

# BOUNDS FOR THE LOSS PROBABILITY IN LARGE LOSS QUEUEING SYSTEMS

VYACHESLAV M. ABRAMOV

ABSTRACT. Let  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$  be the class of all probability distribution functions of positive random variables having the given first two moments  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$ . Let  $G_1(x)$  and  $G_2(x)$  be two probability distribution functions of this class satisfying the condition  $|G_1(x) - G_2(x)| < \epsilon$  for some small positive value  $\epsilon$  and let  $\widehat{G}_1(s)$  and, respectively,  $\widehat{G}_2(s)$  denote their Laplace-Stieltjes transforms. For real  $\mu$  satisfying  $\mu \mathfrak{g}_1 > 1$  let us denote by  $\gamma_{G_1}$  and  $\gamma_{G_2}$  the least positive roots of the equations  $z = \widehat{G}_1(\mu - \mu z)$  and  $z = \widehat{G}_2(\mu - \mu z)$  respectively. In the paper, the upper bound for  $|\gamma_{G_1} - \gamma_{G_2}|$  is derived. This upper bound is then used to find lower and upper bounds for the loss probabilities in different large loss queueing systems.

## 1. INTRODUCTION

In most of stochastic models studied analytically in the literature the probability distribution functions of their random characteristics are assumed to be known. In queueing problems, for example, the input characteristics are the distributions of interarrival and service times, and they are clearly described in the formulation of a problem. For example in the case of an  $M/G/1$  queueing system, the arrival process is usually assumed to be Poisson of rate  $\lambda$ , and service time distribution is assumed to be a given function  $B(x)$ , with mean  $1/\mu$  and other moments if required. This enables us to use the techniques of the Laplace-Stieltjes transforms or generating functions to obtain the desired output characteristics.

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1991 *Mathematics Subject Classification.* 60E15, 60K25.

*Key words and phrases.* Classes of probability distributions, Inequalities, Kolmogorov's metric, Loss queueing systems, Empirical distribution function, Continuity of queueing systems.

In practice, however, the distribution of an interarrival or service time is unknown. It can be only *approximated* by available information about that distribution, and the accuracy of that approximation can be obtained by analysis of real observations.

The problems of modeling, approximating and estimating the output characteristics of queueing systems are a well-known, and it is a well-established and distinguished area of queueing theory. There is a wide literature related to this subject. To mention only a few papers that use different approaches, we refer Bareche and Aissani [11], Kalashnikov [12] and van Dijk and Miyazawa [20]. Bareche and Aissani [11] used the strong stability method to study the error of approximation of  $GI/M/1$  or  $M/GI/1$  queueing systems by that  $M/M/1$ , when the distribution of inter-arrival time or, respectively, service time is unknown but in the certain sense (that defined in that paper) close to the exponential distribution. Kalashnikov [12] studied stochastic sequences satisfying the recurrence relation  $V_{n+1} = F(V_n, \xi_n)$ , where  $\xi_n$  was a sequence of independent and identically distributed finite-dimensional random vectors. By replacing the original sequence  $\{\xi_n\}$  by “perturbed” sequence  $\{\xi'_n\}$ , under the assumption that a specially defined weighted distance between  $\xi_n$  and  $\xi'_n$  is given, that weighted distance between  $V_n$  and  $V'_n$ , where  $V'_{n+1} = F(V'_n, \xi'_n)$ , has been studied. Van Dijk and Miyazawa [20] studied non-exponential queues such as  $GI/GI/1/n$ ,  $M/GI/c/n$  and  $GI/M/c/n$ . For the  $GI/GI/1/n$  queueing system they demonstrated the influence of an error in the service time distribution on the resulting error in different performance measures such as the throughput of the system. They also established the error bounds for the throughput of the  $M/GI/c/n$  queue, and obtained similar results for the  $GI/M/c/n$  queue with a perturbation of the interarrival time distribution.

In the present paper, we study the class  $\mathcal{G}(\mathbf{g}_1, \mathbf{g}_2)$  of probability distribution functions of positive random variables having the given first two moments  $\mathbf{g}_1$  and  $\mathbf{g}_2$ . We establish the bounds for the least positive root of the functional equation  $z = \widehat{G}(\mu - \mu z)$ , where  $\widehat{G}(s)$  is the Laplace-Stieltjes transform of an unknown probability distribution function  $G(x)$  belonging to the class  $\mathcal{G}(\mathbf{g}_1, \mathbf{g}_2)$ , and  $\mu$  is a positive parameter satisfying the condition  $\mu \mathbf{g}_1 > 1$ . The additional information

characterizing  $G(x)$  is that

$$(1.1) \quad \mathcal{K}(G, F) := \sup_{x>0} |G(x) - F(x)| < \epsilon,$$

where  $F(x)$  is known probability distribution function of a positive random variable having the same two moments (i.e. belonging to the class  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$  as well), and the least positive root,  $\gamma_F$ , of the functional equation  $z = \widehat{F}(\mu - \mu z)$  is therefore known ( $\widehat{F}(s)$  denotes the Laplace-Stieltjes transform of  $F(x)$ ). The metric  $\mathcal{K}(G, F)$  is known as the uniform (Kolmogorov's) metric (e.g. [13], [16]).

The aforementioned bounds for the least positive root  $\gamma_G$  of the functional equation  $z = \widehat{G}(\mu - \mu z)$  (or similar functional equations) are then used in asymptotic analysis of the loss probability in certain queueing systems with the large number of waiting places.

There are two areas of applications where these bounds are used. They are *statistics* of queueing systems and *continuity* of queueing systems.

In statistical problems, the empirical probability distribution  $G_{\text{emp}}(x, N)$  ( $N$  is the number of observations) is assumed to be known. If the number of observations increases to infinity, then for any given positive value  $\epsilon$  the probability  $\text{P} \left\{ \sup_{x>0} |G_{\text{emp}}(x, N) - G(x)| < \epsilon \right\}$  approaches 1.

More exact information about this probability is given by Kolmogorov's theorem (see Kolmogoroff [15] or Takács [19], p.170). Namely,

$$\lim_{N \rightarrow \infty} \text{P} \left\{ \sup_{x \geq 0} |G_{\text{emp}}(x, N) - G(x)| \leq \frac{z}{\sqrt{N}} \right\} = K(z),$$

where

$$K(z) = \begin{cases} \sum_{j=-\infty}^{+\infty} (-1)^j e^{-2j^2 z^2}, & \text{for } z > 0, \\ 0, & \text{for } z \leq 0. \end{cases}$$

So, the probability of

$$(1.2) \quad \sup_{x \geq 0} |G(x) - G_{\text{emp}}(x, N)| < \epsilon,$$

can be asymptotically evaluated when  $N$  is large and  $\epsilon$  is small. (For relevant studies associated with statistics (1.2) or other related statistics see the book of Takács [19].)

The first and second moments of  $G(x)$  are usually unknown either. However, for large  $N$ , they can be taken approximately to the empirical moments of  $G_{\text{emp}}(x, N)$

with some error. It will be shown below (see Theorem 2.3, rel. (2.31) and Remark 2.4) that if in relation (1.2) the value  $\epsilon$  is small enough, then the bounds for the least positive root  $\gamma_G$  are expressed via the first moment  $\mathfrak{g}_1$  only. In this case the only error of the empirical mean is to be taken into account. Thus, in the motivation of assumption (1.2), the value  $\epsilon$  is assumed to be chosen such small that the probability of (1.2) should be large on the one hand, and the bounds for the empirical mean should be small on the other hand.

In continuity problems, we assume that the unknown probability distribution  $G(x) := \mathbb{P}\{\zeta \leq x\}$  with the expectation  $\mathfrak{g}_1 := \frac{1}{\lambda}$  satisfies some specific properties such as

$$\sup_{x>0, y>0} |G(x) - \mathbb{P}\{\zeta \leq x + y | \zeta > y\}| < \epsilon.$$

Then, according to the known characterization theorem of Azlarov and Volodin (see [10] or [6]), we have

$$\sup_{x>0} |G(x) - (1 - e^{-\lambda x})| < 2\epsilon.$$

For other related continuity problems see [6], where Kolmogorov's metric is used for continuity analysis of the  $M/M/1/n$  queueing system.

The class of probability distributions functions  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$  itself, i.e. without metrical condition (1.1), has been studied by Vasilyev and Kozlov [21] and Rolski [17]. Rolski [17] has established the bounds for the least positive root of the functional equation  $z = \widehat{G}(\mu - \mu z)$ .

In the present paper, we show that additional condition (1.1) nontrivially improves the earlier bounds obtained by Rolski [17]. The new bounds have various applications. For example, the upper and lower asymptotic bounds can be obtain for the loss probabilities in  $M/GI/1/n$ ,  $GI/M/1/n$  and  $GI/M/m/n$  queueing systems with large capacity  $n$  as well as in many related models of telecommunication systems (see [2], [3], [4], [5], [7] and [8]). We demonstrate application of this theory for the  $GI/M/1/n$  queueing system with large buffer capacity  $n$ , for the  $M/GI/1$  buffer system with two types of losses [4] and then for the special buffers model with batch service and priorities [7]. The last two of the mentioned applications have especial importance for telecommunication systems. We also establish new continuity results for the loss probability in the  $M/M/1/n$  queueing systems with large

capacity  $n$  under special assumptions related to interarrival times. The continuity theorems for  $M/M/1/n$  queueing systems, where the buffer capacity  $n$  is fixed, have been established in [6]. Statistical analysis of  $M/GI/1/n$  and  $GI/M/1/n$  loss systems with fixed buffer capacity  $n$  based on Kolmogorov's statistics has been provided in [9].

The paper is structured as follows. In Section 2, properties of distributions belonging to the class  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$  and satisfying additional condition (1.1) are studied. Let  $G_1(x)$  and  $G_2(x)$  be arbitrary probability distribution functions of this class satisfying (1.1), i.e.  $G_1, G_2 \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$ , and  $\mathcal{K}(G_1, G_2) < \epsilon$ . Denote by  $\widehat{G}_1(s)$  and, respectively, by  $\widehat{G}_2(s)$  ( $s \geq 0$ ) their Laplace-Stieltjes transforms. Let  $\gamma_{G_1}$  and  $\gamma_{G_2}$  be the corresponding solutions of the functional equations  $z = \widehat{G}_1(\mu - \mu z)$  and  $z = \widehat{G}_2(\mu - \mu z)$  both belonging to the interval  $(0,1)$ . (Recall that according to the well-known theorem of Takács [18], under the assumption  $\mu \mathfrak{g}_1 > 1$  the least positive roots  $\gamma_{G_1}$  and  $\gamma_{G_2}$  of the equations  $z = \widehat{G}_1(\mu - \mu z)$  and  $z = \widehat{G}_2(\mu - \mu z)$  are unique in the interval  $(0,1)$ .) An upper bound for  $|\gamma_{G_1} - \gamma_{G_2}|$  is obtained in Section 2. In Sections 3 and 4, applications of the results of Section 2 are given for different loss queueing systems. Specifically, in Section 3.1 lower and upper asymptotic bounds are established for loss probabilities in the  $GI/M/1/n$  queueing system as  $n$  increases to infinity; in Section 3.2, bounds for the loss probability in  $M/GI/1$  buffer model with two types of losses, which has been studied in [4], are obtained, and in Section 3.3, bounds for the loss probabilities in the buffers model with priorities, which has been studied in [7], are established. In Section 4, the continuity analysis of the loss probability in the  $M/M/1/n$  queueing system is provided. The continuity analysis of Section 4 is based on the bounds obtained in Section 2, the results for the loss probabilities obtained in Section 3.1 and characterization properties of the exponential distribution.

## 2. PROPERTIES OF PROBABILITY DISTRIBUTION FUNCTIONS OF THE CLASS $\mathcal{G}$

In this section we establish an inequality for  $|\gamma_{G_1} - \gamma_{G_2}|$  for probability distribution functions  $G_1(x)$  and  $G_2(x)$  belonging to the class  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$  and satisfying the condition

$$(2.1) \quad \sup_{x>0} |G_1(x) - G_2(x)| < \epsilon.$$

We start from the known inequalities for probability distribution functions of the class  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$ . Vasilyev and Kozlov [21] proved,

$$(2.2) \quad \inf_{G \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)} \int_0^\infty e^{-sx} dG(x) = e^{-s\mathfrak{g}_1}, \quad s \geq 0$$

and

$$(2.3) \quad \max_{G \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)} \int_0^\infty e^{-sx} dG(x) = 1 - \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2} + \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2} \exp\left(-\frac{\mathfrak{g}_2}{\mathfrak{g}_1} s\right), \quad s \geq 0,$$

where the maximum is obtained for

$$(2.4) \quad G(x) = G_{\max}(x) = \begin{cases} 0, & \text{if } t < 0; \\ 1 - \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}, & \text{if } 0 \leq t < \frac{\mathfrak{g}_2}{\mathfrak{g}_1}; \\ 1, & \text{if } t \geq \frac{\mathfrak{g}_2}{\mathfrak{g}_1}. \end{cases}$$

The lower and upper bounds given by (2.2) and (2.3) are tight. If  $\mathfrak{g}_2 = \mathfrak{g}_1^2$ , then these bounds coincide.

It is pointed out in Rolski [17] that (2.2) and (2.3) could be obtained immediately by the method of reduction to the Tchebycheff system [14] if one takes into account that  $\{1, t, t^2\}$  and  $\{1, t, t^2, e^{-st}\}$  form Tchebycheff systems on  $[0, \infty)$ . Rolski [17] has established as follows. For  $\gamma_G$ , the least positive root of the functional equation  $z = \widehat{G}(\mu - \mu z)$ , it was shown

$$(2.5) \quad \inf_{G \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)} \gamma_G = \ell,$$

and

$$(2.6) \quad \max_{G \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)} \gamma_G = \gamma_{G_{\max}} = 1 + \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}(\ell - 1),$$

where  $\ell$  in (2.5) and (2.6) is the least root of the equation:

$$(2.7) \quad z = e^{-\mu\mathfrak{g}_1 + \mu\mathfrak{g}_1 z}.$$

The proof of (2.5) and (2.6) given in [17] is based on the convexity of the function  $\widehat{G}(\mu - \mu z) - z$ .

From (2.2) and (2.3) we also have as follows. Let  $G_1(x)$  and  $G_2(x)$  be arbitrary probability distribution functions of the class  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$ , and let  $\widehat{G}_1(s)$  and, correspondingly,  $\widehat{G}_2(s)$  be their Laplace-Stieltjes transforms ( $s \geq 0$ ). Then,

$$(2.8) \quad \sup_{G_1, G_2 \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)} \sup_{s \geq 0} |\widehat{G}_1(s) - \widehat{G}_2(s)| = 1 - \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}.$$

Indeed, for the derivative of the difference between the right-hand side of (2.3) and that of (2.2) we have

$$(2.9) \quad \begin{aligned} & \frac{d}{ds} \left[ 1 - \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2} + \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2} \exp\left(-\frac{\mathfrak{g}_2}{\mathfrak{g}_1}s\right) - e^{-s\mathfrak{g}_1} \right] \\ &= \mathfrak{g}_1 \left( \exp(-\mathfrak{g}_1 s) - \exp\left(-\frac{\mathfrak{g}_2}{\mathfrak{g}_1}s\right) \right). \end{aligned}$$

This derivative is equal to zero for  $s = 0$  (minimum) and  $s = +\infty$  (maximum). (The trivial case  $\mathfrak{g}_2 = \mathfrak{g}_1^2$ , leading to the identity to zero of the right-hand side of (2.9) for all  $s \geq 0$ , is not considered.)

Therefore, from (2.9) as well as from (2.2) and (2.3) we arrive at (2.8).

In turn, from (2.5) and (2.6) we have the following inequality for  $|\gamma_{G_1} - \gamma_{G_2}|$ :

$$(2.10) \quad |\gamma_{G_1} - \gamma_{G_2}| \leq 1 + \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}(\ell - 1) - \ell.$$

The inequality (2.10) follows from the results of Rolski [17]. Under additional condition (2.1) we will establish an improved inequality for  $|\gamma_{G_1} - \gamma_{G_2}|$ .

Prior studying the properties of the class of probability distribution functions  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$  under additional condition (2.1), note that inequalities (2.2), (2.3), (2.5), and (2.6) hold true for a wider class of probability distribution functions than  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$ . We will prove that the above inequalities remain correct for the class of probability distribution functions  $\bigcup_{(m_1, m_2) \in \mathcal{M}(\mathfrak{g}_1, \mathfrak{g}_2)} \mathcal{G}(m_1, m_2)$ , where the set of pairs  $\{(m_1, m_2)\}$  contains the pair  $(\mathfrak{g}_1, \mathfrak{g}_2)$  (this set of pairs denoted by  $\mathcal{M}(\mathfrak{g}_1, \mathfrak{g}_2)$  will be defined below).

Let  $\mathfrak{m} > \frac{1}{\mu}$  be such the boundary value, that the least root of the equation

$$z = e^{-\mu\mathfrak{m} + \mu\mathfrak{m}z}$$

is equal to the right-hand side of (2.6), and let  $m_1$  and  $m_2$  are the values satisfying the inequalities  $\mathfrak{m} \leq m_1 \leq \mathfrak{g}_1$ , and  $\frac{m_1^2}{m_2} \geq \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}$  ( $m_1^2 \leq m_2$ ). Then, we have the same bounds (2.2) and (2.3) for the probability distribution functions and (2.5) and (2.6) for the roots  $\gamma_G$  but now for the wider class of probability distribution functions belonging to  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2) \cup \mathcal{G}(m_1, m_2)$ .

Indeed, for any  $m_1$  satisfying the inequality  $\mathfrak{m} \leq m_1 \leq \mathfrak{g}_1$ , and any  $m_2$  for which  $\frac{m_1^2}{m_2} \geq \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}$ , according to [21] we have

$$(2.11) \quad \inf_{\mathcal{G}(m_1, m_2)} \int_0^\infty e^{-sx} dG(x) = e^{-sm_1} \geq e^{-s\mathfrak{g}_1}, \quad s \geq 0,$$

and, taking into account that  $\frac{m_1^2}{m_2} \geq \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}$  and  $m_1 \leq \mathfrak{g}_1$  together lead to  $\frac{m_1}{m_2} \geq \frac{\mathfrak{g}_1}{\mathfrak{g}_2}$ , we also have

$$(2.12) \quad \begin{aligned} \max_{G \in \mathcal{G}(m_1, m_2)} \int_0^\infty e^{-sx} dG(x) &= 1 - \frac{m_1^2}{m_2} + \frac{m_1^2}{m_2} \exp\left(-\frac{m_2}{m_1}s\right) \\ &\leq 1 - \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2} + \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2} \exp\left(-\frac{\mathfrak{g}_2}{\mathfrak{g}_1}s\right), \quad s \geq 0, \end{aligned}$$

where the equality in the right-hand side of the first line of (2.12) is a replacement of the initial probability distribution function (given by (2.4)) by another one, where the parameters  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are correspondingly replaced with  $m_1$  and  $m_2$ .

According to the result of Rolski [17], we respectively have:

$$(2.13) \quad \inf_{G \in \mathcal{G}(m_1, m_2)} \gamma_G = \ell^* \geq \ell,$$

and

$$(2.14) \quad \max_{G \in \mathcal{G}(m_1, m_2)} \gamma_G = 1 + \frac{m_1^2}{m_2}(\ell^* - 1) \leq 1 + \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}(\ell - 1).$$

From the above inequalities of (2.11) - (2.14), one can conclude as follows. Let

$$\mathcal{M}(\mathfrak{g}_1, \mathfrak{g}_2) = \left\{ (m_1, m_2) : \mathfrak{m} \leq m_1 \leq \mathfrak{g}_1; \quad \frac{m_1^2}{m_2} \geq \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}; \quad m_1^2 \leq m_2 \right\}.$$

(Recall that  $\mathfrak{m} > \frac{1}{\mu}$  is such the boundary value that the least root of the equation  $z = e^{-\mu m + \mu m z}$  is equal to the right-hand side of (2.6).) Denote

$$\mathcal{G}(\mathcal{M}) = \bigcup_{(m_1, m_2) \in \mathcal{M}(\mathfrak{g}_1, \mathfrak{g}_2)} \mathcal{G}(m_1, m_2).$$

Then we have the following elementary generalization of (2.5) and (2.6):

$$(2.15) \quad \inf_{G \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)} \gamma_G = \inf_{G \in \mathcal{G}(\mathcal{M})} \gamma_G = \ell,$$

and

$$(2.16) \quad \max_{G \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)} \gamma_G = \max_{G \in \mathcal{G}(\mathcal{M})} \gamma_G = \gamma_{G_{\max}} = 1 + \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}(\ell - 1).$$

Notice that if  $m_1 = \mathfrak{m}$ , then we have  $\ell^* = 1 + \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}(\ell - 1)$ , where  $\ell^*$  is defined by (2.13). On the other hand, according to (2.14) we obtain  $\frac{m_1^2}{m_2} = 1$ , i.e. in this case  $m_2 = \mathfrak{m}^2$ . Thus the set  $\mathcal{M}(\mathfrak{g}_1, \mathfrak{g}_2)$  and, consequently, the class  $\mathcal{G}(\mathcal{M})$  are defined correctly.

We start now to work with (2.1). We have the following elementary property:

$$\begin{aligned}
(2.17) \quad \sup_{s \geq 0} |\widehat{G}_1(s) - \widehat{G}_2(s)| &= \sup_{s > 0} \left| \int_0^\infty e^{-sx} dG_1(x) - \int_0^\infty e^{-sx} dG_2(x) \right| \\
&\leq \sup_{s > 0} \int_0^\infty s e^{-sx} \underbrace{\sup_{y \geq 0} |G_1(y) - G_2(y)|}_{\leq \epsilon \text{ according to (2.1)}} dx \\
&\leq \epsilon.
\end{aligned}$$

Thus under the assumption of (2.1), the difference in absolute value between the Laplace-Stieltjes transforms  $\widehat{G}_1(s)$  and  $\widehat{G}_2(s)$  is not greater than  $\epsilon$ .

It follows from (2.17) that

$$(2.18) \quad \sup_{\substack{G_1, G_2 \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2) \\ \mathcal{K}(G_1, G_2) \leq \epsilon}} \sup_{s \geq 0} |\widehat{G}_1(s) - \widehat{G}_2(s)| = \epsilon_1 \leq \epsilon.$$

(We do not know whether or not the value  $\epsilon_1$  can be found. However, the exact value of  $\epsilon_1$  is not important for our further considerations. Relation (2.18) will be used later in this section.)

On the other hand, according to (2.8) for two arbitrary probability distribution functions of the class  $\mathcal{G}(\mathcal{M})$  the difference in absolute value between their Laplace-Stieltjes transforms is not greater than  $1 - \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}$ . Therefore, if  $\epsilon \geq 1 - \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}$ , then the condition (1.1) is not meaningful. Therefore, it will be assumed in the further consideration that  $\epsilon < 1 - \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}$ .

The lemma below is the statement on the dense of the class  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$ .

**Lemma 2.1.** *For any probability distribution function  $G(x) \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$  ( $\mathfrak{g}_1^2 \neq \mathfrak{g}_2$ ) there exists another probability distribution function  $\widetilde{G}(x) \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$ , which distinguishes from  $G(x)$  at least in one point, such that for any  $\delta > 0$ ,*

$$\mathcal{K}(\widetilde{G}, G) < \delta.$$

*Proof.* Under the assumption that the class  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$  is not trivial, i.e.  $\mathfrak{g}_2 \neq \mathfrak{g}_1^2$ , one can take two distinct probability distribution functions  $G(x)$  and  $F(x)$  of this class. For any  $p \in (0, 1)$ , let  $G_p(x) = pF(x) + (1-p)G(x)$ . Apparently,  $G_p(x) \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$  as well. Therefore, choosing  $p < \frac{\delta}{2}$ , by the triangle inequality we obtain:

$$|G_p(x) - G(x)| = |pF(x) + (1-p)G(x) - G(x)| \leq pF(x) + pG(x) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Hence, for  $p < \frac{\delta}{2}$  one can set  $\widetilde{G}(x) = G_p(x)$ .  $\square$

An extended version of Lemma 2.1 is given in the following lemma.

**Lemma 2.2.** *Let  $(m_1, m_2) \in \mathcal{M}$  and  $(m'_1, m'_2) \in \mathcal{M}$  ( $m_2 \neq m_1^2$ ,  $m'_2 \neq (m'_1)^2$ ), and let  $G(x) \in \mathcal{G}(m_1, m_2)$ . Then for any  $\delta > 0$  there exists a probability distribution function  $\tilde{G}(x) \in \mathcal{G}(m'_1, m'_2)$  such that  $\sup_{x>0} |G(x) - \tilde{G}(x)| < \delta$ .*

*Proof.* Assume that  $m'_2 \geq m_2$ . Then take a probability distribution function  $F(x)$  satisfying the properties:

$$(2.19) \quad \int_{-\infty}^{\infty} x dF(x) = \frac{2m'_1 - (2 - \delta)m_1}{\delta},$$

and

$$(2.20) \quad \int_{-\infty}^{\infty} x^2 dF(x) = \frac{2m'_2 - (2 - \delta)m_2}{\delta}.$$

Then, the probability distribution function

$$(2.21) \quad \tilde{G}(x) = \left(1 - \frac{\delta}{2}\right) G(x) + \frac{\delta}{2} F(x)$$

belongs to the class  $\mathcal{G}(m'_1, m'_2)$ , and, according to the triangle inequality

$$|\tilde{G}(x) - G(x)| \leq \frac{\delta}{2} G(x) + \frac{\delta}{2} F(x) < \delta.$$

In the opposite case where  $m'_2 < m_2$  take the probability distribution function  $F(x)$  satisfying the properties

$$\int_{-\infty}^{\infty} x dF(x) = \frac{2m_1 - (2 - \delta)m'_1}{\delta},$$

and

$$\int_{-\infty}^{\infty} x^2 dF(x) = \frac{2m_2 - (2 - \delta)m'_2}{\delta}.$$

Then, instead of (2.21) we set

$$\tilde{G}(x) = \frac{2G(x)}{2 - \delta} - \frac{\delta}{2} F(x),$$

or

$$G(x) = \left(1 - \frac{\delta}{2}\right) \tilde{G}(x) + \frac{\delta}{2} F(x).$$

Apparently,  $\tilde{G}(x) \in \mathcal{G}(m'_1, m'_2)$ , and, assuming that  $0 < F(x) < 1$  for all  $x > 0$  according to the triangle inequality we obtain:

$$|\tilde{G}(x) - G(x)| \leq \frac{\delta}{2} \tilde{G}(x) + \frac{\delta}{2} F(x) < \delta.$$

The lemma is proved. □

With the aid of Lemmas 2.1 and 2.2 we will solve the following problem. Let  $G_1(x)$  and  $G_2(x)$  be two probability distributions belonging to the class  $\mathcal{G}(\mathcal{M})$ . Under the assumption that  $\sup_{x>0} |G_1(x) - G_2(x)| < \epsilon$  we will find an estimate for the supremum of  $|\gamma_{G_1} - \gamma_{G_2}|$  (the supremum between the corresponding least positive roots of the functional equations  $z = \widehat{G}_1(\mu - \mu z)$  and  $z = \widehat{G}_2(\mu - \mu z)$ .) Solution of this problem, in particular, addresses the case when the probability distribution functions  $G_1(x)$  and  $G_2(x)$  belong to the class  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$ . In our analysis below the estimate of (2.18) is used.

The analysis uses Lemma 2.1. The Laplace-Stieltjes transform  $\widehat{G}_1(s) = e^{-\mathfrak{g}_1 s}$  contains only the parameter  $\mathfrak{g}_1$  and does not contain the second one  $\mathfrak{g}_2$ . Hence similarly to (2.2) one can write

$$(2.22) \quad \inf_{G \in \mathcal{G}(\mathfrak{g}_1, m_2)} \int_0^\infty e^{-sx} dG(x) = \inf_{G \in \mathcal{G}(\mathcal{M})} \int_0^\infty e^{-sx} dG(x) = e^{-s\mathfrak{g}_1}, \quad s \geq 0,$$

where  $m_2$  is a fictive parameter, which is assumed to be unknown. Let us find this unknown parameter  $m_2$  in the Laplace-Stieltjes transform  $\widehat{G}_2(s) = 1 - \frac{\mathfrak{g}_1^2}{m_2} + \frac{\mathfrak{g}_1^2}{m_2} \exp\left(-\frac{m_2}{\mathfrak{g}_1} s\right)$  taking into account that (cf. (2.18))

$$(2.23) \quad \sup_{\substack{G_2 \in \mathcal{G}(\mathfrak{g}_1, m_2) \\ \mathcal{K}(G_1, G_2) \leq \epsilon}} \sup_{s>0} |\widehat{G}_1(s) - \widehat{G}_2(s)| = \epsilon_1.$$

Relation (2.23) holds true, because  $G_1(x) \in \mathcal{G}(\mathcal{M})$ , and according to Lemma 2.2 for any  $\epsilon > 0$  there exists a probability distribution function  $\widetilde{G}_1(x) \in \mathcal{G}(\mathcal{M})$  such that  $|G_1(x) - \widetilde{G}_1(x)| < \epsilon$ . On the other hand, the class of probability distribution functions  $\mathcal{G}(\mathfrak{g}_1, m_2)$  is dense, so  $\widetilde{G}_1(x)$  can be chosen belonging to the same class  $\mathcal{G}(\mathfrak{g}_1, m_2)$  as the probability distribution function  $G_1(x)$ .

The real distance between the Laplace-Stieltjes transforms  $\widehat{G}_1(s)$  and  $\widehat{G}_2(s)$  ( $G_1 \in \mathcal{G}(\mathfrak{g}_1, m_2), G_2 \in \mathcal{G}(\mathfrak{g}_1, m_2)$ ) is

$$(2.24) \quad \sup_{G_1, G_2 \in \mathcal{G}(\mathfrak{g}_1, m_2)} \sup_{s \geq 0} |\widehat{G}_1(s) - \widehat{G}_2(s)| = 1 - \frac{\mathfrak{g}_1^2}{m_2}.$$

(cf. relation (2.8)). Therefore, equating the right hand side of (2.24) to  $\epsilon_1$  we have

$$1 - \frac{\mathfrak{g}_1^2}{m_2} = \epsilon_1,$$

and hence

$$(2.25) \quad m_2 = \frac{\mathfrak{g}_1^2}{1 - \epsilon_1}.$$

The meaning of the parameter  $m_2$  given by (2.25) is as follows. If the distance between two Laplace-Stieltjes transforms is  $\epsilon_1$  in the sense of relation (2.24), then it remains the same for all distributions  $G_1(x)$  and  $G_2(x)$  belonging to the family  $\mathcal{G}(\mathfrak{g}_1, g_2)$ ,  $m_2 \leq g_2 \leq \mathfrak{g}_2$ , where the class  $\mathcal{G}(\mathfrak{g}_1, m_2)$  is a marginal class of this family. In this case (2.23) can be simplified as

$$(2.26) \quad \sup_{\substack{G_2 \in \mathcal{G}(\mathfrak{g}_1, m_2) \\ \mathcal{K}(G_1, G_2) \leq \epsilon}} \sup_{s > 0} |\widehat{G}_1(s) - \widehat{G}_2(s)| = \sup_{G_2 \in \mathcal{G}(\mathfrak{g}_1, m_2)} \sup_{s > 0} |\widehat{G}_1(s) - \widehat{G}_2(s)| = \epsilon_1.$$

Hence, in this case we have the bounds coinciding with the class of all distributions of positive random variables having the moments  $m_1 = \mathfrak{g}_1$  and  $m_2 = \frac{\mathfrak{g}_1^2}{1 - \epsilon_1}$ , i.e. with the class  $\mathcal{G}\left(\mathfrak{g}_1, \frac{\mathfrak{g}_1^2}{1 - \epsilon_1}\right)$ . We also have as follows:

$$(2.27) \quad \begin{aligned} \sup_{G_1, G_2 \in \mathcal{G}\left(\mathfrak{g}_1, \frac{\mathfrak{g}_1^2}{1 - \epsilon_1}\right)} |\gamma_{G_1} - \gamma_{G_2}| &= 1 + \frac{\mathfrak{g}_1^2}{1 - \epsilon_1}(\ell - 1) - \ell \\ &= 1 + (1 - \epsilon_1)(\ell - 1) - \ell \\ &= \epsilon_1 - \epsilon_1 \ell \\ &\leq \epsilon - \epsilon \ell. \end{aligned}$$

Let us consider another case, where  $m_1 = \mathfrak{g}_1 - \delta \geq \mathfrak{m}$ ,  $\delta > 0$ . Let  $\widehat{G}_1(s) = e^{-(\mathfrak{g}_1 - \delta)s}$ , and let  $\widehat{G}_2(s) = 1 - \frac{(\mathfrak{g}_1 - \delta)^2}{m_2} + \frac{(\mathfrak{g}_1 - \delta)^2}{m_2} \exp\left(-\frac{m_2}{\mathfrak{g}_1 - \delta}s\right)$  with an unknown parameter  $m_2$ . In this case,

$$\sup_{\substack{G_2 \in \mathcal{G}(\mathfrak{g}_1 - \delta, m_2) \\ \mathcal{K}(G_1, G_2) \leq \epsilon}} \sup_{s > 0} |\widehat{G}_1(s) - \widehat{G}_2(s)|$$

cannot be greater than  $\epsilon$  (see relations (2.17) and (2.18)).

For example, taking  $m_1 = \mathfrak{m}$  we arrive at  $\widehat{G}_1(s) \equiv \widehat{G}_2(s)$ , and therefore

$$\sup_{\substack{G_2 \in \mathcal{G}(\mathfrak{m}, m_2) \\ \mathcal{K}(G_1, G_2) \leq \epsilon}} \sup_{s > 0} |\widehat{G}_1(s) - \widehat{G}_2(s)| = 0.$$

For an arbitrary choice of  $m_1 = \mathfrak{g}_1 - \delta \geq \mathfrak{m}$ , one have

$$\sup_{\substack{G_2 \in \mathcal{G}(\mathfrak{g}_1 - \delta, m_2) \\ \mathcal{K}(G_1, G_2) \leq \epsilon}} \sup_{s > 0} |\widehat{G}_1(s) - \widehat{G}_2(s)| = \epsilon_2 \leq \epsilon.$$

(The exact value of  $\epsilon_2$  is not important.) In this case, similarly to (2.25)

$$(2.28) \quad m_2 = \frac{(\mathfrak{g}_1 - \delta)^2}{1 - \epsilon_2},$$

and similarly to (2.27),

$$(2.29) \quad \sup_{G_1, G_2 \in \mathcal{G}\left(\mathfrak{g}_1 - \delta, \frac{(\mathfrak{g}_1 - \delta)^2}{1 - \epsilon}\right)} |\gamma_{G_1} - \gamma_{G_2}| = \epsilon_2 - \epsilon_2 \ell^*,$$

where  $\ell^*$  is the solution of the equation  $z = e^{-\mu(\mathfrak{g}_1 - \delta) + \mu(\mathfrak{g}_1 - \delta)z}$ . It is readily seen that  $\ell^* > \ell$ . (The presence of positive  $\delta$  yields the value of the root of functional equation greater compared to the case where  $\delta$  is not presented (i.e.  $\delta = 0$ ).)

Keeping in mind that  $\ell^* > \ell$  and  $\epsilon_2 \leq \epsilon$ , from (2.29) we have:

$$(2.30) \quad \sup_{G_1, G_2 \in \mathcal{G}\left(\mathfrak{g}_1 - \delta, \frac{(\mathfrak{g}_1 - \delta)^2}{1 - \epsilon}\right)} |\gamma_{G_1} - \gamma_{G_2}| \leq \epsilon - \epsilon \ell.$$

Hence, from relations (2.27) and (2.30) we arrive at the following theorem.

**Theorem 2.3.** *For any probability distribution functions  $G_1(x)$  and  $G_2(x)$  belonging to the class  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$  and satisfying condition (2.1) we have as follows.*

*If  $\epsilon < 1 - \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}$ , then*

$$(2.31) \quad \sup_{\substack{G_1, G_2 \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2) \\ \mathcal{K}(G_1, G_2) \leq \epsilon}} |\gamma_{G_1} - \gamma_{G_2}| \leq \epsilon - \epsilon \ell.$$

*Otherwise,*

$$\sup_{\substack{G_1, G_2 \in \mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2) \\ \mathcal{K}(G_1, G_2) \leq \epsilon}} |\gamma_{G_1} - \gamma_{G_2}| = 1 + \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2} (\ell - 1) - \ell,$$

*where  $\ell$  is the least root of the equation*

$$z = e^{-\mu \mathfrak{g}_1 + \mu \mathfrak{g}_1 z}.$$

*Remark 2.4.* Theorem 2.3 is formulated for the class of distributions  $\mathcal{G}(\mathfrak{g}_1, \mathfrak{g}_2)$ . As it was mentioned, the parameters  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are usually unknown. For practical applications, the errors for these parameters should be taken into account. Let us assume that ranges of these parameters are known. For example,  $g_1^{\text{lower}} \leq \mathfrak{g}_1 \leq g_1^{\text{upper}}$  and  $g_2^{\text{lower}} \leq \mathfrak{g}_2 \leq g_2^{\text{upper}}$  are assumed to be satisfied with a given confidence probability  $P$ . It follows from Theorem 2.3 that if  $g_2^{\text{lower}} > (g_1^{\text{upper}})^2$  and  $\epsilon < 1 - \frac{(g_1^{\text{upper}})^2}{g_2^{\text{lower}}}$ , then the lower bound  $\ell$  (the least root of equation (2.7)) for the least positive root  $\gamma_G$ , should be replaced by the smaller value given by the least root of the equation

$$z = e^{-\mu g_1^{\text{upper}} + \mu g_1^{\text{upper}} z}.$$

This new value should replace  $\ell$  in (2.31) to be used in real applications.

For a nontrivial class  $\mathcal{G}(\mathbf{g}_1, \mathbf{g}_2)$ , and large enough volume of observations  $N$ , the above condition  $g_2^{\text{lower}} > (g_1^{\text{upper}})^2$  is natural.

### 3. ASYMPTOTIC BOUNDS FOR CHARACTERISTICS IN LARGE LOSS QUEUEING SYSTEMS

**3.1. Loss probability in the  $GI/M/1/n$  queueing system.** In this section we apply the results of Section 2 to large loss  $GI/M/1/n$  queueing systems (the parameter  $n$  is assumed to be large). The results of this section are elementary. However, they serve as a basis for the analysis of the more realistic queueing systems, which are studied in Sections 3.2 and 3.3. The bounds for the loss probability obtained for this elementary system are then also used for a more delicate continuity analysis of the loss probability in  $M/M/1/n$  queueing systems in Section 4.

Recall the known asymptotic result for the loss probability in the  $GI/M/1/n$  queueing system as  $n \rightarrow \infty$ .

Let  $\widehat{A}(s)$  denote the Laplace-Stieltjes transform of the interarrival time probability distribution function  $A(x)$ , let  $\mu$  denote the reciprocal of the expected service time, let  $\rho$  denote the load,  $\rho = -\frac{1}{\mu \widehat{A}'(0)}$ , which is assumed to be less than 1, and let  $\alpha$  denote the positive least root of the functional equation  $z = \widehat{A}(\mu - \mu z)$ . It has been shown in [3] that, as  $n \rightarrow \infty$ , the loss probability  $P_{\text{loss}}(n)$  is asymptotically represented as follows:

$$(3.1) \quad P_{\text{loss}}(n) = \frac{(1 - \rho)[1 + \mu \widehat{A}'(\mu - \mu\alpha)]\alpha^n}{1 - \rho - \rho[1 + \mu \widehat{A}'(\mu - \mu\alpha)]\alpha^n} + o(\alpha^{2n}).$$

Notice, that the function  $\Psi(z) = \widehat{A}(\mu - \mu z) - z$  is a convex function in variable  $z$ . There are two roots  $z = \alpha$  and  $z = 1$  in the interval  $[0, 1]$ , and  $\Psi'(\alpha) = -\mu \widehat{A}'(\mu - \mu\alpha) - 1 > -1$ . Therefore, according to convexity we have the inequality:

$$(3.2) \quad \Psi'(\alpha) \leq -\frac{\Psi(0)}{\alpha}.$$

From (3.2) we obtain:

$$1 + \mu \widehat{A}'(\mu - \mu\alpha) \geq \frac{\widehat{A}(\mu)}{\alpha},$$

and therefore

$$(3.3) \quad \frac{\widehat{A}(\mu)}{\alpha} \leq 1 + \mu \widehat{A}'(\mu - \mu\alpha) \leq 1.$$

Assume that  $A(x) \in \mathcal{G}(\mathbf{g}_1, \mathbf{g}_2)$  is unknown, but the first two moments  $\mathbf{g}_1$  and  $\mathbf{g}_2$  are given. In this and the following examples we do not discuss the statistical bounds for these moments such as those considered in Remark 2.4. So, all our examples are built on the basis of the moments  $\mathbf{g}_1$  and  $\mathbf{g}_2$  only.

Assume that  $A_{\text{emp}}(x)$  is an empirical probability distribution function of this class, its Laplace-Stieltjes transform is  $\widehat{A}_{\text{emp}}(s)$ , the root of the corresponding functional equation  $z = \widehat{A}_{\text{emp}}(\mu - \mu z)$  is  $\alpha^*$ , and according to available information, Kolmogorov's distance between  $A_{\text{emp}}(x)$  and  $A(x)$  is  $\mathcal{K}(A, A_{\text{emp}}) \leq \epsilon$ .

Consider the case  $\epsilon < 1 - \frac{\mathbf{g}_1^2}{\mathbf{g}_2}$  ( $\mathbf{g}_1^2 \neq \mathbf{g}_2$ ). Since  $A(x)$  is unknown,  $\widehat{A}(s)$  will be replaced by  $\widehat{A}_{\text{emp}}(s)$  in (3.3). The numerator of the left-hand side of (3.3) is replaced by the extremal element  $e^{-\mu \mathbf{g}_1}$ , which is not greater than that original. The corresponding denominator is replaced by  $(\alpha^* + \epsilon - \epsilon \ell)$ , which is not smaller than that original  $\alpha^*$ . Assume that  $\epsilon$  is such small that  $\alpha^* - \epsilon + \epsilon \ell > \ell$  and  $\alpha^* + \epsilon - \epsilon \ell < 1 + \frac{\mathbf{g}_1^2}{\mathbf{g}_2}(\ell - 1)$ . Then we have:

$$(3.4) \quad \frac{e^{-\mu \mathbf{g}_1}}{\alpha^* + \epsilon - \epsilon \ell} \leq 1 + \mu \widehat{A}'_{\text{emp}}(\mu - \mu \alpha^*) \leq 1.$$

Note, that the assumption on  $\epsilon$  under which (3.4) is satisfied can be written as

$$(3.5) \quad \epsilon < \min \left\{ 1 - \frac{\mathbf{g}_1^2}{\mathbf{g}_2}, \frac{\alpha^* - \ell}{1 - \ell}, \frac{\mathbf{g}_2(1 - \alpha^*) - \mathbf{g}_1^2(1 - \ell)}{\mathbf{g}_2(1 - \ell)} \right\}.$$

Using (3.4), in the case of small  $\epsilon$  satisfying (3.5), according to Theorem 2.3 for  $n$  large enough we have the following two inequalities for lower  $\underline{P}(n)$  and upper  $\overline{P}(n)$  levels of the loss probability:

$$(3.6) \quad \underline{P}(n) = \frac{(1 - \rho)e^{-\mu \mathbf{g}_1}(\alpha^* - \epsilon + \epsilon \ell)^n}{(1 - \rho)(\alpha^* + \epsilon - \epsilon \ell) - \rho e^{-\mu \mathbf{g}_1}(\alpha^* - \epsilon + \epsilon \ell)^n},$$

$$(3.7) \quad \overline{P}(n) = \frac{(1 - \rho)(\alpha^* + \epsilon - \epsilon \ell)^n}{1 - \rho - \rho(\alpha^* + \epsilon - \epsilon \ell)^n}.$$

Therefore, for large  $n$  we have the following asymptotic bounds for  $P_{\text{loss}}(n)$ :

$$(3.8) \quad \begin{aligned} & \frac{(1 - \rho)e^{-\mu \mathbf{g}_1}(\alpha^* - \epsilon + \epsilon \ell)^n}{(1 - \rho)(\alpha^* + \epsilon - \epsilon \ell) - \rho e^{-\mu \mathbf{g}_1}(\alpha^* - \epsilon + \epsilon \ell)^n} \\ & \leq P_{\text{loss}}(n) \\ & \leq \frac{(1 - \rho)(\alpha^* + \epsilon - \epsilon \ell)^n}{1 - \rho - \rho(\alpha^* + \epsilon - \epsilon \ell)^n}. \end{aligned}$$

If  $\epsilon \geq 1 - \frac{g_1^2}{g_2^2}$ , then the terms  $(\alpha^* + \epsilon - \epsilon\ell)$  in (3.6), (3.7) and (3.8) should be replaced by these  $\left[1 + \frac{g_1^2}{g_2^2}(\ell - 1)\right]$ , and the terms  $(\alpha^* - \epsilon + \epsilon\ell)$  in (3.6) and (3.8) should be replaced by  $\ell$ .

**3.2. Losses from the  $M/GI/1$  buffer model.** In this section we obtain lower and upper bounds for the loss probability of the following  $M/GI/1$  buffer model [4]. Assume that messages (units) arrive in the buffer of large capacity  $N$ . Units arrive by batches, the sizes of which are independent and identically distributed positive integer random variables  $\nu_i$  with expectation  $c$ . In addition, the random variables  $\nu_i$  are assumed to be bounded, i.e.  $\mathbb{P}\{\nu^{\text{lower}} \leq \nu_i \leq \nu^{\text{upper}}\} = 1$ . Interarrival times of batches are exponentially distributed with parameter  $\lambda$ , and the service (processing) times of these batches are independent and identically distributed random variables with the probability distribution function  $B(x)$  and expectation  $b$ . If upon arrival of a batch the number of units in the system exceeds the buffer capacity, then the entire batch loses from the system. In addition, there is probability  $p$  that an arrival batch of units does not join the system due to transmission error. In all other situations an arrival batch of units joins the system and waits for its processing.

Assume that  $\rho = \lambda b > 1$ . Then, for the lost probability the following representation has been derived in [4] (see relation (5.3) on page 757):

$$(3.9) \quad \pi_N = \frac{p + \rho - 1}{\rho} \cdot \frac{(\rho - 1) + p[1 + \lambda\widehat{B}(\lambda - \lambda\beta)]\mathbb{E}\beta^{\zeta(N)}}{(\rho - 1) + [1 + \lambda\widehat{B}(\lambda - \lambda\beta)]\mathbb{E}\beta^{\zeta(N)}} + o(\mathbb{E}\beta^{\zeta(N)}),$$

where  $\widehat{B}(s)$  denotes the Laplace-Stieltjes transform of the probability distribution function  $B(x)$ ,  $\beta$  is the least in absolute value root of the functional equation  $z = \widehat{B}(\lambda - \lambda z)$ , and

$$\zeta(N) = \sup \left\{ m : \sum_{i=1}^m \nu_i \leq N \right\}.$$

Notice that since  $\mathbb{P}\{\nu^{\text{lower}} \leq \nu_i \leq \nu^{\text{upper}}\} = 1$ , then the similar property for the random variable  $\zeta(N)$  is satisfied:  $\mathbb{P}\{\zeta^{\text{lower}}(N) \leq \zeta(N) \leq \zeta^{\text{upper}}(N)\} = 1$ , and, in addition, since as  $N \rightarrow \infty$

$$\mathbb{P} \left\{ \lim_{N \rightarrow \infty} \frac{\zeta(N)}{N} = \frac{1}{c} \right\} = 1,$$

then, as  $N \rightarrow \infty$ ,

$$(3.10) \quad \mathbb{E}\beta^{\zeta(N)} = \beta^{\frac{N}{c}} [1 + o(1)].$$

Substituting (3.10) into (3.9) we obtain:

$$(3.11) \quad \pi_N = \frac{p + \rho - 1}{\rho} \cdot \frac{(\rho - 1) + p[1 + \lambda\widehat{B}(\lambda - \lambda\beta)]\beta^{\frac{N}{c}}}{(\rho - 1) + [1 + \lambda\widehat{B}(\lambda - \lambda\beta)]\beta^{\frac{N}{c}}} + o(\beta^{\frac{N}{c}}).$$

Similarly to (3.3) for the term  $1 + \lambda\widehat{B}'(\lambda - \lambda\beta)$  we have the inequalities:

$$(3.12) \quad \frac{\widehat{B}(\lambda)}{\beta} \leq 1 + \lambda\widehat{B}'(\lambda - \lambda\beta) \leq 1.$$

Assume now that  $B(x) \in \mathcal{G}_2(\mathfrak{g}_1, \mathfrak{g}_2)$  is unknown, but with the first two moments  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are given, assume that  $B_{\text{emp}}(x)$  is an empirical probability distribution function of this class, its Laplace-Stieltjes transform is  $\widehat{B}_{\text{emp}}(s)$ , the least positive root of the corresponding functional equation  $z = \widehat{B}_{\text{emp}}(\lambda - \lambda z)$  is  $\beta^*$ , and assume that according to an available information the Kolmogorov distance between  $B_{\text{emp}}(x)$  and  $B(x)$  is  $\mathcal{K}(B, B_{\text{emp}}) \leq \epsilon$ . Similarly to (3.5) assume that  $\epsilon$  satisfy the inequality

$$(3.13) \quad \epsilon < \min \left\{ 1 - \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}, \frac{\beta^* - \ell}{1 - \ell}, \frac{\mathfrak{g}_2(1 - \beta^*) - \mathfrak{g}_1^2(1 - \ell)}{\mathfrak{g}_2(1 - \ell)} \right\}.$$

Then similarly to (3.4) we have

$$(3.14) \quad \frac{e^{-\lambda\mathfrak{g}_1}}{\beta^* + \epsilon - \epsilon\ell} \leq 1 + \lambda\widehat{B}'_{\text{emp}}(\lambda - \lambda\beta^*) \leq 1.$$

Under same assumption (3.13), using (3.14) for large  $N$  we arrive at the inequalities for lower,  $\underline{\pi}_N$ , and upper,  $\overline{\pi}_N$ , levels of this loss probability:

$$\begin{aligned} \underline{\pi}_N &\geq \frac{p + \rho - 1}{\rho} \cdot \frac{(\rho - 1)(\beta^* + \epsilon - \epsilon\ell) + pe^{-\lambda\mathfrak{g}_1}(\beta^* - \epsilon + \epsilon\ell)^{\frac{N}{c}}}{(\rho - 1)(\beta^* + \epsilon - \epsilon\ell) + (\beta^* + \epsilon - \epsilon\ell)^{\frac{N}{c}+1}}, \\ \overline{\pi}_N &\leq \frac{p + \rho - 1}{\rho} \cdot \frac{(\rho - 1)(\beta^* + \epsilon - \epsilon\ell) + p(\beta^* + \epsilon - \epsilon\ell)^{\frac{N}{c}+1}}{(\rho - 1)(\beta^* + \epsilon - \epsilon\ell) + e^{-\lambda\mathfrak{g}_1}(\beta^* - \epsilon + \epsilon\ell)^{\frac{N}{c}}}. \end{aligned}$$

Hence, we arrive at the following statement.

**Proposition 3.1.** *Under the above assumptions given in this section for the M/GI/1 buffer model with large parameter  $N$ , in the case*

$$\epsilon < \min \left\{ 1 - \frac{\mathfrak{g}_1^2}{\mathfrak{g}_2}, \frac{\beta^* - \ell}{1 - \ell}, \frac{\mathfrak{g}_2(1 - \beta^*) - \mathfrak{g}_1^2(1 - \ell)}{\mathfrak{g}_2(1 - \ell)} \right\}$$

for the loss probabilities  $\pi_N$  we have the inequalities:

$$\begin{aligned} & \frac{p + \rho - 1}{\rho} \cdot \frac{(\rho - 1)(\beta^* + \epsilon - \epsilon\ell) + pe^{-\lambda\mathfrak{q}_1}(\beta^* - \epsilon + \epsilon\ell)^{\frac{N}{c}}}{(\rho - 1)(\beta^* + \epsilon - \epsilon\ell) + (\beta^* + \epsilon - \epsilon\ell)^{\frac{N}{c}+1}} \\ & \leq \pi_N \\ & \leq \frac{p + \rho - 1}{\rho} \cdot \frac{(\rho - 1)(\beta^* + \epsilon - \epsilon\ell) + p(\beta^* + \epsilon - \epsilon\ell)^{\frac{N}{c}+1}}{(\rho - 1)(\beta^* + \epsilon - \epsilon\ell) + e^{-\lambda\mathfrak{q}_1}(\beta^* - \epsilon + \epsilon\ell)^{\frac{N}{c}}}. \end{aligned}$$

**3.3. The buffers system with priorities.** In this section we study the following special model considered in [7] (see also [8]). This model describes processing messages in priority queueing systems with large buffers, and the effective bandwidth problem. Another interpretation of this system is a transportation system, in which vehicles pass by with some time intervals to pick up  $C$  passengers, at the most, with accordance of their priority status (different types of passenger are supposed to be). The formal description of the problem, given in terms of the buffers system with priorities, is as follows.

Suppose that arrival process of customers in the system is a renewal process  $\mathcal{A}(t)$ , with the expected value of a renewal period  $\frac{1}{\lambda}$ . There are  $l$  types of customers, and there is the probability  $p_j > 0$  that an arriving customer belongs to the type  $j$   $\left(\sum_{j=1}^l p^{(j)} = 1\right)$ . Therefore, the time intervals between arrivals of the type  $j$  customers are independent and identically distributed random variables with expectation  $\frac{1}{\lambda p^{(j)}}$ .

Assume that for  $i < j$ , the customers of the type  $i$  have higher priority than the customers of the type  $j$ , so customers of type 1 are those of the highest priority and customers of the type  $l$  have the lowest priority. Assume that customers leave the system by groups of  $C$  as follows. If the number of customers in the system is not greater than  $C$ , then all (remaining) customers leave the system. Otherwise, if the number of customers in the system exceeds the value  $C$ , then customers leave according to their priority: a higher priority customer has an advantage to leave earlier. For example if  $C = 5$ ,  $l = 3$ , and immediately before departure moment there are three customers of type 1, three customer of type 2 and one customers of type 3 (i.e. seven customers in total), then after the departure there will only remain one customer of type 2 and one customer of type 3 in the system. Times

between departures are assumed to be exponentially distributed with parameter  $\mu$ . Assume that  $\frac{\lambda}{C\mu} < 1$ .

The buffer capacities for the type  $j$  customers is denoted  $N^{(j)}$ . Assume that all of the capacities  $N^{(j)}$ ,  $j = 1, 2, \dots, l$  are large enough, i.e. they are assumed to increase to infinity according to the rule that roughly is explained as follows. For specific numbers  $0 < \alpha_1 < \alpha_2 < \dots < \alpha_l < 1$ , the meaning of which is explained later, it is assumed that for any  $j < k$

$$(3.15) \quad \alpha_j^{N_j} = o\left(\alpha_k^{N_k}\right),$$

where  $N_j := \sum_{i=1}^j N^{(i)}$ ,  $j = 1, 2, \dots, l$ , is the cumulative buffer content of customers of the first  $j$  types.

Let  $p_k := \sum_{j=1}^k p^{(j)}$  be the probability of arrival of a customer of one of the first  $k$  types ( $p_l \equiv 1$ ). Then the times between arrivals of customers, who are related to one of the first  $k$  types,  $k = 1, 2, \dots, l$ , are independent and identically distributed with expectation  $\frac{1}{\lambda p_k}$ .

Since  $\frac{\lambda}{C\mu} < 1$ , then  $\rho_k = \frac{\lambda p_k}{C\mu} < 1$  for all  $k = 1, 2, \dots, l$ . Let  $A_k(x)$  denote the probability distribution function of an interarrival time of the cumulative arrival process generated by customers of the first  $k$  types, and let  $\widehat{A}_k(s)$  ( $s \geq 0$ ) denote the Laplace-Stieltjes transform of  $A_k(x)$ . For the Laplace-Stieltjes transform  $\widehat{A}_k(s)$  we have:

$$(3.16) \quad \begin{aligned} \widehat{A}_k(s) &= \sum_{i=1}^{\infty} p_k (1 - p_k)^{i-1} [\widehat{A}_l(s)]^i \\ &= p_k \widehat{A}_l(s) \frac{1}{1 - (1 - p_k) \widehat{A}_l(s)}. \end{aligned}$$

Let  $\alpha_k$  denote the least positive root of the functional equation

$$(3.17) \quad z = \widehat{A}_k(\mu - \mu z^C)$$

(There is a unique root of this functional equation in the interval  $(0,1)$ , see [7].)

Since  $\rho_1 < \rho_2 < \dots < \rho_l$ , then we also have  $\alpha_1 < \alpha_2 < \dots < \alpha_l$ . It is shown in [7] and [8] that under assumptions (3.15), the loss probability of type  $k$  customers

is given by the asymptotic formula

$$(3.18) \quad \pi_k = \frac{(1 - \rho_k)[1 + C\mu\widehat{A}'_k(\mu - \mu\alpha_k^C)]\alpha_k^{N_k}}{(1 - \rho_k)(1 + \alpha_k + \alpha_k^2 + \dots + \alpha_k^{C-1}) - \rho_k[1 + C\mu\widehat{A}'_k(\mu - \mu\alpha_k^C)]\alpha_k^{N_k}} + o\left(\alpha_k^{2N_k}\right)$$

(The assumption  $\alpha_j^{N_j} = o\left(\alpha_k^{N_k}\right)$ ,  $j < k$ , given in (3.15) actually means that the losses of higher priority customers occur much more rarely compared to the losses of lower priority customers.)

Our task is to find lower and upper bounds for  $\pi_k$ . Note that asymptotic relation (3.18) is similar to that (3.1) of the stationary loss probability in the  $GI/M/1/n$  queueing system with large  $n$ . The functional equation (3.17) is a more general than that considered before in Sections 1 and 2 (that functional equation is a particular case when  $C = 1$ ). For this functional equation, the lower and upper bounds are as follows. Let  $\mathfrak{g}_1 := \frac{1}{\lambda}$  and  $\mathfrak{g}_2$  denote the first and, respectively, the second moments of the probability distribution function  $A_l(x)$ . (In the sequel, the notation  $\mathfrak{g}_1$  is used instead of  $\frac{1}{\lambda}$ .) Then, for  $k = 1, 2, \dots, l$ , from the representation of (3.16) one can obtain the first and second moments of the probability distribution function  $A_k(x)$ :

$$\int_0^\infty x dA_k(x) = \frac{\mathfrak{g}_1}{p_k},$$

and, respectively,

$$\int_0^\infty x^2 dA_k(x) = \frac{2(1 - p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2}{p_k^2}.$$

Furthermore, let  $\ell$  denote the least positive root of the functional equation

$$(3.19) \quad z = \exp\left(-\frac{\mu\mathfrak{g}_1 + \mu\mathfrak{g}_1 z^C}{p_k}\right).$$

(We use the same notation  $\ell$  as it was used for the root of the simpler functional equation in Sections 1 and 2, because the consideration of the more general functional equation (3.19) leads to an elementary extension of the result of Rolski [17] and consequently to elementary extension of the results in Sections 1 and 2). Following this, we have:

$$(3.20) \quad \inf \alpha_k = \inf_{A_k \in \mathcal{G}\left(\frac{\mathfrak{g}_1}{p_k}, \frac{2(1-p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2}{p_k^2}\right)} \alpha_{A_k} = \ell,$$

and

$$(3.21) \quad \sup \alpha_k = \sup_{A_k \in \mathcal{G} \left( \frac{\mathfrak{g}_1}{p_k}, \frac{2(1-p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2}{p_k^2} \right)} \alpha_{A_k} = 1 + \frac{\mathfrak{g}_1^2}{2(1-p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2} (\ell - 1),$$

where  $\alpha_{A_k}$  is the notation for the root of the above functional equation associated with the probability distribution  $A_k(x)$ . (Along with the earlier notation  $\alpha_k$ , this notation is required for our purposes because it is spoken about the upper and lower bounds associated with the class of probability distribution functions defined in (3.20), (3.21) and the equations appearing later in this section.)

Let us assume now that  $\epsilon < 1 - \frac{\mathfrak{g}_1^2}{2(1-p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2}$ . Then according to the modified version of Theorem 2.3 related to this case we have the following:

$$(3.22) \quad \sup_{\substack{A'_k, A''_k \in \mathcal{G} \left( \frac{\mathfrak{g}_1}{p_k}, \frac{2(1-p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2}{p_k^2} \right) \\ \mathcal{K}(A'_k, A''_k) \leq \epsilon}} |\alpha_{A'_k} - \alpha_{A''_k}| \leq \epsilon - \epsilon\ell,$$

where  $\alpha_{A'_k}$  and  $\alpha_{A''_k}$  are the versions of  $\alpha_k$  corresponding the probability distribution functions  $A'_k(x)$  and  $A''_k(x)$  of the class  $\mathcal{G} \left( \frac{\mathfrak{g}_1}{p_k}, \frac{2(1-p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2}{p_k^2} \right)$ .

Similarly to inequality (3.3), we have:

$$(3.23) \quad \frac{\widehat{A}_k(\mu)}{\alpha_k} \leq 1 + C\mu\widehat{A}'_k(\mu - \mu\alpha_k^C) \leq 1,$$

where  $\widehat{A}'_k(\cdot)$  in (3.23) denotes the derivative of  $\widehat{A}_k(\cdot)$ .

Assume now that  $A_k(x) \in \mathcal{G} \left( \frac{\mathfrak{g}_1}{p_k}, \frac{2(1-p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2}{p_k^2} \right)$  is unknown, but the first two moments  $\frac{\mathfrak{g}_1}{p_k}$  and  $\frac{2(1-p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2}{p_k^2}$  are given. Assume that  $A_{\text{emp},k}(x)$  is the empirical probability distribution function corresponding the theoretical probability distribution function  $A_k(x)$ , and the Laplace-Stieltjes transform of  $A_{\text{emp},k}(x)$  is denoted by  $\widehat{A}_{\text{emp},k}(s)$ ,  $s \geq 0$ . Let  $\alpha_k^*$  denote the least positive root of the functional equation  $z = \widehat{A}_{\text{emp},k}(\mu - \mu z^C)$ . Assume also that according to available information, Kolmogorov's distance between  $A_{\text{emp},k}(x)$  and  $A_k(x)$  is  $\mathcal{K}(A_{\text{emp},k}, A_k) \leq \epsilon$ , where similarly to (3.5)  $\epsilon$  is assumed to satisfy the inequality

$$\epsilon < \min \left\{ 1 - \frac{\mathfrak{g}_1^2}{2(1-p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2}, \frac{\alpha_k^* - \ell}{1 - \ell}, \frac{[2(1-p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2](1 - \alpha_k^*) - \mathfrak{g}_1^2(1 - \ell)}{[2(1-p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2](1 - \ell)} \right\}.$$

Then similarly to (3.4) we have

$$(3.24) \quad \frac{\exp\left(-\frac{\mu\mathfrak{g}_1}{p_k}\right)}{\alpha_k^* + \epsilon - \epsilon\ell} \leq 1 + C\mu\widehat{A}'_{\text{emp},k}(\mu - \mu(\alpha_k^*)^C) \leq 1.$$

Therefore, taking into account (3.23) and (3.24) for sufficiently large  $N_k$  we arrive at the following lower (denoted by  $\underline{\pi}_k(N_k)$ ) and upper (denoted by  $\overline{\pi}_k(N_k)$ ) values for probability  $\pi_k$ :

$$\underline{\pi}_k(N_k) = \frac{(1 - \rho_k) \exp\left(-\frac{\mu \mathfrak{g}_1}{p_k}\right) (\alpha_k^* - \epsilon + \epsilon \ell)^{N_k}}{(1 - \rho_k) \sum_{i=0}^C (\alpha_k^* + \epsilon - \epsilon \ell)^i - \rho_k \exp\left(-\frac{\mu \mathfrak{g}_1}{p_k}\right) (\alpha_k^* - \epsilon + \epsilon \ell)^{N_k}},$$

$$\overline{\pi}_k(N_k) = \frac{(1 - \rho_k) (\alpha_k^* + \epsilon - \epsilon \ell)^{N_k}}{(1 - \rho_k) \sum_{i=0}^C (\alpha_k^* - \epsilon + \epsilon \ell)^i - \rho_k (\alpha_k^* + \epsilon - \epsilon \ell)^{N_k}}.$$

Hence, we arrive at the following statement.

**Proposition 3.2.** *Under the above assumptions given in this section, in the case where*

$$\epsilon < \min \left\{ 1 - \frac{\mathfrak{g}_1^2}{2(1 - p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2}, \frac{\alpha_k^* - \ell}{1 - \ell}, \frac{[2(1 - p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2](1 - \alpha_k^*) - \mathfrak{g}_1^2(1 - \ell)}{[2(1 - p_k)\mathfrak{g}_1^2 + p_k\mathfrak{g}_2](1 - \ell)} \right\}.$$

for the loss probabilities  $\pi_k$ ,  $k = 1, 2, \dots, l$ , we have:

$$\begin{aligned} & \frac{(1 - \rho_k) \exp\left(-\frac{\mu \mathfrak{g}_1}{p_k}\right) (\alpha_k^* - \epsilon + \epsilon \ell)^{N_k}}{(1 - \rho_k) \sum_{i=0}^C (\alpha_k^* + \epsilon - \epsilon \ell)^i - \rho_k \exp\left(-\frac{\mu \mathfrak{g}_1}{p_k}\right) (\alpha_k^* - \epsilon + \epsilon \ell)^{N_k}} \\ & \leq \pi_k \leq \frac{(1 - \rho_k) (\alpha_k^* + \epsilon - \epsilon \ell)^{N_k}}{(1 - \rho_k) \sum_{i=0}^C (\alpha_k^* - \epsilon + \epsilon \ell)^i - \rho_k (\alpha_k^* + \epsilon - \epsilon \ell)^{N_k}}. \end{aligned}$$

#### 4. CONTINUITY OF THE LOSS PROBABILITY IN THE $M/M/1/n$ QUEUEING SYSTEM

The results of Section 2 enable us to establish continuity of the  $M/M/1/n$  queueing system when  $n$  is large. The continuity of the  $M/M/1/n$  queueing system was studied in [6]. In contrast to [6] where by continuity of  $M/M/1/n$  queueing system it is meant the continuity of a  $M/GI/1/n$  queueing system, which is close to the  $M/M/1/n$  queueing system, in the present paper by continuity of the  $M/M/1/n$  queueing system it is meant the continuity of a  $GI/M/1/n$  queueing system, which is close to that  $M/M/1/n$  queueing system. Then, in the case when parameter  $n$  is large, the analysis becomes much simpler compared to the case when  $n$  is not assumed to be large. (In [6] Conditions (A) and (B) mentioned below are applied to the probability distribution function of a service time.)

Our assumptions here are similar to those of [6]. Let  $A(x)$  denote probability distribution function of interarrival time, which slightly differs from the exponential distribution  $E_\lambda(x) = 1 - e^{-\lambda x}$  as indicated in the cases below.

- Condition (A). The probability distribution function  $A(x)$  has the representation

$$(4.1) \quad A(x) = pF(x) + (1-p)E_\lambda(x), \quad 0 < p \leq 1,$$

where  $F(x) = \Pr\{\zeta \leq x\}$  is a probability distribution function of a nonnegative random variable having the expectation  $\frac{1}{\lambda}$ , and

$$(4.2) \quad \sup_{x,y \geq 0} |F_y(x) - F(x)| < \epsilon, \quad \epsilon > 0,$$

where  $F_y(x) = \Pr\{\zeta \leq x + y | \zeta > y\}$ . Relation (4.2) says that the distance in Kolmogorov's metric between  $F(x)$  and  $E_\lambda(x)$ , according to the characterization theorem of Azlarov and Volodin [10] (see also [6]), is not greater than  $2\epsilon$ .

- Condition (B). Along with (4.1) and (4.2) it is assumed that  $F(x)$  belongs either to the class NBU or to the class NWU.

Recall that a probability distribution function  $\Xi(x)$  of a nonnegative random variable is said to belong to the class NBU if for all  $x \geq 0$  and  $y \geq 0$  we have  $\bar{\Xi}(x+y) \leq \bar{\Xi}(x)\bar{\Xi}(y)$ , where  $\bar{\Xi}(x) = 1 - \Xi(x)$ . If the opposite inequality holds, i.e.  $\bar{\Xi}(x+y) \geq \bar{\Xi}(x)\bar{\Xi}(y)$ , then  $\Xi(x)$  is said to belong to the class NWU.

Under both of these Conditions (A) and (B) we assume that  $E\zeta^2 < \infty$  is given.

Under Condition (A), we have

$$(4.3) \quad \begin{aligned} \sup_{x>0} |A(x) - E_\lambda(x)| &= \sup_{x>0} |pF(x) - (1-p)E_\lambda(x) - E_\lambda(x)| \\ &= p \sup_{x>0} |F(x) - E_\lambda(x)|. \end{aligned}$$

According to the aforementioned characterization theorem of Azlarov and Volodin,

$$\sup_{x>0} |F(x) - E_\lambda(x)| < 2\epsilon.$$

Therefore, from (4.3) we obtain

$$(4.4) \quad \sup_{x>0} |A(x) - E_\lambda(x)| < 2p\epsilon.$$

We also have:

$$\int_0^\infty x^2 dA(x) = pE\zeta^2 + \frac{2(1-p)}{\lambda^2}.$$

Apparently,  $E\zeta^2 \geq (E\zeta)^2 = \frac{1}{\lambda^2}$ . Denote  $E\zeta^2 = \sigma^2 + \frac{1}{\lambda^2}$ , assuming that  $\sigma^2 > \frac{1}{\lambda^2}$ . Thus, it is assumed that  $E\zeta^2 > \frac{2}{\lambda^2}$ .

Now one can apply the estimate given by (2.31) in Theorem 2.3, to obtain continuity bounds for the loss probability in the case of large  $n$ . In this estimate,  $\ell$  is the least positive root of the equation  $z = \exp\left(-\frac{\mu}{\lambda} + \frac{\mu}{\lambda}z\right)$ . It is not difficult to check that the least positive root of this functional equation is  $\rho = \frac{\lambda}{\mu}$ , and, because of the assumption  $\sigma^2 > \frac{1}{\lambda^2}$ , the value  $\rho$  is within the bounds:

$$(4.5) \quad \ell \leq \rho \leq 1 + \frac{\frac{1}{\lambda^2}}{\sigma^2 + \frac{1}{\lambda^2}}(\ell - 1).$$

So, keeping in mind the relation (3.8) for the loss probability, where in the given case  $\alpha^*$  is replaced by  $\rho$ , we arrive at the following statement.

**Proposition 4.1.** *Under Condition (A) and under the assumption  $\sigma^2 > \frac{1}{\lambda^2}$  the following inequalities for the loss probability, as  $n \rightarrow \infty$ , hold:*

$$\begin{aligned} & \frac{(1-\rho)e^{-\frac{1}{\rho}}(\rho - 2p\epsilon_1 + 2p\epsilon_1\ell)^n}{(1-\rho)(\rho + 2p\epsilon_1 - 2p\epsilon_1\ell) - \rho e^{-\frac{1}{\rho}}(\rho - 2p\epsilon_1 + 2p\epsilon_1\ell)^n} \\ & \leq P_{\text{loss}}(n) \\ & \leq \frac{(1-\rho)(\rho + 2p\epsilon_2 - 2p\epsilon_2\ell)^n}{1 - \rho - \rho(\rho + 2p\epsilon_2 - 2p\epsilon_2\ell)^n}, \end{aligned}$$

where

$$\epsilon_1 = \min\{\rho - \ell, 2p\epsilon(1 - \ell)\},$$

and

$$\epsilon_2 = \min\left\{1 + \frac{1}{1 + \lambda^2\sigma^2}(\ell - 1) - \rho, 2p\epsilon(1 - \ell)\right\}.$$

Under Condition (B) we have (4.3), where under the additional assumption that  $F(x)$  belongs either to the class NBU or to the class NWU one should apply Lemma 3.1 of [6] rather than the characterization theorem of Azlarov and Volodin [10], [6].

In this case we have

$$\sup_{x>0} |F(x) - E_\lambda(x)| < \epsilon.$$

Therefore, from (4.3) we obtain

$$\sup_{x>0} |A(x) - E_\lambda(x)| < p\epsilon.$$

In this case we have the following statement.

**Proposition 4.2.** *Under Condition (B) and under the assumption  $\sigma^2 > \frac{1}{\lambda^2}$  the following inequalities for the loss probability, as  $n \rightarrow \infty$ , hold:*

$$\begin{aligned} & \frac{(1-\rho)e^{-\frac{1}{\rho}}(\rho - p\epsilon_3 + p\epsilon_3\ell)^n}{(1-\rho)(\rho + p\epsilon_3 - p\epsilon_3\ell) - \rho e^{-\frac{1}{\rho}}(\rho - p\epsilon_3 + p\epsilon_3\ell)^n} \\ & \leq P_{\text{loss}}(n) \\ & \leq \frac{(1-\rho)(\rho + p\epsilon_4 - p\epsilon_4\ell)^n}{1 - \rho - \rho(\rho + p\epsilon_4 - p\epsilon_4\ell)^n}, \end{aligned}$$

where

$$\epsilon_3 = \min \{ \rho - \ell, p\epsilon(1 - \ell) \},$$

and

$$\epsilon_4 = \min \left\{ 1 + \frac{1}{1 + \lambda^2 \sigma^2} (\ell - 1) - \rho, p\epsilon(1 - \ell) \right\}.$$

#### ACKNOWLEDGEMENT

The author acknowledges with thanks the support of the Australian Research Council, grant #DP0771338. The author thanks Prof. Kais Hamza for a fruitful discussion of the results and bringing to the attention of the author an alternative way of the proof of Lemmas 2.1 and 2.2.

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SCHOOL OF MATHEMATICAL SCIENCES, MONASH UNIVERSITY, CLAYTON CAMPUS, WELLINGTON ROAD, VICTORIA-3800, AUSTRALIA

*E-mail address:* vyacheslav.abramov@sci.monash.edu.au