \leq 1 \right\} = 1. \end{aligned}$$

Hence, according to (2.13)

$$(2.14) \quad 1 - (\lambda - \mu)\delta \stackrel{a.s.}{\leq} \overline{q}(t) \stackrel{a.s.}{\leq} 1$$

for all  $t \in [0, \delta)$ . Note, that

$$\lim_{\delta \downarrow 0} \frac{1 - \overline{q}(\delta)}{\delta} = \lambda - \mu.$$

So, the value  $\delta$  can be chosen such small that  $\overline{q}(\delta)$  is asymptotically close to  $1 - (\lambda - \mu)\delta$ .

Let us consider the second interval  $[\delta, 2\delta)$ . As  $N$  is large enough, for small  $\delta$  the number of units in the hub at time  $\delta$ ,  $\overline{Q}_N(\delta)$ , behaves as  $N[1 - (\lambda - \mu)\delta]$ , i.e

we have approximately this number of busy servers in the hub. Therefore, at that time moment  $\delta$  the instantaneous rate of arrival of units in the satellite station is asymptotically equivalent to  $\lambda N[1 - (\lambda - \mu)\delta]$  and the service rate  $\mu N$ . Assuming the inequality  $\mu < \lambda[1 - (\lambda - \mu)\delta]$  satisfied, one can use the same arguments as above. First,  $a_N[\delta, 2\delta)$  is asymptotically equivalent to  $\lambda N[1 - (\lambda - \mu)\delta]$ . As  $N \rightarrow \infty$ ,

$$(2.15) \quad \Pr \left\{ \limsup_{N \rightarrow \infty} \frac{\mathcal{A}_N[\delta, 2\delta)}{N} \leq \lambda\delta[1 - (\lambda - \mu)\delta] \right\} = 1,$$

and

$$(2.16) \quad \Pr \left\{ \limsup_{N \rightarrow \infty} \frac{\mathcal{S}_N[\delta, 2\delta)}{N} = \mu\delta \right\} = 1,$$

and at the moment  $2\delta$  the queue-length in the satellite station,  $Q_N(2\delta)$ , is asymptotically evaluated as  $\frac{\lambda - \mu}{\lambda}[2(\lambda\delta) - (\lambda\delta)^2]N$ . Therefore,

$$(2.17) \quad \begin{aligned} & \Pr \left\{ 1 - \limsup_{N \rightarrow \infty} \left( \frac{\mathcal{A}_N[0, 2\delta)}{N} - \frac{\mathcal{S}_N[0, 2\delta)}{N} \right) \leq \lim_{N \rightarrow \infty} \bar{q}_N(t) \right. \\ & \leq 1 - \limsup_{N \rightarrow \infty} \left( \frac{\mathcal{A}_N[0, \delta)}{N} - \frac{\mathcal{S}_N[0, \delta)}{N} \right) \left. \right\} \\ & = \Pr \left\{ 1 - \frac{\lambda - \mu}{\lambda}[2(\lambda\delta) - (\lambda\delta)^2] \leq \lim_{N \rightarrow \infty} \bar{q}_N(t) \leq 1 - (\lambda - \mu)\delta \right\} = 1. \end{aligned}$$

Hence, according to (2.17)

$$(2.18) \quad 1 - \frac{\lambda - \mu}{\lambda}[2(\lambda\delta) - (\lambda\delta)^2] \stackrel{a.s.}{\leq} \bar{q}(t) \stackrel{a.s.}{\leq} 1 - (\lambda - \mu)\delta$$

for all  $t \in [\delta, 2\delta)$ .

Then, considering the third interval  $[2\delta, 3\delta)$ , at the end of this interval the queue-length in the satellite station,  $Q_N(3\delta)$ , is evaluated as  $\frac{\lambda - \mu}{\lambda}[3(\lambda\delta) - 3(\lambda\delta)^2 + (\lambda\delta)^3]N$ . In this case, similarly to (2.18) we arrive at the inequality

$$(2.19) \quad 1 - \frac{\lambda - \mu}{\lambda}[3(\lambda\delta) - 3(\lambda\delta)^2 + (\lambda\delta)^3] \stackrel{a.s.}{\leq} \bar{q}(t) \stackrel{a.s.}{\leq} 1 - \frac{\lambda - \mu}{\lambda}[2(\lambda\delta) - (\lambda\delta)^2]$$

for all  $t \in [2\delta, 3\delta)$ .

Thus, considering the sequence of intervals, at the end of the  $j$ th interval, for the queue-length in the satellite station,  $Q_N(j\delta)$ , we obtain the expansion:

$$\begin{aligned} & \frac{\lambda - \mu}{\lambda} \left[ j(\delta\lambda) - \binom{j}{2} (\delta\lambda)^2 + \dots \right. \\ & \left. + (-1)^i \binom{j}{i} (\delta\lambda)^{i+1} + \dots \right. \\ & \left. + (-1)^j (\delta\lambda)^j \right]. \end{aligned}$$

From this expansion we have

$$\begin{aligned}
(2.20) \quad & 1 - \frac{\lambda - \mu}{\lambda} \left[ j(\delta\lambda) - \binom{j}{2} (\delta\lambda)^2 \right. \\
& \quad \left. + \dots + (-1)^i \binom{j}{i} (\delta\lambda)^{i+1} + \dots + (-1)^j (\delta\lambda)^j \right] \\
& \stackrel{a.s.}{\leq} \bar{q}(t) \stackrel{a.s.}{\leq} 1 - \frac{\lambda - \mu}{\lambda} \left[ (j-1)(\delta\lambda) - \binom{j-1}{2} (\delta\lambda)^2 + \dots \right. \\
& \quad \left. + (-1)^i \binom{j-1}{i} (\delta\lambda)^{i+1} + \dots + (-1)^{j-1} (\delta\lambda)^{j-1} \right]
\end{aligned}$$

for all  $t \in [(j-1)\delta, j\delta)$ .

Assume now that  $\delta \rightarrow 0$ . Then taking  $j = \lfloor \frac{t}{\delta} \rfloor$ , where  $\lfloor \cdot \rfloor$  denotes the integer part of the number, and passing from the sequence of sums to the limits, for the  $\bar{q}(t)$  we arrive at

$$\begin{aligned}
(2.21) \quad \bar{q}(t) &= 1 - \frac{\lambda - \mu}{\lambda} \left[ 1 - \lim_{\delta \rightarrow 0} (1 - \lambda\delta)^{\frac{t}{\delta}} \right] \\
&= 1 - \frac{\lambda - \mu}{\lambda} (1 - e^{-\lambda t}),
\end{aligned}$$

where  $t \in [0, \infty)$ . Thus, in view that  $q(t) = 1 - \bar{q}(t)$ , (2.21) implies (2.10), and the desired relationship is proved.

### 3. A GENERAL CLOSED NETWORK CONTAINING SEVERAL SATELLITE STATIONS, ONE OF WHICH IS BOTTLENECK

Consider now a more general case of the model, where there are  $k$  satellite stations numbered  $1, 2, \dots, k$ , and only the  $k$ th satellite station is bottleneck, while the first  $k-1$  satellite stations are non-bottleneck. In this case as in [2] we use the level crossing method. Note, that the approach of [2] based on straightforward generalization of the level crossing method related to Markovian queues was mistaken. The error of [2] was corrected in another paper [4] (see also [8]), and here we follow by the improved method. Let us recall this method in the case of the standard  $GI/M/1$  queueing system and then explain how the method can be adapted to this network with bottleneck.

Consider the  $GI/M/1$  queueing system, where an interarrival time has the probability distribution function  $G(x)$  with the expectation  $\frac{1}{\lambda}$ , a service time is exponentially distributed with parameter  $\mu$ , and  $\lambda < \mu$ . For a busy period of this system, let  $f(i)$  denote the number of cases during that busy period where a customer at the moment of his/her arrival finds exactly  $i$  other customers in the system. Clearly, that  $f(0) = 1$  with probability 1. Let  $t_{i,1}, t_{i,2}, \dots, t_{i,f(i)}$  denote the time moments of these arrivals, and let  $s_{i,1}, s_{i,2}, \dots, s_{i,f(i)}$  denote the service completions during the aforementioned busy period, at which there remain exactly  $i$  customers in the system.

Consider the intervals

$$(3.1) \quad [t_{i,1}, s_{i,1}), [t_{i,2}, s_{i,2}), \dots, [t_{i,f(i)}, s_{i,f(i)})$$

and

$$(3.2) \quad \begin{aligned} & [t_{i+1,1}, s_{i+1,1}), [t_{i+1,2}, s_{i+1,2}), \dots, \\ & [t_{i+1,f(i+1)}, s_{i+1,f(i+1)}) \end{aligned}$$

The intervals of (3.2) all are contained in the intervals of (3.1). Let us delete the intervals of (3.2) from those of (3.1) and merge the ends. Then we obtain a special type of branching process  $\{f(i)\}$ , which according to its construction is *not* a standard branching process, because the number of offspring generated by particles of different generations are not independent random variables. (Only they are independent in the case of the  $M/M/1$  system.) However, the process satisfies the property:  $Ef(i) = \varphi^i$ ,  $i = 0, 1, \dots$ , which is similar to the respective property of a standard Galton-Watson branching process. The branching process  $\{f(i)\}$  is called *GI/M/1 type branching process*. The properties of this branching process are discussed in [4] and [8].

The value  $\varphi := Ef(1)$  is calculated as follows. We have the equation

$$Ef(1) = \varphi = \sum_{i=0}^{\infty} \varphi^i \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^i}{i!} dG(x),$$

where  $\varphi$  is the least positive root of the functional equation  $z = \widehat{G}(\mu - \mu z)$ ;  $\widehat{G}(s)$ ,  $s \geq 0$ , is the Laplace-Stieljies transform of the probability distribution function  $G(x)$ .

Let us now consider the closed queueing network with  $k$  satellite stations. The first  $k-1$  satellite stations are non-bottleneck, i.e. the relation  $\mu_j > \lambda p_j$  is satisfied, while the  $k$ th satellite station is a bottleneck station, and then the relation  $\mu_k < \lambda p_k$  is satisfied (Recall that by  $p_j$ ,  $j = 1, 2, \dots, k$ , we denote the routing probabilities.) The input rate to the  $j$ th satellite station at time  $t$  is  $\lambda p_j \mathcal{N}_t$ , where the random variable  $\mathcal{N}_t$  ( $\mathcal{N}_t \leq N$  with probability 1) is the number of units in the hub at time  $t$ .

By the level crossing method one can study the queue-length process in any satellite station  $j$ ,  $j = 1, 2, \dots, k-1$ . Consider first the time interval  $[0, \delta)$ , where  $\delta$  is a fixed sufficiently small value. Then, similarly to the construction in Section 2 in the case of only one satellite station, for  $N$  increasing to infinity with probability approaching 1 the queue-length in the hub at time moment  $t = \delta$  is asymptotically equal to  $N[1 - (\lambda p_k - \mu_k)\delta]$ . If there are several non-bottleneck satellite stations, then the queue-length in the hub at time moment  $\delta$  is asymptotically equal to the same aforementioned value  $N[1 - (\lambda p_k - \mu_k)\delta]$  because the queue-lengths in non-bottleneck satellite stations all are finite with probability 1, and their contribution is therefore negligible. More detailed arguments are as follows.

In any satellite station  $j < k$  the length of a busy period is finite with probability 1. Hence, as  $N$  increases unboundedly, the number of busy periods in the interval  $[0, \delta)$  increases to infinity as well, and at time moment  $\delta$  the total number of units in all of satellite stations  $j < k$  with probability 1 is bounded, i.e. it is negligible compared to  $N$ . Therefore, the number of customers in the hub is asymptotically evaluated as  $N[1 - (\lambda p_k - \mu_k)\delta]$  as well.

For the  $j$ th satellite station,  $j < k$ , let us consider the last busy period that finished before time moment  $\delta$ . Let  $f_j(i)$  denotes the number of cases during that busy period that a customer at the moment of his/her arrival find  $i$  other customers in the system, let  $t_{j,i,1}, t_{j,i,2}, \dots, t_{j,i,f_j(i)}$  be the moments of these arrivals, and let

$s_{j,i,1}, s_{j,i,2}, \dots, s_{j,i,f_j(i)}$  be the moments of service completions that there remain exactly  $i$  units in the satellite station. We have the intervals

$$(3.3) \quad [t_{j,i,1}, s_{j,i,1}), [t_{j,i,2}, s_{j,i,2}), \dots, [t_{j,i,f_j(i)}, s_{j,i,f_j(i)})$$

and

$$(3.4) \quad [t_{j,i+1,1}, s_{j,i+1,1}), [t_{j,i+1,2}, s_{j,i+1,2}), \dots, \\ [t_{j,i+1,f_j(i+1)}, s_{j,i+1,f_j(i+1)})$$

which are similar to the intervals (3.1) and (3.2) considered before. (The only difference in the additional index  $j$  indicating the  $j$ th satellite station.) Delete the intervals (3.4) from those of (3.3) and merge the ends. As  $N$  increases unboundedly, the process  $\{f_j(i)\}_{i \geq 0}$  in a random sequence in  $i$ . Each of the random variables  $f_j(i)$  converges a.s. (as  $N \rightarrow \infty$ ) to the limiting random variable, which is the number of offspring in the  $i$ th generation of the  $GI/M/1$  type branching process. So, we have the a.s. convergence to the  $GI/M/1$  type branching process in the sense that for all  $i$  this a.s. convergence holds. Let us find  $Ef_j(1)$ . Similarly to the above case of the  $GI/M/1$  queueing system for sufficiently large  $N$  we have the equation

$$(3.5) \quad z = \frac{p_j \widehat{F}_{[N[1-(\lambda p_k - \mu_k)\delta]]}(\mu_j N - \mu_j N z)}{1 - (1 - p_j) \widehat{F}_{[N[1-(\lambda p_k - \mu_k)\delta]]}(\mu_j N - \mu_j N z)} [1 + o(1)] \\ = \frac{p_j \widehat{G}_{[N[1-(\lambda p_k - \mu_k)\delta]]} \left( \frac{\mu_j N - \mu_j N z}{[N[1-(\lambda p_k - \mu_k)\delta]]} \right)}{1 - (1 - p_j) \widehat{G}_{[N[1-(\lambda p_k - \mu_k)\delta]]} \left( \frac{\mu_j N - \mu_j N z}{[N[1-(\lambda p_k - \mu_k)\delta]]} \right)} [1 + o(1)],$$

where  $\widehat{F}_N(s)$  denotes the Laplace-Stieltjes transform of the probability distribution function  $F_N(x) = G_N(Nx) = G(Nx)$ . For details of derivation of (3.5) see [2] or [8]. As  $N \rightarrow \infty$ , in limit we have the equation

$$(3.6) \quad z = \frac{p_j \widehat{G} \left( \frac{\mu_j - \mu_j z}{1 - (\lambda p_k - \mu_k)\delta} \right)}{1 - (1 - p_j) \widehat{G} \left( \frac{\mu_j - \mu_j z}{1 - (\lambda p_k - \mu_k)\delta} \right)}.$$

Considering now the interval  $[\delta, 2\delta)$ , in the endpoint  $2\delta$ , due to the arguments of Section 2 and the arguments above in this section, the queue-length is asymptotically equal to  $N \frac{\lambda p_k - \mu_k}{\lambda p_k} [2(\lambda p_k \delta) - (\lambda p_k \delta)^2]$ . Therefore, similarly to (3.6) for the  $j$ th satellite station,  $j < k$ , we have the equation

$$(3.7) \quad z = \frac{p_j \widehat{G} \left( \frac{\mu_j - \mu_j z}{1 - \frac{\lambda p_k - \mu_k}{\lambda p_k} [2\lambda p_k \delta - (\lambda p_k \delta)^2]} \right)}{1 - (1 - p_j) \widehat{G} \left( \frac{\mu_j - \mu_j z}{1 - \frac{\lambda p_k - \mu_k}{\lambda p_k} [2\lambda p_k \delta - (\lambda p_k \delta)^2]} \right)}.$$

In an arbitrary interval  $[(i-1)\delta, i\delta)$ , in its endpoint  $i\delta$  we correspondingly have the equation:

$$(3.8) \quad z = \frac{p_j \widehat{G} \left( \frac{\mu_j - \mu_j z}{U(i, \delta)} \right)}{1 - (1 - p_j) \widehat{G} \left( \frac{\mu_j - \mu_j z}{U(i, \delta)} \right)},$$

where

$$\begin{aligned} U(i, \delta) &= 1 - \frac{\lambda p_k - \mu_k}{\lambda p_k} [i \delta \lambda p_k - \binom{i}{2} (\lambda p_k \delta)^2 + \dots \\ &\quad + (-1)^l \binom{i}{l} (\lambda p_k \delta)^l + \dots \\ &\quad + (-1)^i (\lambda p_k \delta)^i]. \end{aligned}$$

Taking  $i = \lfloor \frac{t}{\delta} \rfloor$ , in the limit as  $\delta \rightarrow 0$  we obtain:

$$(3.9) \quad z = \frac{p_j \widehat{G} \left( \frac{\mu_j - \mu_j z}{\bar{q}(t)} \right)}{1 - (1 - p_j) \widehat{G} \left( \frac{\mu_j - \mu_j z}{\bar{q}(t)} \right)},$$

where  $\bar{q}(t) = 1 - \frac{\lambda p_k - \mu_k}{\lambda p_k} (1 - e^{-\lambda p_k t})$ .

Thus, denoting by  $\varphi_j(t)$  the root of the functional equation (3.9), we arrive at the following statement.

**Theorem 3.1.** *The queue-length distribution in the non-bottleneck satellite station  $j < k$  is*

$$(3.10) \quad \begin{aligned} \mathbb{P}\{Q_j(t) = i\} &= \rho_j(t) [\varphi_j(t)]^{i-1} [1 - \varphi_j(t)], \\ i &= 1, 2, \dots, \end{aligned}$$

where

$$\begin{aligned} \rho_j(t) &= \frac{\lambda \bar{q}(t) p_j}{\mu_j}, \\ \bar{q}(t) &= 1 - \frac{\lambda p_k - \mu_k}{\lambda p_k} (1 - e^{-\lambda p_k t}), \end{aligned}$$

and  $\varphi_j(t)$  the root of the functional equation (3.9).

#### 4. CONTINUITY OF THE QUEUE-LENGTH PROCESSES

##### 4.1. Formulation of the main results.

**Theorem 4.1.** *For small positive  $\epsilon$  assume that Condition (1.4) is satisfied. Then, for  $\varphi_j(t)$  in (3.10) the following bounds are true:*

$$\rho_j(t) - \epsilon_1(t) \leq \varphi_j(t) \leq \rho_j(t) + \epsilon_2(t),$$

where

$$\begin{aligned} \epsilon_1(t) &= \min \left\{ \rho_j(t) - \ell_j(t), \frac{2\epsilon}{p_j} [1 - \ell_j(t)] \right\}, \\ \epsilon_2(t) &= \min \left\{ 1 + \frac{[a_j(t)]^2}{b_j(t)} [\ell_j(t) - 1] - \rho_j(t), \frac{2\epsilon}{p_j} [1 - \ell_j(t)] \right\}, \end{aligned}$$

$\ell_j(t)$  is the least root of the equation

$$\begin{aligned} z &= e^{-a_j(t)\mu_j + a_j(t)\mu_j z}, \\ a_j(t) &= \frac{1}{\lambda p_j \bar{q}(t)}, \end{aligned}$$

and

$$b_j(t) = \frac{1}{p_j[\bar{q}(t)]^2} \left( r - \frac{1}{\lambda^2} \right) + \left( \frac{1}{1-p_j} + \frac{1-2p_j}{p_j^2} \right) \frac{1}{\lambda^2[\bar{q}(t)]^2} + \frac{1}{\lambda^2 p_j^2 [\bar{q}(t)]^2}.$$

**Theorem 4.2.** *Under the assumptions made in Theorem 4.1 assume additionally that the probability distribution function  $G(x)$  belongs either to the class NBU or to the class NWU. Then, for  $\varphi_j(t)$  in (3.10) the following bounds are true:*

$$\rho_j(t) - \epsilon_3(t) \leq \varphi_j(t) \leq \rho_j(t) + \epsilon_4(t),$$

where

$$\epsilon_3(t) = \min \left\{ \rho_j(t) - \ell_j(t), \frac{\epsilon}{p_j} [1 - \ell_j(t)] \right\},$$

$$\epsilon_4(t) = \min \left\{ 1 + \frac{[a_j(t)]^2}{b_j(t)} [\ell_j(t) - 1] - \rho_j(t), \frac{\epsilon}{p_j} [1 - \ell_j(t)] \right\},$$

and  $\ell_j(t)$ ,  $a_j(t)$  and  $b_j(t)$  are as in Theorem 4.1.

Note, that the difference between  $\epsilon_1(t)$  (or  $\epsilon_2(t)$ ) in Theorem 4.1 and  $\epsilon_3(t)$  (or  $\epsilon_4(t)$ ) in Theorem 4.2 is only in one term in the min function. While  $\epsilon_1(t)$  (or  $\epsilon_2(t)$ ) contains the term  $\frac{2\epsilon}{p_j} [1 - \ell_j(t)]$ , the corresponding term in  $\epsilon_3(t)$  (or  $\epsilon_4(t)$ ) is  $\frac{\epsilon}{p_j} [1 - \ell_j(t)]$ .

**4.2. Background derivations and the proof of the theorems.** The results of continuity used to prove Theorems 4.1 and 4.2 are based on a recent result obtained in [7]. Recall it. Let  $G_1(x)$  and  $G_2(x)$  be two probability distribution functions belonging to the class  $\mathcal{G}(a, b)$  of probability distributions functions of positive random variables having the first and second moments  $a$  and  $b$  respectively,  $b > a^2$ . Assume that

$$(4.1) \quad \sup_{x>0} |G_1(x) - G_2(x)| < \kappa,$$

where  $\kappa < 1 - \frac{a^2}{b}$ . Let  $\widehat{G}_1(s)$  and  $\widehat{G}_2(s)$  be the corresponding Laplace-Stieltjes transforms of  $G_1(x)$  and  $G_2(x)$  and let  $\gamma_{G_1}$  and  $\gamma_{G_2}$  be the least positive roots of corresponding functional equations  $z = \widehat{G}_1(\mu - \mu z)$  and  $z = \widehat{G}_2(\mu - \mu z)$ , where  $\mu > \frac{1}{a}$  is some real number. Then  $|\gamma_{G_1} - \gamma_{G_2}| < \kappa(1 - \ell)$ , where  $\ell$  is the least root of the equation

$$x = e^{-a\mu + a\mu x}.$$

On the other hand, according to the results of Rolski [23], the guaranteed bounds for any of probability distribution functions  $G_1(x)$  and  $G_2(x)$  (i.e. not necessarily satisfying (4.1)) having the first two moments  $a$  and  $b$  respectively

$$(4.2) \quad \ell \leq \gamma_{G_i} \leq 1 + \frac{a^2}{b}(\ell - 1), \quad i = 1, 2.$$

In our case, (3.9) can be formally rewritten as

$$z = \widehat{H}_j(\mu_j - \mu_j z),$$

where

$$\widehat{H}_j(s) = \frac{p_j \widehat{G}\left(\frac{s}{\overline{q}(t)}\right)}{1 - (1 - p_j) \widehat{G}\left(\frac{s}{\overline{q}(t)}\right)}$$

is the Laplace-Stieltjes transform of a positive random variable having the probability distribution  $H_j(x)$ ,

$$H_j(x) = \sum_{i=1}^{\infty} p_j (1 - p_j)^{i-1} G^{*i}(\overline{q}(t)x),$$

$G^{*i}(x)$  denotes  $i$ -fold convolution of the probability distribution function  $G(x)$  with itself, and  $\overline{q}(t) = 1 - \frac{\lambda p_k - \mu_k}{\lambda p_k} (1 - e^{-\lambda p_k t})$ . Our challenge now is to find the bounds for the least positive root  $\gamma_{H_j}$  of the functional equation in (3.9) similar to those given in (4.2) and then, applying the aforementioned result of [7], find the continuity bounds for non-stationary distributions in non-bottleneck stations. (The notation  $\gamma_{H_j}$  is used here (along with the other notation  $\varphi_j(t)$  in the formulations of Theorems 3.1, 4.1 and 4.2) for consistency with the notation such as  $\gamma_{G_1}$  and  $\gamma_{G_2}$  that introduced before.)

The probability distribution function  $G(x)$  is assumed to have the expectation  $\frac{1}{\lambda}$  and the second moment  $r > \frac{1}{\lambda^2}$ . Let us find the expectation and second moment of the probability distribution  $H_j(x)$ . The best way for deriving these numerical characteristics is to use Wald's identities as follows. Let  $\xi_1, \xi_2, \dots$  be a sequence of independent and identically distributed random variables, let  $\tau$  be an integer random variable independent of the sequence  $\xi_1, \xi_2, \dots$ . Denote  $S_\tau = \xi_1 + \xi_2 + \dots + \xi_\tau$ . Then,

$$(4.3) \quad \mathbb{E}S_\tau = \mathbb{E}\xi_1 \mathbb{E}\tau,$$

$$(4.4) \quad \text{Var}(S_\tau) = \text{Var}(\xi_1) \mathbb{E}\tau + \text{Var}(\tau) (\mathbb{E}\xi_1)^2.$$

In our case, the random variable  $\tau$  is a geometrically distributed random variable,  $\Pr\{\tau = n\} = p_j (1 - p_j)^{n-1}$ ,  $n \geq 1$ . Therefore,

$$\mathbb{E}\tau = \frac{1}{p_j},$$

and

$$\text{Var}(\tau) = \frac{1}{1 - p_j} + \frac{1 - 2p_j}{p_j^2}.$$

The random variable  $\xi_1$  has the probability distribution function  $G(\overline{q}(t)x)$ . Hence,

$$\mathbb{E}\xi_1 = \frac{1}{\lambda \overline{q}(t)},$$

and

$$\text{Var}(\xi_1) = \frac{1}{[\overline{q}(t)]^2} \left( r - \frac{1}{\lambda^2} \right).$$

So, according to Wald's equations (4.3) and (4.4) we obtain:

$$(4.5) \quad \int_0^\infty x dH_j(x) = \frac{1}{\lambda p_j \bar{q}(t)},$$

$$(4.6) \quad \int_0^\infty x^2 dH_j(x) = \frac{1}{p_j [\bar{q}(t)]^2} \left( r - \frac{1}{\lambda^2} \right) + \left( \frac{1}{1-p_j} + \frac{1-2p_j}{p_j^2} \right) \frac{1}{\lambda^2 [\bar{q}(t)]^2} + \frac{1}{\lambda^2 p_j^2 [\bar{q}(t)]^2}.$$

*Proof of Theorem 4.1.* It follows from (4.5) and (4.6) and the aforementioned result by Rolski [23] that  $\gamma_{H_j}$  satisfies the inequalities

$$(4.7) \quad \ell_j(t) \leq \gamma_{H_j} \leq 1 + \frac{[a_j(t)]^2}{b_j(t)} [\ell_j(t) - 1],$$

where  $\ell_j(t)$  is the least root of the equation

$$(4.8) \quad z = e^{-a_j(t)\mu_j + a_j(t)\mu_j z},$$

$$(4.9) \quad a_j(t) = \frac{1}{\lambda p_j \bar{q}(t)},$$

and

$$(4.10) \quad b_j(t) = \frac{1}{p_j [\bar{q}(t)]^2} \left( r - \frac{1}{\lambda^2} \right) + \left( \frac{1}{1-p_j} + \frac{1-2p_j}{p_j^2} \right) \frac{1}{\lambda^2 [\bar{q}(t)]^2} + \frac{1}{\lambda^2 p_j^2 [\bar{q}(t)]^2}.$$

Assume that (1.4) is satisfied. Then according to the characterization theorem of Azlarov and Volodin [9], [6] we have (1.6), and then

$$(4.11) \quad \begin{aligned} \sup_{s \geq 0} \left| \widehat{G} \left( \frac{s}{\bar{q}(t)} \right) - \frac{\lambda \bar{q}(t)}{\lambda \bar{q}(t) + s} \right| &= \sup_{s > 0} \left| \int_0^\infty e^{-sx} dG(\bar{q}(t)x) - \int_0^\infty e^{-sx} d(1 - e^{-\lambda \bar{q}(t)x}) \right| \\ &\leq \sup_{s > 0} \int_0^\infty s e^{-sx} \sup_{x \geq 0} |G(\bar{q}(t)x) - (1 - e^{-\lambda \bar{q}(t)x})| dx \\ &= \sup_{s > 0} \int_0^\infty s e^{-sx} \underbrace{\sup_{x \geq 0} |G(x) - (1 - e^{-\lambda x})|}_{\leq 2\epsilon} dx \\ &\leq 2\epsilon. \end{aligned}$$

In turn, from (4.11) we obtain

$$\begin{aligned}
(4.12) \quad & \sup_{s \geq 0} \left| \widehat{H}_j(s) - \frac{\lambda p_j \bar{q}(t)}{\lambda p_j \bar{q}(t) + s} \right| \\
&= \sup_{s \geq 0} \left| \sum_{i=1}^{\infty} p_j (1-p_j)^{i-1} \left[ \widehat{G}^i \left( \frac{s}{\bar{q}(t)} \right) - \left( \frac{\lambda \bar{q}(t)}{\lambda \bar{q}(t) + s} \right)^i \right] \right| \\
&\leq \sum_{i=1}^{\infty} p_j (1-p_j)^{i-1} \underbrace{\sup_{s \geq 0} \left| \widehat{G}^i \left( \frac{s}{\bar{q}(t)} \right) - \left( \frac{\lambda \bar{q}(t)}{\lambda \bar{q}(t) + s} \right)^i \right|}_{\leq 2i\epsilon} \\
&\leq 2\epsilon \sum_{i=1}^{\infty} i p_j (1-p_j)^{i-1} \\
&= \frac{2\epsilon}{p_j}.
\end{aligned}$$

Then, the results of [7] enables us to conclude as follows. Let us consider the functional equation

$$(4.13) \quad z = \frac{\lambda p_j \bar{q}(t)}{\lambda p_j \bar{q}(t) + \mu_j - \mu_j z}.$$

The right-hand side of (4.13) can be written as  $\widehat{\Pi}(\mu_j - \mu_j z)$ , where  $\Pi(s)$  is the Laplace-Stieltjes transform of exponential distribution with the parameter  $\lambda p_j \bar{q}(t)$

The least positive root of equation (4.13) is

$$\rho_j(t) = \frac{\lambda p_j \bar{q}(t)}{\mu_j}.$$

Apparently,

$$(4.14) \quad \ell_j(t) \leq \gamma_{H_j} \leq 1 + \frac{[a_j(t)]^2}{b_j(t)} [\ell_j(t) - 1],$$

where  $\ell_j(t)$ ,  $a_j(t)$  and  $b_j(t)$  are defined in (4.8)-(4.10). The lower and upper bounds in (4.14) are natural bounds obtained from the results by Rolski [23].

Along with (4.14) we have also the following inequality

$$(4.15) \quad \ell_j(t) \leq \rho_j(t) \leq 1 + \frac{[a_j(t)]^2}{b_j(t)} [\ell_j(t) - 1],$$

which holds true because of Condition (1.2).

On the other hand, following the results in [7] and (4.12),

$$|\gamma_{H_j} - \rho_j(t)| \leq \frac{2\epsilon}{p_j} [1 - \ell_j(t)].$$

So,

$$(4.16) \quad \rho_j(t) - \frac{2\epsilon}{p_j} [1 - \ell_j(t)] \leq \gamma_{H_j} \leq \rho_j(t) + \frac{2\epsilon}{p_j} [1 - \ell_j(t)].$$

Amalgamating (4.14) and (4.16) we arrive at the following bounds:

$$(4.17) \quad \rho_j(t) - \epsilon_1(t) \leq \gamma_{H_j} \leq \rho_j(t) + \epsilon_2(t),$$

where

$$\epsilon_1(t) = \min \left\{ \rho_j(t) - \ell_j(t), \frac{2\epsilon}{p_j} [1 - \ell_j(t)] \right\},$$

and

$$\epsilon_2(t) = \min \left\{ 1 + \frac{[a_j(t)]^2}{b_j(t)} [\ell_j(t) - 1] - \rho_j(t), \frac{2\epsilon}{p_j} [1 - \ell_j(t)] \right\}.$$

The theorem is proved.

*Proof of Theorem 4.2.* The proof of this theorem is similar to that of Theorem 4.1. The only difference that instead of (1.6) given by characterization theorem of Azlarov and Volodin [9], [6] one should use (1.8) from [6]. In this case, instead of estimate (4.11) we arrive at

$$\sup_{s \geq 0} \left| \widehat{G} \left( \frac{s}{\bar{q}(t)} \right) - \frac{\lambda \bar{q}(t)}{\lambda \bar{q}(t) + s} \right| \leq \epsilon,$$

and then instead of (4.12) at

$$\sup_{s \geq 0} \left| \widehat{H}_j(s) - \frac{\lambda p_j \bar{q}(t)}{\lambda p_j \bar{q}(t) + s} \right| \leq \frac{\epsilon}{p_j}.$$

Then instead of (4.16) we have

$$\rho_j(t) - \frac{\epsilon}{p_j} [1 - \ell_j(t)] \leq \gamma_{H_j} \leq \rho_j(t) - \frac{\epsilon}{p_j} [1 - \ell_j(t)],$$

and instead of (4.17) we obtain

$$\rho_j(t) - \epsilon_3(t) \leq \gamma_{H_j} \leq \rho_j(t) + \epsilon_4(t),$$

with

$$\epsilon_3(t) = \min \left\{ \rho_j(t) - \ell_j(t), \frac{\epsilon}{p_j} [1 - \ell_j(t)] \right\},$$

and

$$\epsilon_4(t) = \min \left\{ 1 + \frac{[a_j(t)]^2}{b_j(t)} [\ell_j(t) - 1] - \rho_j(t), \frac{\epsilon}{p_j} [1 - \ell_j(t)] \right\}.$$

The theorem is proved.

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