

OPTIMAL CONTROL OF A LARGE DAM, TAKING INTO ACCOUNT THE WATER COSTS [NEW EDITION]

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ABSTRACT. This paper studies large dam models where the difference between lower and upper levels L is assumed to be large. Passage across the levels leads to damage, and the damage costs of crossing the lower or upper level are proportional to the large parameter L . Input stream of water is described by compound Poisson process, and the water cost depends upon current level of water in the dam. The aim of the paper is to choose the parameters of output stream (specifically defined in the paper) minimizing the long-run expenses. The particular problem, where input stream is Poisson and water costs are not taken into account has been studied in [Abramov, *J. Appl. Prob.*, 44 (2007), 249-258]. The present paper partially answers the question *How does the structure of water costs affect the optimal solution?* In particular the case of linear costs is studied.

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1. INTRODUCTION

A large dam is defined by the parameters L^{lower} and L^{upper} , which are, respectively, the lower and upper levels of the dam. If the current level is between these bounds, the dam is assumed to be in a normal state. The difference $L = L^{\text{upper}} - L^{\text{lower}}$ is large, and this is the reason for calling the dam *large*. This property enables us to use asymptotic analysis as $L \rightarrow \infty$ and solve easier different problems of optimal control than we would were the dam not large.

Let L_t denote the water level at time t . If $L^{\text{lower}} < L_t \leq L^{\text{upper}}$, then the state of the dam is called *normal*. Passage across lower or upper level leads to damage. The costs per time unit of this damage is $J_1 = j_1 L$ for lower level and, respectively, $J_2 = j_2 L$ for upper level, where j_1 and j_2 are given real constants. The water inflow is described by the compound Poisson process. Namely, the probability generative function of input amount of water (which is assumed to be an integer-valued random variable) in an interval t is given by

$$(1.1) \quad f_t(z) = \exp \left\{ -\lambda t \left(1 - \sum_{i=1}^{\infty} r_i z^i \right) \right\},$$

where r_i is the probability that at a specified moment of Poisson arrival the amount of water will increase by i units. In practice this means that the arrival of water is registered at random instants t_1, t_2, \dots ; the times between consecutive instants are mutually independent and exponentially distributed with parameter λ , and quantities of water (number of water units) of input flow are specified as a quantity i with probability r_i ($r_1 + r_2 + \dots = 1$). Clearly that this assumption is more applicable to real world problems than the assumption of [4] that the arrival of water units is registered by counter at random instants t_1, t_2, \dots , and the times between consecutive instants are mutually independent and exponentially distributed with parameter λ . For example, the assumption made in the present paper enables us to approach a continuous dam model, assuming that the water levels L_t take the discrete values $\{j\Delta\}$, where j is a positive integer and step Δ is a positive small real constant. In the paper, however, the water levels L_t are assumed to be integer-valued.

The outflow of water is state-dependent as follows. If the level of water is between L^{lower} and L^{upper} , then an interval between departures of units of water (inverse

output flow) has the probability distribution function $B_1(x)$. If level of water exceeds L^{upper} , then an inverse output flow has the probability distribution function $B_2(x)$. The probability distribution function $B_2(x)$ is assumed to obey the condition $\int_0^\infty x dB_2(x) < \frac{1}{\lambda}$. If the level of water is L^{lower} exactly, then output of water is frozen, and it resumes again as soon as the level of water exceeds the level L^{lower} . (The exact mathematical formulation of the problem taking into account some specific details is given below.)

Let c_{L_t} denote the cost of water at level L_t . The sequence c_i is assumed to be positive and non-increasing. The problem of the present paper is to choose the parameter $\int_0^\infty x dB_1(x)$ of the dam in the normal state minimizing the objective function

$$(1.2) \quad J = p_1 J_1 + p_2 J_2 + \sum_{i=L^{\text{lower}}+1}^{L^{\text{upper}}} c_i q_i,$$

where

$$(1.3) \quad p_1 = \lim_{t \rightarrow \infty} \Pr\{L_t = L^{\text{lower}}\},$$

$$(1.4) \quad p_2 = \lim_{t \rightarrow \infty} \Pr\{L_t > L^{\text{upper}}\},$$

$$(1.5) \quad q_i = \lim_{t \rightarrow \infty} \Pr\{L_t = L^{\text{lower}} + i\}, \quad i = 1, 2, \dots, L.$$

Usually the level L^{lower} is identified with an empty queue (i.e. $L^{\text{lower}} := 0$ and $L^{\text{upper}} := L$), and the dam model is the following queueing system with service depending on queue-length. If immediately before a service beginning the queue-length exceeds the level L , then the customer is served by the probability distribution function $B_2(x)$. Otherwise, the service time distribution is $B_1(x)$. The value p_1 is the stationary probability of empty system, the value p_2 is the stationary probability that a customer is served by probability distribution $B_2(x)$, and q_i , $i = 1, 2, \dots, L$, are the stationary probabilities of the queue-length process, so $p_1 + p_2 + \sum_{i=1}^L q_i = 1$. (For the described queueing system, the right-hand side limits in relations (1.3)-(1.5) do exist.)

In our study, the parameter L increases unboundedly, and we deal with the series of queueing systems. The parameters above, such as p_1 , p_2 , J_1 , J_2 as well as other parameters are functions of L . The argument L will be often omitted in these functions.

Similarly to [4], it is assumed that the input parameter λ , the probabilities r_1, r_2, \dots and probability distribution function $B_2(x)$ are given, while the appropriate probability function $B_1(x)$ should be chosen from the specified parametric family of functions $B_1(x, C)$. (Actually, we deal with the family of probability distributions $B_1(x)$ depending on two parameters δ and L in series, i.e. $B_1(x, \delta, L)$). Then the parametric family of distributions $B_1(x, C)$ is described in the limiting scheme as $\delta L \rightarrow C$, so the parameter C belongs to the family of possible limits of δL as $\delta \rightarrow 0$ and $L \rightarrow \infty$.

The outflow rate, should be chosen such that to minimize the objective function of (1.2) with respect to the parameter C , which results in choice of the corresponding probability distribution function $B_1(x, C)$ of that family.

More particular problems have been studied in [4] and [5] (see also [7]). The simplest model with Poisson input stream and the objective function having the form $J = p_1 J_1 + p_2 J_2$ (i.e. the water costs are not taken into account), has been studied in [4]. It was shown in [4] that the solution to the control problem is unique and has one of the following three forms. Denote $\rho_2 = \lambda \int_0^\infty x dB_2(x)$ and $\rho_1 = \rho_1(C) = \lambda \int_0^\infty x dB_1(x, C)$. In the case $j_1 = j_2 \frac{\rho_2}{1-\rho_2}$ the optimal solution is $\rho_1 = 1$. In the case $j_1 > j_2 \frac{\rho_2}{1-\rho_2}$ the optimal solution has the form $\rho_1 = 1 + \delta$, where $\delta(L)$ is a small positive parameter, and $\delta(L)L \rightarrow C$ as $L \rightarrow \infty$. In the case $j_1 < j_2 \frac{\rho_2}{1-\rho_2}$, the optimal strategy has the form $\rho_1 = 1 - \delta$, and $\delta(L)L \rightarrow C$ as $L \rightarrow \infty$. The parameter C is a unique solution of a specific minimization problem precisely formulated in [4]. It has been also shown in [4] that the solution to the control problem is insensitive to the type of probability distributions $B_1(x)$ and $B_2(x)$. Specifically, it is expressed via the first moment of $B_2(x)$ and the first two moments of $B_1(x)$. The aforementioned cases fall into the category of heavy traffic analysis in queueing theory. There are many papers related to this subject. For some recent references, we mention the papers of Whitt [23], [24] and books of Chen and Yao [8] and Whitt [22], where a reader can find many other references.

A more general model, taking into account the water costs has been reported in unpublished manuscript [5]. The results of [4] and [5] have also been discussed (without detailed proofs) in review paper [7], where the problem of the present paper has been formulated as future research problem.

Similarly to [4], we use the notation $\rho_{1,l} = \lambda^l \int_0^\infty x^l dB_1(x)$, $l = 2, 3$. The existence of a moment of the order corresponding to $\rho_{1,l}$ will be specially assumed in formulations of statements corresponding to case studies.

Compared to the earlier studies, the solution of the problems in the present paper requires a much deeper and delicate analysis. For example, asymptotic methods of [4] do not longer work, and one should use more delicate techniques instead. Essential difficulty of the control problem in the present formulation is to prove a uniqueness of the optimal solution, while in the case of the particular problem of [4], the uniqueness of the solution followed automatically from the explicit representations of the functionals obtained there.

It is assumed in the present paper that c_i is a not increasing sequence. If the cost sequence c_i were an arbitrary bounded sequence, then a richer class of possible cases could be studied. However, in the case of arbitrary cost sequence, the solution need not be unique, and arbitrary costs c_i , say increasing in i , seem not to be useful and, therefore, are not considered here. Note, that a not increasing sequence c_i depends on L in series. This means that as L changes (increasing to infinity) we have different not increasing sequences (see example in Section 7). Taking c_1 and c_L fixed for all L , then for all L we have not increasing bounded sequences c_i .

More realistic models arising in practice assume that the probability distribution function $B_1(x)$ should also depend on i , i.e have representation $B_{1,i}(x)$. The model of the present paper, where $B_1(x)$ is the same for all i , under appropriate additional information can approximate those more general models. Namely, one can suppose the stationary service time distribution $B_1(x)$ has the representation $B_1(x) = \sum_{i=1}^L q_i B_{1,i}(x)$ (q_i , $i = 1, 2, \dots, L$ are the state probabilities), and the solution to the control problem for $B_1(x)$ enables us to find then the approximate solutions to the control problem for $B_{1,i}(x)$, $i = 1, 2, \dots, L$ by using the Bayes rule. For example, the simplest model can be of the form $B_1(x) = aB_1^*(x) + bB_1^{**}(x)$, where $a := q_i$ and $B_1^*(x) := B_{1,i}(x)$ for $i = 1, 2, \dots, L^0 < L$, and, respectively, $b := q_i$ and $B_1^{**}(x) := B_{1,i}(x)$ for $i = L^0 + 1, L^0 + 2, \dots, L$.

Similarly to the solution of the control problem in [4], the solution of the present problem with extended criteria (1.2) is related to the same class of solutions as in [4]. That is, it must be either $\rho_1 = 1$ or one of two limits of $\rho_1 = 1 + \delta$, $\rho_1 = 1 - \delta$ for

positive small vanishing δ as L increases indefinitely, and $L\delta \rightarrow C$. The reason for this is, that the penalties upon reaching upper or lower level are of order $O(L)$ (i.e. they increase to infinity as $L \rightarrow \infty$ with proportion to L), while the water costs are assumed to be bounded as L tends to infinity, and although the water costs affect the solution of the control problem, this influence remains in the framework of the same class of solutions mentioned above.

The following questions are of special interest here.

1. What is the structure of an optimal solution? Is an optimal solution unique?

The answer on these questions is the main result of the paper. These questions are answered by Theorem 6.4. We prove that a solution to the control problem does exist uniquely, however there are some additional mild assumptions related to the class of probability distributions $\{B_1(x)\}$. The proof of the existence and uniqueness of a solution is based on special techniques of Mathematical Analysis. Specifically, we use the known techniques of majorization inequalities [10], [11] in order to prove the monotonicity of specified functions. This property of monotonicity together with standard theorems of the theory of analytic functions is then used to prove a uniqueness of a solution.

2. Under what relation between j_1 , j_2 , ρ_2 , c_i (and maybe other parameters of the model) the optimal strategy is $\rho_1 = 1$?

For the simplest model studied in [4], the condition for $\rho_1 = 1$ is $j_1 = j_2 \frac{\rho_2}{1-\rho_2}$. This result has a simple intuitive explanation and is a consequence of the well-known property of the stream of lost calls during a busy period of $M/GI/1/n$ queues, under the assumption that the expected interarrival and service times are equal (see Abramov [1] as well as Righter [13] or Wolff [27]). For the same model, taking into account the structure of water costs generally changes this condition for the aforementioned optimal solution $\rho_1 = 1$. Specifically, the optimal solution $\rho_1 = 1$ is achieved under the condition $j_1 \leq j_2 \frac{\rho_2}{1-\rho_2}$, and the equality in this relation holds if and only if the water costs are the same at all levels of water. This result only partially answers the question. More exact answers can be obtained in particular cases, and one of them is the case of linearly decreasing costs as the level of water increases (for brevity, this case is called *linear costs*). In the case of linear costs we derive more exact and useful representations, which enable us

to calculate numerically the relation between j_1 and j_2 to have finally the optimal solution $\rho_1 = 1$. The relevant numerical results are provided for special values of the parameters of the model.

The rest of the paper is organized as follows. In Section 2 the main ideas and methods of asymptotic analysis are given. In Section 2.1, the basic methods related to state dependent queueing system with ordinary Poisson input that have been used in [4] are recalled. In Section 2.2, some extensions of these methods for the state dependent queueing system with compound Poisson input are explained. Specifically, a method of constructing linear representations between characteristics of the system given in a busy period is explained.

In Section 3, the asymptotic behavior of the stationary probabilities is studied. In Section 3.1 known Tauberian theorems that used in asymptotic analysis in the paper are recalled. In Section 3.2, exact formulae for the stationary probabilities p_1 and p_2 are derived. On the basis of these exact formulae, in Sections 3.3 and 3.4 asymptotic theorems for the stationary probabilities p_1 and p_2 have been established.

Section 4 is devoted to asymptotic analysis of the stationary probabilities q_{L-i} , $i = 1, 2, \dots$. In Section 4.1 the explicit representation for the stationary probabilities q_i is derived. On the basis of that explicit representation and Tauberian theorems, in following Sections 4.2, 4.3 and 4.4 asymptotic theorems for these stationary probabilities are established in the cases $\rho_1 = 1$, $\rho_1 = 1 + \delta$ and $\rho_1 = 1 - \delta$ correspondingly, where positive δ is assumed to vanish such that $\delta L \rightarrow C$ as $L \rightarrow \infty$.

In Section 5 the objective function given in 1.2 is studied. In following Sections 5.1, 5.2 and 5.3, asymptotics theorems for this objective function are established in the aforementioned cases $\rho_1 = 1$, $\rho_1 = 1 + \delta$ and $\rho_1 = 1 - \delta$ correspondingly.

In Section 6, a solution to the control problem is discussed. A theorem on existence and uniqueness of a solution is proved.

In Section 7, the case of linear costs is studied and relevant numerical results are given.

2. METHODOLOGY OF ANALYSIS

In this section we describe general ideas that are used in the present paper. We start from the simplest models with Poisson input, and then we explain how these

ideas are developed for more complicated models where input process is compound Poisson.

2.1. State dependent queueing system with Poisson input and its characteristics.

In this section we consider the simplest model in which arrival flow is Poisson with parameter λ . Let T_L denote the length of a busy period of this system, and let $T_L^{(1)}, T_L^{(2)}$ denote the cumulative times spent for service of customers arrived during that busy period with probability distribution functions $B_1(x)$ and $B_2(x)$ correspondingly. For $k = 1, 2$, the expectations of service times will be denoted $\frac{1}{\mu_k} = \int_0^\infty x dB_k(x)$, and $\rho_k = \frac{\lambda}{\mu_k}$. Let $\nu_L, \nu_L^{(1)}$ and $\nu_L^{(2)}$ denote correspondingly the number of served customers during a busy period, and the numbers of those customers served with probability distribution functions $B_1(x)$ and $B_2(x)$. The random variable $T_L^{(1)}$ coincides in distribution with a busy period of the $M/GI/1/L$ queueing system (L is the number of waiting places not including the place in server). The elementary explanation of this fact is based on a property of level crossings and the property of the lack of memory of exponential distribution (e.g. [4]), so the analytic representation for $ET_L^{(1)}$ is the same as this for the expected busy period of the $M/GI/1/L$ queueing system. The recurrence relation for the Laplace-Stieltjes transform and consequently that for the expected busy period of the $M/GI/1/L$ queueing system has been derived by Tomko [21]. So, for $ET_L^{(1)}$ the following recurrence relation is satisfied:

$$(2.1) \quad ET_L^{(1)} = \sum_{i=0}^L ET_{L-i+1}^{(1)} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB_1(x),$$

where $ET_0^{(1)} = \frac{1}{\mu_1}$. (The random variable $T_i^{(1)}$ is defined similarly to that of $T_L^{(1)}$ for the system only having difference in the state parameter i .)

Using the obvious system of equations:

$$(2.2) \quad ET_L = ET_L^{(1)} + ET_L^{(2)},$$

$$(2.3) \quad E\nu_L = E\nu_L^{(1)} + E\nu_L^{(2)},$$

and Wald's equations (see [9], p.384)

$$(2.4) \quad ET_L^{(1)} = \frac{1}{\mu_1} E\nu_L^{(1)},$$

$$(2.5) \quad ET_L^{(2)} = \frac{1}{\mu_2} E\nu_L^{(2)},$$

one can express the quantities ET_L , $E\nu_L$, $ET_L^{(2)}$, $E\nu_L^{(1)}$ and $E\nu_L^{(2)}$ all via $ET_L^{(1)}$ as the linear functions. Indeed, taking into account that the number of arrivals during a busy cycle coincides with the total number of customers served during a busy period we have $\lambda ET_L + 1 = E\nu_L$, which together with (2.2)-(2.5) yields the linear representations required.

For example,

$$E\nu_L^{(2)} = \frac{1}{1 - \rho_2} - \frac{1}{\mu_1} \cdot \frac{1 - \rho_1}{1 - \rho_2} ET_L^{(1)},$$

and

$$ET_L^{(2)} = \frac{\rho_2}{\lambda(1 - \rho_2)} - \frac{\rho_2}{\lambda} \cdot \frac{1 - \rho_1}{1 - \rho_2} ET_L^{(1)}.$$

As a result, the stationary probabilities p_1 and p_2 both are expressed via $E\nu_L^{(1)}$ as follows:

$$p_1 = \frac{1 - \rho_2}{1 + (\rho_1 - \rho_2)E\nu_L^{(1)}},$$

$$p_2 = \frac{\rho_2 + \rho_2(\rho_1 - 1)E\nu_L^{(1)}}{1 + (\rho_1 - \rho_2)E\nu_L^{(1)}}$$

(see Section 2 of [4] for further details). It is interesting to note that the coefficients in linear representation all are insensitive to the probability distribution functions $B_1(x)$ and $B_2(x)$ and are only expressed via parameters such as μ_1 , μ_2 and λ .

Next, the representation for $ET_L^{(1)}$ given by (2.1) falls into the category of convolution type recurrence relations. Asymptotic results as $L \rightarrow \infty$ of the recurrence relations of this type are well-known (see [20], p.22, [12] as well as recent paper [7]). So, asymptotic results can be established for all required characteristics of this queueing system. A more detailed information about this type of recurrence relations and its further applications will be given in Section 3.1.

2.2. State dependent queueing system with compound Poisson input and its characteristics.

Certain characteristics associated with busy periods of the queueing system $M^X/GI/1/L$ have been studied by Rosenlund [14]. Developing the results by Tomko [21], Rosenlund [14] has derived the recurrence relations for the joint Laplace-Stieltjes and z -transform of two-dimensional distributions of a generalized busy period and the number of customers served during that period. In turn, both of these approaches [21] and [14] are based on well-known Takács' method (see [18] or [19]).

For further analysis [14] used matrix-analytic techniques of complex analysis. This type of analysis is very hard and seems cannot be easily adapted for the purposes of the present paper, where a more general model than that from a paper [14] is studied.

In this section we explain how the method of Section 2.1 can be extended, and how the characteristics of the system can be expressed via the similar convolution type recurrence relations.

Notice first, that the linear representations similar to those derived for the state dependent queueing system with ordinary Poisson input are satisfied for the present system as well. Indeed, equations (2.2)-(2.5) all hold in the case of the present queueing system. The only difference is that the relation between ET_L and $E\nu_L$ is

$$(2.6) \quad \lambda E_{\varsigma} E T_L + E_{\varsigma} = E \nu_L,$$

where ς denotes a batch size. (It has the distribution $\Pr\{\varsigma = i\} = r_i$.) This leads to a slight change of linear representations mentioned in Section 2.1. The main difficulty of the immediate extension of the earlier results related to the case of ordinary Poisson arrivals is that the recurrence relation for $ET_L^{(1)}$ (or for the corresponding quantity $E\nu_L^{(1)}$) is not longer satisfied to the recurrence relation of convolution type as (2.1), and asymptotic analysis becomes very hard. So, we should use another type of analysis, which is explained below.

For this system, let \tilde{T}_j , $j = 1, 2, \dots, L$, denote the time interval starting from the moment when there are $L - j + 1$ customers in the system until the moment when there remain $L - j$ customers for the first time since its beginning. Similarly to the notation used in Section 2.1, let us introduce the random variables $\tilde{T}_j^{(1)}$, $\tilde{T}_j^{(2)}$, $\tilde{\nu}_j$, $\tilde{\nu}_j^{(1)}$, $\tilde{\nu}_j^{(2)}$, $j = 1, 2, \dots, L$, which have the same meaning as before. Specifically, \tilde{T}_L is the length a busy period starting from a single customer (1-busy period); $\tilde{\nu}_L$ is the number of customers served during a 1-busy period, and so on.

With the aid of the aforementioned Takács' method [18], [19], one can derive the recurrence relation similar to that of (2.1). Namely,

$$(2.7) \quad E\tilde{T}_L^{(1)} = \sum_{i=0}^L E\tilde{T}_{L-i+1}^{(1)} \int_0^{\infty} \frac{1}{i!} \frac{d^i f_x(z)}{dz^i} \Big|_{z=0} dB_1(x),$$

where $E\tilde{T}_0^{(1)} = \frac{1}{\mu_1}$, and the function $f_x(z)$ is given by (1.1). So, the only difference between (2.1) and (2.7) is in their integrands, and in particular case $r_1 = 1$, $r_i = 0$, $i \geq 2$ we clearly arrive at the same expressions. The explicit results associated with recurrence relation (2.7) will be given later in the paper. Apparently, the similar system of equations as (2.1) - (2.5) are satisfied for the characteristics of the state dependent queueing system $M^X/GI/1$. Namely,

$$(2.8) \quad E\tilde{T}_L = E\tilde{T}_L^{(1)} + E\tilde{T}_L^{(2)},$$

$$(2.9) \quad E\tilde{\nu}_L = E\tilde{\nu}_L^{(1)} + E\tilde{\nu}_L^{(2)},$$

$$(2.10) \quad E\tilde{T}_L^{(1)} = \frac{1}{\mu_1} E\tilde{\nu}_L^{(1)},$$

$$(2.11) \quad E\tilde{T}_L^{(2)} = \frac{1}{\mu_2} E\tilde{\nu}_L^{(2)}.$$

Therefore, the same linear representations via $E\tilde{T}_L^{(1)}$ hold for characteristics of these systems, where by ρ_1 and ρ_2 one now should mean the expected numbers of arrived customers per service time having the probability distribution function $B_1(x)$ and, respectively, $B_2(x)$.

Let us now consider the length of a busy period T_L and associated random variables $T_L^{(1)}$, $T_L^{(2)}$, ν_L , $\nu_L^{(1)}$ and $\nu_L^{(2)}$. Let ς_1 denote a size of batch that starts a busy period. (An integer random variable ς_1 has the distribution $\Pr\{\varsigma = i\} = r_i$.) Then T_L can be represented

$$(2.12) \quad T_L = \sum_{i=1}^{\varsigma_1 \wedge (L+1)} \tilde{T}_{L-i+1} + \sum_{i=1}^{\varsigma_1 - (L+1)} \tilde{T}_{0,i},$$

where 1-busy periods \tilde{T}_{L-i+1} , $i = 1, 2, \dots, L$ are mutually independent;

\tilde{T}_0 denotes a special 1-busy period that starts from a service time having the probability distribution function $B_1(x)$ and all other service times are mutually independent and identically distributed having the probability distribution $B_2(x)$, and the distributions of interarrival times and batch sizes are the same as in the original state dependent queueing system;

$\tilde{T}_{0,i}$, $i = 1, 2, \dots$, is a sequence of independent and identically distributed 1-busy periods of the $M^X/G/1$ queueing system, the service times of which all are independent and identically distributed random variables having the probability

distribution function $B_2(x)$, and the distributions of interarrival times and batch sizes are the same as in the original state dependent queueing system;

$a \wedge b$ denotes $\min\{a, b\}$;

in the case where $\varsigma_1 - (L + 1) \leq 0$, the empty sum in (2.12) is assumed to be zero.

In turn, the representation for $T_L^{(1)}$ is as follows:

$$(2.13) \quad T_L^{(1)} = \sum_{i=1}^{\varsigma_1 \wedge (L+1)} \tilde{T}_{L-i+1}^{(1)},$$

where $\tilde{T}_0^{(1)}$ denotes a single service time having the probability distribution function $B_1(x)$.

Under the assumption that the batch size ς_1 is bounded by L (i.e. instead of ς_1 the other random variable $\varsigma_1 \wedge L$ is considered), the linear representations hold in the special sense which is explained below. That representation will be called *semi-linear*.

Write $E\tilde{T}_L = a + bE\tilde{T}_L^{(1)}$, where a and b are specified constants. Then, by the total probability formula,

$$(2.14) \quad \begin{aligned} EE\{T_L|\varsigma_1 \wedge L\} &= \sum_{i=1}^L \Pr\{\varsigma_1 \wedge L = i\} \sum_{j=1}^i E\tilde{T}_{L-j+1} \\ &= \sum_{i=1}^L \Pr\{\varsigma_1 \wedge L = i\} \sum_{j=1}^i (a + bE\tilde{T}_{L-i+1}^{(1)}) \\ &= a \sum_{i=1}^L i \Pr\{\varsigma_1 \wedge L = i\} + b \sum_{i=1}^L \Pr\{\varsigma_1 \wedge L = i\} \sum_{j=1}^i E\tilde{T}_{L-i+1}^{(1)} \\ &= aE(\varsigma_1 \wedge L) + bEE\{T_L^{(1)}|\varsigma_1 \wedge L\}. \end{aligned}$$

This representation is semi-linear in the sense that only for all $J \geq L$

$$EE\{T_J|\varsigma_1 \wedge L\} = aE(\varsigma_1 \wedge L) + bEE\{T_J^{(1)}|\varsigma_1 \wedge L\}.$$

Apparently the type of semi-linear representation similar to that of (2.14) is satisfied for other characteristics such as $EE\{\nu_L|\varsigma_1 \wedge L\}$, $EE\{T_L^{(2)}|\varsigma_1 \wedge L\}$, $EE\{\nu_L^{(1)}|\varsigma_1 \wedge L\}$ and $EE\{\nu_L^{(2)}|\varsigma_1 \wedge L\}$ all via $EE\{T_L^{(1)}|\varsigma_1 \wedge L\}$. The specific coefficients of these semi-linear representations will be derived in the next section.

In the limiting scheme, as $L \rightarrow \infty$, we obviously have $\lim_{L \rightarrow \infty} \Pr\{\varsigma_1 \wedge L = i\} = \Pr\{\varsigma_1 = i\}$, as well as $\lim_{L \rightarrow \infty} \mathbb{E}\mathbb{E}\{T_L | \varsigma_1 \wedge L\} = \mathbb{E}T_L$ and similarly for the other aforementioned characteristics (where by limit we mean finite or infinite limit). So asymptotic behavior of probabilities p_1 and p_2 as $L \rightarrow \infty$ can be established similarly to that in [4]. Asymptotic analysis of these probabilities as well as a more delicate analysis of stationary probabilities q_i is given in the next sections.

3. ASYMPTOTIC THEOREMS FOR THE STATIONARY PROBABILITIES p_1 AND p_2

In this section, the explicit expressions are derived for the stationary probabilities, and their asymptotic behavior is studied. These results will be used in our further findings of the optimal solution.

3.1. Preliminaries. Recurrence relations (2.1) and (2.7) that have presented in Section 2.1 and, respectively, in Section 2.2 are special cases of the general convolution type recurrence relation

$$(3.1) \quad Q_n = \sum_{j=0}^n Q_{n-j+1} f_j,$$

with $f_0 > 0$, $f_j \geq 0$ for all $j \geq 1$, and $f_0 + f_1 + \dots = 1$, and $Q_0 \neq 0$. The detailed theory of these recurrence relations can be found in Takács [20]. For the generating function $Q(z) = \sum_{j=0}^{\infty} Q_j z^j$, $|z| \leq 1$ we have

$$(3.2) \quad Q(z) = \frac{Q_0 F(z)}{F(z) - z},$$

where $F(z) = \sum_{j=0}^{\infty} f_j z^j$.

Asymptotic behavior of Q_n as $n \rightarrow \infty$ has been studied by Takács [20] and Postnikov [12]. Recall the theorems that we need in this paper.

Denote $\gamma_m = \lim_{z \uparrow 1} \frac{d^m F(z)}{dz^m}$.

Lemma 3.1. (Takács [20], p.22-23). *If $\gamma_1 < 1$ then*

$$(3.3) \quad \lim_{n \rightarrow \infty} Q_n = \frac{Q_0}{1 - \gamma_1}.$$

If $\gamma_1 = 1$ and $\gamma_2 < \infty$, then

$$\lim_{n \rightarrow \infty} \frac{Q_n}{n} = \frac{2Q_0}{\gamma_2}.$$

If $\gamma_1 > 1$, then

$$(3.4) \quad \lim_{n \rightarrow \infty} \left(Q_n - \frac{Q_0}{\delta^n [1 - F'(\delta)]} \right) = \frac{Q_0}{1 - \gamma_1},$$

where δ is the least in absolute value root of the functional equation $z = F(z)$.

Lemma 3.2. (Postnikov [12], Sect.25). Let $\gamma_1 = 1$, $\gamma_2 < \infty$ and $f_0 + f_1 < 1$. Then, as $n \rightarrow \infty$,

$$(3.5) \quad Q_{n+1} - Q_n = \frac{2Q_0}{\gamma_2} + o(1).$$

3.2. Exact formulae for p_1 and p_2 . In this section we derive exact representations for p_1 and p_2 that expressed via $E\nu_L^{(1)}$. We also obtain some preliminary asymptotic representations that easily follow from explicit results and then will be used in the sequel.

We first start from the linear representations for $E\tilde{\nu}_L^{(2)}$ in terms $E\tilde{\nu}_L^{(1)}$, which will be substantially used later. Namely, we have the following lemma.

Lemma 3.3. For $E\tilde{\nu}_L^{(2)}$ for any $L \geq 1$ the following representation

$$(3.6) \quad E\tilde{\nu}_L^{(2)} = \frac{1}{1 - \rho_2} - \frac{1 - \rho_1}{1 - \rho_2} E\tilde{\nu}_L^{(1)}$$

is satisfied, where $\rho_1 = \frac{\lambda E\zeta}{\mu_1}$ and $\rho_2 = \frac{\lambda E\zeta}{\mu_2} < 1$, and $E\tilde{\nu}_L^{(1)}$ is given by

$$(3.7) \quad E\tilde{\nu}_L^{(1)} = \sum_{i=0}^L E\tilde{\nu}_{L-i+1}^{(1)} \int_0^\infty \frac{1}{i!} \left. \frac{d^i f_x(z)}{dz^i} \right|_{z=0} dB_1(x),$$

$$E\tilde{\nu}_0^{(1)} = 1.$$

Proof. Taking into account that the number of arrivals during 1-busy cycle (1-busy period plus idle period) coincides with the number of customers served during the same 1-busy period, according to Wald's identity we have:

$$\lambda \left(E\tilde{T}_L + \frac{1}{\lambda} \right) = \lambda E\tilde{T}_L + 1 = E\tilde{\nu}_L = E\tilde{\nu}_L^{(1)} + E\tilde{\nu}_L^{(2)}.$$

This equality together with (2.8)-(2.11) yields the desired statement of the lemma, where (3.7) in turn follows from (2.7) and (2.10). \square

The next step is to derive representations for $EE\{\nu_L^{(1)} | \varsigma_1 \wedge L\}$ and $EE\{\nu_L^{(2)} | \varsigma_1 \wedge L\}$. We have the following lemma.

Lemma 3.4. For $EE\{\nu_L^{(2)} | \varsigma_1 \wedge L\}$ we have

$$(3.8) \quad EE\{\nu_L^{(2)} | \varsigma_1 \wedge L\} = \frac{E(\varsigma_1 \wedge L)}{1 - \rho_2} - \frac{1 - \rho_1}{1 - \rho_2} EE\{\nu_L^{(1)} | \varsigma_1 \wedge L\},$$

where

$$\mathbb{E}\mathbb{E}\{\nu_L^{(1)} | \varsigma_1 \wedge L\} = \sum_{i=1}^L \Pr\{\varsigma_1 \wedge L = i\} \sum_{j=1}^i \mathbb{E}\tilde{\nu}_{L-j+1}^{(1)},$$

and $\mathbb{E}\tilde{\nu}_{L-j+1}^{(1)}$, $j = 1, 2, \dots, L$, are given by (3.7).

Proof. Following the same arguments as in (2.14), one can write

$$\mathbb{E}\mathbb{E}\{\nu_L^{(2)} | \varsigma_1 \wedge L\} = a\mathbb{E}(\varsigma_1 \wedge L) + b\mathbb{E}\mathbb{E}\{\nu_L^{(1)} | \varsigma_1 \wedge L\}$$

for specified constants a and b for which the linear representation $\mathbb{E}\tilde{\nu}_L^{(2)} = a + b\mathbb{E}\tilde{\nu}_L^{(1)}$ is satisfied. Therefore, the statement of Lemma 3.4 follows immediately from that of Lemma 3.3. \square

The following lemma yields exact estimates for the difference $\mathbb{E}\nu_L^{(1)} - \mathbb{E}\mathbb{E}\{\nu_L^{(1)} | \varsigma_1 \wedge L\}$.

Lemma 3.5. *We have the following estimate:*

$$(3.9) \quad \mathbb{E}\nu_L^{(1)} - \mathbb{E}\mathbb{E}\{\nu_L^{(1)} | \varsigma_1 \wedge L\} = \Pr\{\varsigma_1 > L\},$$

Proof. Similarly to (2.13) we have

$$\nu_L^{(1)} = \sum_{i=1}^{\varsigma_1 \wedge (L+1)} \tilde{\nu}_{L-i+1}^{(1)},$$

where $\tilde{\nu}_{L-i+1}^{(1)}$, $i = 1, 2, \dots, L$ are mutually independent, and $\tilde{\nu}_0^{(1)} = 1$. Therefore,

$$(3.10) \quad \mathbb{E}\nu_L^{(1)} = \sum_{i=1}^{L+1} \Pr\{\varsigma_1 \wedge (L+1) = i\} \sum_{j=1}^i \mathbb{E}\tilde{\nu}_{L-j+1}^{(1)}.$$

In turn, the representation for $\mathbb{E}\mathbb{E}\{\nu_L^{(1)} | \varsigma_1 \wedge L\}$ is

$$(3.11) \quad \mathbb{E}\mathbb{E}\{\nu_L^{(1)} | \varsigma_1 \wedge L\} = \sum_{i=1}^L \Pr\{\varsigma_1 \wedge L = i\} \sum_{j=1}^i \mathbb{E}\tilde{\nu}_{L-j+1}^{(1)}.$$

Subtracting (3.11) from (3.10) we obtain:

$$\begin{aligned} \mathbb{E}\nu_L^{(1)} - \mathbb{E}\mathbb{E}\{\nu_L^{(1)} | \varsigma_1 \wedge L\} &= \Pr\{\varsigma_1 = L\} \sum_{j=1}^L \mathbb{E}\tilde{\nu}_j^{(1)} + \Pr\{\varsigma_1 > L\} \sum_{j=0}^L \mathbb{E}\tilde{\nu}_j^{(1)} \\ &\quad - \Pr\{\varsigma_1 \geq L\} \sum_{j=1}^L \mathbb{E}\tilde{\nu}_j^{(1)} \\ &= \Pr\{\varsigma_1 > L\}. \end{aligned}$$

Relation (3.9) is proved. \square

From Lemma 3.5 we have the following important corollary.

Corollary 3.6. *As $L \rightarrow \infty$,*

$$(3.12) \quad E\nu_L^{(1)} - EE\{\nu_L^{(1)} | \zeta_1 \wedge L\} = o(1),$$

and

$$(3.13) \quad E\nu_L^{(2)} - EE\{\nu_L^{(2)} | \zeta_1 \wedge L\} = o(1).$$

Proof. Asymptotic relation (3.12) follows immediately from (3.9). In order to show (3.13) let us first derive a linear representation of $E\nu_N^{(2)}$ via $E\nu_N^{(1)}$. From relation (2.6) and equations (2.2)-(2.5) in Section 2.1, which also hold true in the case of the present queueing system with batch arrivals, we obtain:

$$(3.14) \quad E\nu_L^{(2)} = \frac{E\zeta}{1 - \rho_2} - \frac{1 - \rho_1}{1 - \rho_2} E\nu_L^{(1)}.$$

On the other hand, for $EE\{\nu_L^{(2)} | \zeta_1 \wedge L\}$ representation (3.8) holds. Therefore, comparing the terms of (3.14) and (3.8) and taking into account (3.12) we easily arrive at asymptotic relation (3.13). Lemma 3.6 is proved. \square

The following lemma presents exact formulae for the stationary probabilities p_1 and p_2 in terms of $E\nu_L^{(1)}$.

Lemma 3.7. *We have:*

$$(3.15) \quad p_1 = \frac{(1 - \rho_2)E\zeta}{E\zeta + (\rho_1 - \rho_2)E\nu_L^{(1)}},$$

and

$$(3.16) \quad p_2 = \frac{\rho_2 E\zeta + \rho_2(\rho_1 - 1)E\nu_L^{(1)}}{E\zeta + (\rho_1 - \rho_2)E\nu_L^{(1)}}.$$

Proof. Using renewal arguments (e.g. [15]) and relation (2.6), we have:

$$(3.17) \quad p_1 = \frac{\frac{1}{\lambda}}{ET_L^{(1)} + ET_L^{(2)} + \frac{1}{\lambda}} = \frac{E\zeta}{E\nu_L^{(1)} + E\nu_L^{(2)}}$$

and

$$(3.18) \quad p_2 = \frac{ET_L^{(2)}}{ET_L^{(1)} + ET_L^{(2)} + \frac{1}{\lambda}} = \frac{\rho_2 E\nu_L^{(2)}}{E\nu_L^{(1)} + E\nu_L^{(2)}}.$$

Now, substituting (3.14) for the right sides of (3.17) and (3.18) we obtain relations (3.15) and (3.16) of this lemma. \square

3.3. Asymptotic theorems for p_1 and p_2 under usual assumptions. The main result of Section 3.2 is Lemma 3.7, where the stationary probabilities p_1 and p_2 are expressed explicitly via $E\nu_L^{(1)}$. The aim of this section is to obtain statements on asymptotic behavior of p_1 and p_2 as $L \rightarrow \infty$ under different assumptions on ρ_1 such as $\rho_1 < 1$, $\rho_1 = 1$ and $\rho_1 > 1$. That is, the aim is to obtain the analogue of asymptotic Theorem 3.1 of [4]. To this end, we will derive an asymptotic representation for $EE\{\nu_L^{(1)} | \varsigma_1 \wedge L\}$ as $L \rightarrow \infty$.

Let us first study asymptotic behavior of $E\tilde{\nu}_L^{(1)}$ as $L \rightarrow \infty$. For this purpose derive the representation for the generating function $\sum_{j=0}^{\infty} E\tilde{\nu}_j^{(1)} u^j$. Using representation (3.7), we have (see relation (3.2)):

$$(3.19) \quad \begin{aligned} \sum_{j=0}^{\infty} E\tilde{\nu}_j^{(1)} u^j &= \sum_{j=0}^{\infty} u^j \sum_{i=0}^j E\tilde{\nu}_{L-i+1}^{(1)} \int_0^{\infty} \frac{1}{i!} \frac{d^i f_x(z)}{dz^i} \Big|_{z=0} dB_1(x) \\ &= \frac{U(z)}{U(z) - z}, \end{aligned}$$

where

$$(3.20) \quad \begin{aligned} U(z) &= \int_0^{\infty} \exp \left\{ -\lambda x \left(1 - \sum_{i=1}^{\infty} r_i z^i \right) \right\} dB_1(x) \\ &= \widehat{B}_1(\lambda - \lambda \widehat{R}(z)). \end{aligned}$$

($\widehat{B}_1(s)$ denotes the Laplace-Stieltjes transform of $B_1(x)$ ($\Re(s) \geq 0$), and $\widehat{R}(z) = \sum_{i=1}^{\infty} r_i z^i$, $|z| \leq 1$.) Therefore, from (3.20) and (3.19) we obtain:

$$(3.21) \quad \sum_{j=0}^{\infty} E\tilde{\nu}_j^{(1)} z^j = \frac{\widehat{B}_1(\lambda - \lambda \widehat{R}(z))}{\widehat{B}_1(\lambda - \lambda \widehat{R}(z)) - z}.$$

According to Lemmas 3.1 and 3.2, the asymptotic behavior of $E\nu_L^{(1)}$, as $L \rightarrow \infty$, is given by the following statements.

Lemma 3.8. *If $\rho_1 < 1$, then*

$$(3.22) \quad \lim_{L \rightarrow \infty} E\tilde{\nu}_L^{(1)} = \frac{1}{1 - \rho_1}.$$

If $\rho_1 = 1$, and additionally $\rho_{1,2} < \infty$ and $E\varsigma^2 < \infty$, then

$$(3.23) \quad E\tilde{\nu}_L^{(1)} - E\tilde{\nu}_{L-1}^{(1)} = \frac{2}{(\rho_{1,2} - 1)E\varsigma + E\varsigma^2} + o(1).$$

If $\rho_1 > 1$, then

$$(3.24) \quad \lim_{L \rightarrow \infty} \left[\mathbb{E} \tilde{\nu}_L^{(1)} - \frac{1}{\varphi^L [1 + \lambda \widehat{B}'_1(\lambda - \lambda \widehat{R}(\varphi)) \widehat{R}'(\varphi)]} \right] = \frac{1}{1 - \rho_1},$$

where φ is the root of the functional equation $z = \widehat{B}_1(\lambda - \lambda \widehat{R}(z))$ that is least in absolute value.

Proof. Asymptotic relations (3.22) and (3.24) follow by application of those (3.3) and (3.4) respectively of Lemma 3.1.

In order to prove asymptotic relation (3.23) one should apply the Tauberian theorem of Postnikov (Lemma 3.2). Then asymptotic relation (3.23) will follow from (3.5) if we prove that the Tauberian condition $f_0 + f_1 < 1$ of Lemma 3.2 is satisfied. (For the proofs of similar statements see [2], [3] and [7].) In the case of the present model, we must prove that for some $\lambda_0 > 0$ the equality

$$(3.25) \quad \int_0^\infty e^{-\lambda_0 x} (1 + \lambda_0 r_1 x) dB_1(x) = 1$$

is not the case. Without loss of generality r_1 in (3.25) can be set to be equal to 1, since

$$\int_0^\infty e^{-\lambda_0 x} (1 + \lambda_0 r_1 x) dB_1(x) \leq \int_0^\infty e^{-\lambda_0 x} (1 + \lambda_0 x) dB_1(x).$$

Thus, we should prove the inequality

$$\int_0^\infty e^{-\lambda x} (1 + \lambda x) dB_1(x) < 1.$$

Indeed, $\int_0^\infty e^{-\lambda x} (1 + \lambda x) dB_1(x)$ is an analytic function in λ , and therefore, according to the theorem on the maximum module of an analytic function, equality (3.25) where $r_1 = 1$ must hold for *all* $\lambda_0 \geq 0$. This means that (3.25) is valid if and only if

$$\int_0^\infty e^{-\lambda_0 x} \frac{(\lambda_0 x)^i}{i!} dB_1(x) = 0$$

for all $i \geq 2$ and $\lambda_0 \geq 0$. In this case the Laplace-Stieltjes transform $\widehat{B}_1(\lambda)$ must be a linear function in λ , i.e. $\widehat{B}_1(\lambda) = d_0 + d_1 \lambda$, d_0 and d_1 are some constants. However, since $|\widehat{B}_1(\lambda)| \leq 1$, we have $d_0 = 1$ and $d_1 = 0$. This is a trivial case where $B_1(x)$ is concentrated in point 0, and therefore it is not a probability distribution function having a positive mean. Thus (3.25) is not the case, and the aforementioned Tauberian conditions are satisfied. The lemma is proved. \square

With the aid of Lemma 3.8 one can easily obtain the statements on asymptotic behavior of $\mathbb{E}\nu_L^{(1)}$, $\mathbb{E}\mathbb{E}\{\nu_L^{(1)}|\varsigma \wedge L\}$ and, consequently, p_1 and p_2 . The theorem below characterizes asymptotic behavior of the probabilities p_1 and p_2 as $L \rightarrow \infty$.

Theorem 3.9. *If $\rho_1 < 1$, then*

$$(3.26) \quad \lim_{L \rightarrow \infty} p_1(L) = 1 - \rho_1,$$

$$(3.27) \quad \lim_{L \rightarrow \infty} p_2(L) = 0.$$

If $\rho_1 = 1$, and additionally $\rho_{1,2} < \infty$ and $\mathbb{E}\varsigma^2 < \infty$, then

$$(3.28) \quad \lim_{L \rightarrow \infty} Lp_1(L) = \frac{(\rho_{1,2} - 1)\mathbb{E}\varsigma + \mathbb{E}\varsigma^2}{2},$$

$$(3.29) \quad \lim_{L \rightarrow \infty} Lp_2(L) = \frac{\rho_2}{1 - \rho_2} \frac{(\rho_{1,2} - 1)\mathbb{E}\varsigma + \mathbb{E}\varsigma^2}{2}.$$

If $\rho_1 > 1$, then

$$(3.30) \quad \lim_{L \rightarrow \infty} \frac{p_1(L)}{\varphi^L} = \frac{(1 - \rho_2)[1 + \lambda \widehat{B}'_1(\lambda - \lambda \widehat{R}(\varphi))\widehat{R}'(\varphi)](1 - \varphi)\mathbb{E}\varsigma}{(\rho_1 - \rho_2)[1 - \widehat{R}(\varphi)]},$$

$$(3.31) \quad \lim_{L \rightarrow \infty} p_2(L) = \frac{\rho_2(\rho_1 - 1)}{\rho_1 - \rho_2},$$

where φ is defined in the formulation of Lemma 3.8.

Proof. Let us first find asymptotic representation for $\mathbb{E}\mathbb{E}\{\nu_L^{(1)}|\varsigma_1 \wedge L\}$ as $L \rightarrow \infty$. According to Lemma 3.8 and explicit representation (3.11) we obtain as follows.

If $\rho_1 < 1$, then

$$(3.32) \quad \begin{aligned} \lim_{L \rightarrow \infty} \mathbb{E}\mathbb{E}\{\nu_L^{(1)}|\varsigma_1 \wedge L\} &= \frac{1}{1 - \rho_1} \lim_{L \rightarrow \infty} \sum_{i=1}^L i \Pr\{\varsigma_1 \wedge L = i\} \\ &= \frac{\mathbb{E}\varsigma}{1 - \rho_1}. \end{aligned}$$

If $\rho_1 = 1$, and $\rho_{1,2} < \infty$ and $\mathbb{E}\varsigma^2 < \infty$, then

$$(3.33) \quad \begin{aligned} \lim_{L \rightarrow \infty} \frac{\mathbb{E}\mathbb{E}\{\nu_L^{(1)}|\varsigma_1 \wedge L\}}{L} &= \frac{2}{(\rho_{1,2} - 1)\mathbb{E}\varsigma + \mathbb{E}\varsigma^2} \lim_{L \rightarrow \infty} \sum_{i=1}^L i \Pr\{\varsigma_1 \wedge L = i\} \\ &= \frac{2\mathbb{E}\varsigma}{(\rho_{1,2} - 1)\mathbb{E}\varsigma + \mathbb{E}\varsigma^2}. \end{aligned}$$

If $\rho_1 > 1$, then

$$\begin{aligned}
(3.34) \quad \lim_{L \rightarrow \infty} \frac{\mathbb{E}\mathbb{E}\{\nu_L^{(1)} | \varsigma_1 \wedge L\}}{\varphi^L} &= \frac{1}{1 + \lambda \widehat{B}'_1(\lambda - \lambda \widehat{R}(\varphi)) \widehat{R}'(\varphi)} \\
&\times \lim_{L \rightarrow \infty} \sum_{i=1}^L \Pr\{\varsigma_1 \wedge L = i\} \sum_{j=0}^{i-1} \varphi^j \\
&= \frac{1}{[1 + \lambda \widehat{B}'_1(\lambda - \lambda \widehat{R}(\varphi)) \widehat{R}'(\varphi)](1 - \varphi)} \\
&\times \lim_{L \rightarrow \infty} \sum_{i=1}^L \Pr\{\varsigma_1 \wedge L = i\} (1 - \varphi^i) \\
&= \frac{1 - \widehat{R}(\varphi)}{[1 + \lambda \widehat{B}'_1(\lambda - \lambda \widehat{R}(\varphi)) \widehat{R}'(\varphi)](1 - \varphi)}.
\end{aligned}$$

Therefore, taking into account these limiting relations (3.32), (3.33) and (3.34) by virtue of (3.12) (Corollary 3.6) and explicit representations (3.15) and (3.16) (Lemma 3.7) for p_1 and p_2 , we finally arrive at the statements of the theorem. The theorem is proved. \square

3.4. Asymptotic theorems for p_1 and p_2 under special heavy load conditions. In this section we establish asymptotic theorems for p_1 and p_2 under heavy load assumptions where (i) $\rho_1 = 1 + \delta$ or (ii) $\rho_1 = 1 - \delta$, and δ is a vanishing positive parameter as $L \rightarrow \infty$. The theorems presented in this section are analogues of the theorems [4] given in Section 4 of that paper. The conditions are special, because these heavy load conditions include change of the parameter ρ_1 as L increases to infinity and δ vanishes, but the other load parameter ρ_2 remains unchanged when the parameters L and δ are changed.

In case (i) we have the following two theorems.

Theorem 3.10. *Assume that $\rho_1 = 1 + \delta$, $\delta > 0$ and that $L\delta \rightarrow C > 0$ as $\delta \rightarrow 0$ and $L \rightarrow \infty$. Assume that $\rho_{1,3}(L)$ is a bounded sequence, assume that $\mathbb{E}\varsigma^3 < \infty$*

and that the limit $\lim_{L \rightarrow \infty} \rho_{1,2}(L) = \tilde{\rho}_{1,2}$ exists. Then,

$$(3.35) \quad p_1 = \frac{\delta}{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right) - 1} [1 + o(1)],$$

$$(3.36) \quad p_2 = \frac{\delta \rho_2 \exp\left(\frac{2C}{(\tilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right)}{(1 - \rho_2) \left[\exp\left(\frac{2C}{(\tilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right) - 1 \right]} [1 + o(1)].$$

Proof. Note first, that under assumptions of the theorem there is the following expansion for φ :

$$(3.37) \quad \varphi = 1 - \frac{2\delta}{(\tilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2} + O(\delta^2).$$

This expansion is similar to that given originally in the book of Subhankulov [16], p.362, and its proof is provided as follows. Write the equation $\varphi = \widehat{B}_1(\lambda - \lambda \widehat{R}(\varphi))$ and expand the right-hand side by Taylor's formula reckoning that $\varphi = 1 - z$, where z is small enough, when δ is small. We obtain:

$$(3.38) \quad 1 - z = 1 - (1 + \delta)z + \frac{((\tilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + (1 + \delta)\mathbb{E}\zeta^2)z^2}{2} + O(z^3).$$

From (3.38) we arrive at the equation

$$(3.39) \quad \delta z + \frac{(\tilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}{2} z^2 + O(z^3) = 0.$$

The positive solution of equation (3.39) is $z = \frac{2}{(\tilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}$ that leads to the expansion given by (3.37).

As well, by virtue of (3.37) we also obtain:

$$(3.40) \quad 1 + \lambda \widehat{B}'_1(\lambda - \lambda \widehat{R}(\varphi)) \widehat{R}'(\varphi) = \delta + O(\delta^2),$$

and in addition, according to the l'Hospitale rule

$$\lim_{u \uparrow 1} \frac{1 - \widehat{R}(u)}{1 - u} = \mathbb{E}\zeta.$$

Hence

$$(3.41) \quad \frac{1 - \widehat{R}(\varphi)}{1 - \varphi} = \mathbb{E}\zeta [1 + o(1)].$$

Substituting (3.37), (3.40) and (3.41) into (3.34) we obtain

$$(3.42) \quad \mathbb{E}\mathbb{E}\{\nu_L^{(1)} | \zeta \wedge L\} = \frac{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right) - 1}{\delta} \mathbb{E}\zeta [1 + o(1)].$$

Hence, relations (3.35) and (3.36) of the theorem follow by virtue of (3.12) (Corollary 3.6) and explicit representations (3.15) and (3.16) (Lemma 3.7) for p_1 and p_2 . \square

Theorem 3.11. *Under the conditions of Theorem 3.10 assume that $C = 0$. Then,*

$$(3.43) \quad \lim_{L \rightarrow \infty} Lp_1(L) = \frac{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2}{2},$$

$$(3.44) \quad \lim_{L \rightarrow \infty} Lp_2(L) = \frac{\rho_2}{1 - \rho_2} \frac{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2}{2}.$$

Proof. The statement of the theorem follows by expanding the main terms of asymptotic relations (3.35) and (3.36) for small C . \square

In case (ii) we have the following two theorems.

Theorem 3.12. *Assume that $\rho_1 = 1 - \delta$, $\delta > 0$ and that $L\delta \rightarrow C > 0$ as $\delta \rightarrow 0$ and $L \rightarrow \infty$. Assume that $\rho_{1,3}(L)$ is a bounded function, assume that $E\zeta^3 < \infty$ and that the limit $\lim_{L \rightarrow \infty} \rho_{1,2}(L) = \tilde{\rho}_{1,2}$ exists. Then,*

$$(3.45) \quad p_1 = \delta \exp\left(\frac{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2}{2C}\right) [1 + o(1)],$$

$$(3.46) \quad p_2 = \frac{\delta \rho_2 \left[\exp\left(\frac{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2}{2C}\right) - 1 \right]}{1 - \rho_2} [1 + o(1)].$$

Proof. The explicit representation for the generating function for $E\tilde{\nu}_j^{(1)}$ is given by (3.21). Since the sequence $\{E\tilde{\nu}_j^{(1)}\}$ is increasing, then the asymptotic behavior of $E\nu_L^{(1)}$ as $L \rightarrow \infty$ under the assumptions $\rho_1 = 1 - \delta$, $L\delta \rightarrow C$ as $L \rightarrow \infty$ can be found according to a Tauberian theorem of Hardy and Littlewood (see e.g. [12], [16], [17], [25], and [20], p.203). Namely, according to that theorem, the behavior of $E\tilde{\nu}_L^{(1)}$ as $L \rightarrow \infty$ and $\delta \rightarrow 0$ such that $\delta L \rightarrow C > 0$ can be found from the asymptotic expansion of

$$(3.47) \quad (1 - z) \frac{\widehat{B}_1(\lambda - \lambda\widehat{R}(z))}{\widehat{B}_1(\lambda - \lambda\widehat{R}(z)) - z}$$

as $z \uparrow 1$. Similarly to the evaluation given in the proof of Theorem 4.3 [4], we have:

$$\begin{aligned}
& (1-z) \frac{\widehat{B}_1(\lambda - \lambda \widehat{R}(z))}{\widehat{B}_1(\lambda - \lambda \widehat{R}(z)) - z} \\
& \asymp \frac{1-z}{1-z - \rho_1(1-z) + \frac{(\widetilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \rho_1\mathbb{E}\zeta^2}{2}(1-z)^2 + O((1-z)^3)} \\
(3.48) \quad & \asymp \frac{1}{\delta + \frac{(\widetilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}{2}(1-z) + O((1-z)^2)} \\
& \asymp \frac{1}{\delta \left[1 + \frac{(\widetilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}{2\delta}(1-z) \right] + O((1-z)^2)} \\
& \asymp \frac{1}{\delta \exp \left[\frac{(\widetilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}{2\delta}(1-z) \right]} [1 + o(1)].
\end{aligned}$$

Therefore, assuming that $z = \frac{L-1}{L} \rightarrow 1$ as $L \rightarrow \infty$, from (3.48) we arrive at the following estimate:

$$(3.49) \quad \mathbb{E}\widetilde{\nu}_L^{(1)} = \frac{1}{\delta} \exp \left(-\frac{(\widetilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}{2C} \right) [1 + o(1)].$$

Comparing (3.24) with (3.34) and taking into account (3.41), which holds true in the case of this theorem as well, we obtain:

$$(3.50) \quad \mathbb{E}\mathbb{E}\{\widetilde{\nu}_L^{(1)} | \zeta_1 \wedge L\} = \frac{\mathbb{E}\zeta}{\delta} \exp \left(-\frac{(\widetilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}{2C} \right) [1 + o(1)].$$

Hence, relations (3.45) and (3.46) of the theorem follow by virtue of (3.12) (Corollary 3.6) and explicit representations (3.15) and (3.16) (Lemma 3.7) for p_1 and p_2 . \square

Theorem 3.13. *Under the conditions of Theorem 3.12 assume that $C = 0$. Then we have (3.43) and (3.44).*

Proof. The proof of the theorem follows by expanding the main terms of the asymptotic relations (3.45) and (3.46) for small C . \square

4. ASYMPTOTIC THEOREMS FOR THE STATIONARY PROBABILITIES q_i

The aim of this section is asymptotic analysis of the stationary probabilities q_i , $i = 1, 2, \dots, L$ as $L \rightarrow \infty$. The challenge is to first obtain the explicit representation for q_i in terms of $\mathbb{E}\nu_i^{(1)}$, and then to study the asymptotic behavior of q_i as $L \rightarrow \infty$

on the basis of the known asymptotic results for $E\nu_i^{(1)}$ as $L \rightarrow \infty$. The asymptotic results are obtained in the following three cases: $\rho_1 = 1$, $\rho_1 = 1 + \delta$ and $\rho_1 = 1 - \delta$, where δ is a positive small value.

4.1. Explicit representation for the stationary probabilities q_i . The aim of this section is to prove the following statement.

Lemma 4.1. *For $i = 1, 2, \dots, L$ we have*

$$(4.1) \quad q_i = \rho_1 p_1 \left(E\nu_i^{(1)} - E\nu_{i-1}^{(1)} \right).$$

Proof. Using renewal arguments (e.g. [15]), relation (2.6) and Wald's identities:

$$ET_i^{(1)} = \frac{\rho_1}{\lambda E\zeta} E\nu_i^{(1)}, \quad i = 1, 2, \dots, L,$$

we have:

$$(4.2) \quad q_i = \frac{ET_i^{(1)} - ET_{i-1}^{(1)}}{ET_L + \frac{1}{\lambda}} = \rho_1 \frac{E\nu_i^{(1)} - E\nu_{i-1}^{(1)}}{E\nu_L}, \quad i = 1, 2, \dots, L.$$

Then, taking into account that $E\nu_L = E\nu_L^{(1)} + E\nu_L^{(2)}$ and applying the linear representation for $E\nu_L^{(2)}$ given by (3.14), from (4.2) we obtain:

$$q_i = \frac{\rho_1(1 - \rho_2)}{E\zeta + (\rho_1 - \rho_2)E\nu_L^{(1)}} \left(E\nu_i^{(1)} - E\nu_{i-1}^{(1)} \right), \quad i = 1, 2, \dots, L.$$

Hence, representation (4.1) follows from (3.15) (Lemma 3.7), and Lemma 4.1 is proved. \square

4.2. Asymptotic analysis of the stationary probabilities q_i : The case $\rho_1 =$

1. Let us study asymptotic behavior of the stationary probabilities q_i . We start from the following modified version of (3.23) (Lemma 3.8):

$$(4.3) \quad E\tilde{\nu}_{L-j}^{(1)} - E\tilde{\nu}_{L-j-1}^{(1)} = \frac{2}{(\rho_{1,2} - 1)E\zeta + E\zeta^2} + o(1),$$

which is assumed to be satisfied under the conditions $\rho_{1,2} < \infty$ and $E\zeta^2 < \infty$.

Under the same conditions, similarly to (3.33) we obtain:

$$(4.4) \quad \begin{aligned} EE\{\nu_{L-j}^{(1)} | \zeta_1 \wedge L\} - EE\{\nu_{L-j-1}^{(1)} | \zeta_1 \wedge L\} &= \frac{2}{(\rho_{1,2} - 1)E\zeta + E\zeta^2} \\ &\times \sum_{i=1}^L i \Pr\{\zeta_1 \wedge L = i\} + o(1) \\ &= \frac{2E\zeta}{(\rho_{1,2} - 1)E\zeta + E\zeta^2} + o(1). \end{aligned}$$

Hence, according to (3.12) (Corollary 3.6) and (4.4) we have the estimate

$$(4.5) \quad \mathbb{E}\nu_{L-j}^{(1)} - \mathbb{E}\nu_{L-j-1}^{(1)} = \frac{2\mathbb{E}\zeta}{(\rho_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2} + o(1).$$

Asymptotic relations (4.5), (3.28) together with explicit relation (4.1) of Lemma 4.1 leads to the following theorem.

Theorem 4.2. *In the case $\rho_1 = 1$ under the additional conditions $\rho_{1,2} < \infty$ and $\mathbb{E}\zeta^2 < \infty$ for any $j \geq 0$ we have*

$$(4.6) \quad \lim_{L \rightarrow \infty} Lq_{L-j} = 1.$$

Note, that the asymptotic relation given by (4.6) does not express via $\mathbb{E}\zeta$ and, therefore, it is the same as for the queueing system with ordinary Poisson arrivals.

4.3. Asymptotic analysis of the stationary probabilities q_i : The case $\rho_1 = 1 + \delta$, $\delta > 0$. In the case $\rho_1 = 1 + \delta$, $\delta > 0$ the asymptotic behavior of q_i is specified by the following theorem.

Theorem 4.3. *Assume that $\rho_1 = 1 + \delta$, $\delta > 0$, and $L\delta \rightarrow C > 0$ as $\delta \rightarrow 0$ and $L \rightarrow \infty$. Assume that $\rho_{1,3}(L)$ is a bounded sequence, assume that $\mathbb{E}\zeta^3 < \infty$ and there exists $\tilde{\rho}_{1,2} = \lim_{L \rightarrow \infty} \rho_{1,2}(L)$. Then, for any $j \geq 0$*

$$(4.7) \quad q_{L-j} = \frac{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right)}{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right) - 1} \times \left(1 - \frac{2\delta}{(\tilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right)^j \frac{2\delta}{(\tilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2} + o(\delta).$$

Proof. Expanding (3.24) for large L , we have:

$$(4.8) \quad \mathbb{E}\tilde{\nu}_{L-j}^{(1)} = \frac{\varphi^j}{\varphi^L [1 + \lambda \widehat{B}'_1(\lambda - \lambda \widehat{R}(\varphi)) \widehat{R}'(\varphi)]} + \frac{1}{1 - \rho_1} + o(1).$$

In turn, from (4.8) for large L we obtain:

$$(4.9) \quad \mathbb{E}\tilde{\nu}_{L-j}^{(1)} - \mathbb{E}\tilde{\nu}_{L-j-1}^{(1)} = \frac{(1 - \varphi)\varphi^j}{\varphi^L [1 + \lambda \widehat{B}'_1(\lambda - \lambda \widehat{R}(\varphi)) \widehat{R}'(\varphi)]} + o(1).$$

From (4.9), similarly to (3.34), we further have:

$$\begin{aligned} & \mathbb{E}\mathbb{E}\{\nu_{L-j}^{(1)} | \varsigma_1 \wedge L\} - \mathbb{E}\mathbb{E}\{\nu_{L-j-1}^{(1)} | \varsigma_1 \wedge L\} \\ &= \frac{(1 - \widehat{R}(\varphi))(1 - \varphi)\varphi^j}{[1 + \lambda \widehat{B}'_1(\lambda - \lambda \widehat{R}(\varphi)) \widehat{R}'(\varphi)](1 - \varphi)} + o(1), \end{aligned}$$

and, according to (3.12) (Corollary 3.6),

$$(4.10) \quad \mathbb{E}\nu_{L-j}^{(1)} - \mathbb{E}\nu_{L-j-1}^{(1)} = \frac{(1 - \widehat{R}(\varphi))(1 - \varphi)\varphi^j}{[1 + \lambda\widehat{B}'_1(\lambda - \lambda\widehat{R}(\varphi))\widehat{R}'(\varphi)](1 - \varphi)} + o(1).$$

Next, under the conditions of the theorem, asymptotic expansions (3.37) (3.40) and (3.41) hold. Taking into consideration these expansions we arrive at the following asymptotic relations for $j = 0, 1, \dots$:

$$\begin{aligned} \mathbb{E}\nu_{L-j}^{(1)} - \mathbb{E}\nu_{L-j-1}^{(1)} &= \exp\left(\frac{2C}{(\widetilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right) \\ &\times \left(1 - \frac{2\delta}{(\widetilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right)^j \frac{2}{(\widetilde{\rho}_{1,2} - 1)\mathbb{E}\zeta + \mathbb{E}\zeta^2} [1 + o(1)]. \end{aligned}$$

Now, taking into account asymptotic relation (3.35) of Theorem 3.10 and the explicit formula given by (4.1) (Lemma 4.1) we arrive at the statement of the theorem. \square

4.4. Asymptotic analysis of the stationary probabilities q_i : The case $\rho_1 = 1 - \delta$, $\delta > 0$. In the case $\rho_1 = 1 + \delta$, $\delta > 0$, the study is more delicate and based on special analysis. The additional assumption of this case is that the class of probability distribution functions $\{B_1(x)\}$ and $\Pr\{\zeta = i\}$ are given such that there exists a unique root $\tau > 1$ of the equation

$$(4.11) \quad z = \widehat{B}_1(\lambda - \lambda\widehat{R}(z)),$$

and there exists the first derivative $\widehat{B}'_1(\lambda - \lambda\widehat{R}(\tau))$.

Under the assumption that $\rho_1 < 1$ the unique root of (4.11) is not necessarily exists. Such type of condition has been considered by Willmot [26] to obtain the asymptotic behavior for high queue-level probabilities in stationary $M/GI/1$ queues. Denote the stationary probabilities in the $M/GI/1$ queueing system by $q_i[M/GI/1]$, $i = 0, 1, \dots$. It was shown in [26] that

$$(4.12) \quad q_i[M/GI/1] = \frac{(1 - \rho_1)(1 - \tau)}{\tau^i [1 + \lambda\widehat{B}'_1(\lambda - \lambda\tau)]} [1 + o(1)] \text{ as } i \rightarrow \infty,$$

where $\widehat{B}_1(s)$ denotes the Laplace-Stieltjes transform of the service time distribution in the $M/G/1$ queueing system, and τ denotes a root of the equation $z = \widehat{B}_1(\lambda - \lambda z)$ greater than 1, which is assumed to be unique. On the other hand, according to

the Pollaczek-Khintchine formula (e.g. Takács [19], p.242), $q_i[M/GI/1]$ can be represented explicitly

$$(4.13) \quad q_i[M/GI/1] = (1 - \rho_1) \left(\mathbb{E}\nu_i^{(1)} - \mathbb{E}\nu_{i-1}^{(1)} \right), i = 1, 2, \dots,$$

where the random variable $\nu_i^{(1)}$ in this formula is associated with the number of served customers during a busy period of the state dependent $M/G/1$ queueing system, where the value of the system parameter, where the service is changed, is i (see Section 2.1). Representation (4.13) can be easily checked, since in that case

$$(4.14) \quad \sum_{j=0}^{\infty} \mathbb{E}\nu_j^{(1)} z^j = \frac{\widehat{B}_1(\lambda - \lambda z)}{\widehat{B}_1(\lambda - \lambda z) - z},$$

and multiplication of the right-hand side of (4.14) by $(1 - \rho_1)(1 - z)$ leads to the well-known Pollaczek-Khintchine formula. Then, from (4.12) and (4.13) there is the asymptotic proportion for large L and any $j \geq 0$:

$$(4.15) \quad \frac{\mathbb{E}\nu_{L-j}^{(1)} - \mathbb{E}\nu_{L-j-1}^{(1)}}{\mathbb{E}\nu_L^{(1)} - \mathbb{E}\nu_{L-1}^{(1)}} = \tau^j [1 + o(1)].$$

In the case of batch arrivals the results are similar. One can prove that the same proportion as (4.15) holds in this case as well, where τ in the case of batch arrivals denotes a unique real root of the equation of (4.11), which is greater than 1. (Recall that our convention was that there is a unique real solution of (4.11) greater than 1.) Indeed, the arguments of [26] are elementary extended for the queueing system with batch arrivals. The simplest way to extend these results straightforwardly is to consider the stationary queueing system with batch Poisson arrivals, in which the first batch in each busy period is equal to 1. Denote this system by $M^{1,X}/G/1$. For this specific system, similarly to (4.12) we obtain:

$$(4.16) \quad q_i[M^{1,X}/GI/1] = \frac{(1 - \rho_1)(1 - \tau)}{\tau^i [1 + \lambda \widehat{B}'_1(\lambda - \lambda \widehat{R}(\tau)) \widehat{R}'(\tau)]} [1 + o(1)] \text{ as } i \rightarrow \infty,$$

where $q_i[M^{1,X}/GI/1]$, $i = 0, 1, \dots$, denotes the stationary probabilities in this system. Then, taking into account (3.21), similarly to (4.13) one can write

$$(4.17) \quad q_i[M^{1,X}/GI/1] = (1 - \rho_1) \left(\mathbb{E}\widetilde{\nu}_i^{(1)} - \mathbb{E}\widetilde{\nu}_{i-1}^{(1)} \right), i = 1, 2, \dots$$

From (4.16) and (4.17) we obtain

$$(4.18) \quad \frac{\mathbb{E}\widetilde{\nu}_{L-j}^{(1)} - \mathbb{E}\widetilde{\nu}_{L-j-1}^{(1)}}{\mathbb{E}\widetilde{\nu}_L^{(1)} - \mathbb{E}\widetilde{\nu}_{L-1}^{(1)}} = \tau^j [1 + o(1)].$$

From (4.18) and the results of Sections 3.2 and 3.3 (see (3.22), (3.32) and (3.12)) we also have the estimate

$$(4.19) \quad \frac{\mathbb{E}\nu_{L-j}^{(1)} - \mathbb{E}\nu_{L-j-1}^{(1)}}{\mathbb{E}\nu_L^{(1)} - \mathbb{E}\nu_{L-1}^{(1)}} = \tau^j [1 + o(1)],$$

which coincides with (4.15).

Now we formulate and prove a theorem on asymptotic behavior of the stationary probabilities q_i in the case $\rho_1 = 1 - \delta$, $\delta > 0$. The special assumption in this theorem is that the class of probability distributions $\{B_1(x)\}$ is defined according to the above convention. More precisely, in the case $\rho_1 = 1 - \delta$, $\delta > 0$, and vanishing δ as $L \rightarrow \infty$ this means that there exists $\epsilon_0 > 0$ (small enough) such that for all $0 \leq \epsilon \leq \epsilon_0$, the above family of probability distribution functions $B_{1,\epsilon}(x)$ (depending now on the parameter ϵ) satisfies the following properties. Let $\widehat{B}_{1,\epsilon}(s)$ denote the Laplace-Stieltjes transform of $B_{1,\epsilon}(x)$. We assume that any $\widehat{B}_{1,\epsilon}(s)$ is an analytic function in a small neighborhood of zero, and

$$(4.20) \quad \widehat{B}'_{1,\epsilon}(s) < \infty.$$

Property (4.20) is required for the existence of the probabilities q_i . Relation (4.16) contains the term $\widehat{B}'_1(\lambda - \lambda\widehat{R}(\tau))$, and this term must be finite. In addition, the term $\widehat{R}'(\tau) < \infty$ must be finite as well, that is, the additional to (4.20) associated assumption is that

$$(4.21) \quad \widehat{R}'(1 + \epsilon) < \infty$$

for any ϵ of the defined neighborhood. Choice of small parameter ϵ is continuously connected with that choice of the parameter δ (or L) in the theorem below.

Theorem 4.4. *Assume that the class of probability distribution functions $\{B_1(x)\}$ and the probabilities r_1, r_2, \dots are defined according to the conventions made and respectively satisfy (4.20) and (4.21), $\rho_1 = 1 - \delta$, $\delta > 0$, and $L\delta \rightarrow C > 0$, as $\delta \rightarrow 0$ and $L \rightarrow \infty$. Assume that $\rho_{1,3} = \rho_{1,3}(L)$ is a bounded sequence, $\mathbb{E}\zeta^3 < \infty$, and*

there exists $\tilde{\rho}_{1,2} = \lim_{L \rightarrow \infty} \rho_{1,2}(L)$. Then,

$$(4.22) \quad q_{L-j} = \frac{1}{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2}\right) - 1} \times \frac{2\delta}{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2} \left(1 + \frac{2\delta}{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2}\right)^j [1 + o(1)]$$

for any $j \geq 0$.

Proof. Under the assumptions of this theorem let us first derive the following asymptotic expansion:

$$(4.23) \quad \tau = 1 + \frac{2\delta}{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2} + O(\delta^2).$$

Asymptotic expansion (4.23) is similar to that of (3.37), and its proof is also similar. Namely, taking into account that the equation $z = \widehat{B}_1(\lambda - \lambda \widehat{R}(z))$ has a unique solution in the set $(1, \infty)$, and this solution approaches 1 as δ vanishes. Therefore, by the Taylor expansion of this equation around the point $z = 1$, we have:

$$(4.24) \quad 1 + z = 1 - (1 + \delta)z + \frac{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2}{2} z^2 + O(z^3).$$

From (4.24) we arrive at exactly the same equation as (3.39) and obtain exactly the same positive solution, which is $z = \frac{2}{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2}$. So, representation (4.23) is proved.

Next, from (4.19), (4.23) and explicit formula (4.1) we obtain

$$(4.25) \quad q_{L-j} = q_L \left(1 + \frac{2\delta}{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2}\right)^j [1 + o(1)].$$

Taking into consideration

$$\begin{aligned} \sum_{j=0}^{L-1} \left(1 + \frac{2\delta}{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2}\right)^j &= \frac{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2}{2\delta} \left[\left(1 + \frac{2\delta}{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2}\right)^L - 1 \right] \\ &= \frac{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2}{2\delta} \left[\exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2}\right) - 1 \right] [1 + o(1)], \end{aligned}$$

from the normalization condition $p_1 + p_2 + \sum_{i=1}^L q_i = 1$ and the fact that both p_1 and p_2 have the order $O(\delta)$, we obtain:

$$(4.26) \quad q_L = \frac{2\delta}{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2} \cdot \frac{1}{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)E_\zeta + E_\zeta^2}\right) - 1} [1 + o(1)].$$

The desired statement of the theorem follows from (4.26). \square

5. OBJECTIVE FUNCTION

In this section we study asymptotic properties of the objective function defined by relation (1.2), which includes the costs c_i . The particular model does not taking into account the water costs c_i has been studied in [4], and Theorem 5.1 of [4] describes the structure of optimal solution under that particular setting.

Apparently, the cases where $\rho_1 > 1$ or $\rho_1 < 1$ do not lead to the optimal solution. According to Theorem 3.9 in these cases the objective function J increases unboundedly as $L \rightarrow \infty$ because one of the limiting probabilities p_1 or p_2 is strictly positive (for details see [4]). On the other hand, the case $\rho_1 = 1$ is not necessary optimal. Therefore, as in [4], the optimal solution can be achieved in one of the cases as (i) $\rho_1 = 1$; (ii) $\rho_1 = 1 + \delta$, $\delta > 0$ and $\delta \rightarrow 0$ as $L \rightarrow \infty$; (iii) $\rho_1 = 1 - \delta$, $\delta > 0$ and $\delta \rightarrow 0$ as $L \rightarrow \infty$. All of these cases are studied below.

5.1. The case $\rho_1 = 1$. In this section we prove the following result.

Proposition 5.1. *In the case $\rho_1 = 1$, under the additional conditions $\rho_{1,2} < \infty$ and $E\zeta^2 < \infty$ we have:*

$$(5.1) \quad \lim_{L \rightarrow \infty} J(L) = j_1 \frac{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2}{2} + j_2 \frac{\rho_2}{1 - \rho_2} \cdot \frac{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2}{2} + c^*,$$

where

$$c^* = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{i=1}^L c_i.$$

Proof. The first two terms in the right-hand side of (5.1) follow from asymptotic relations (3.28) and (3.29) (Theorem 3.9). The last term c^* of the right-hand side of (5.1) follows from (4.6) (Theorem 4.2), since

$$\lim_{L \rightarrow \infty} \sum_{i=1}^L q_i c_i = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{i=1}^L c_i = c^*.$$

□

5.2. The case $\rho = 1 + \delta$, $\delta > 0$. In the case $\rho = 1 + \delta$, $\delta > 0$ we have the following statement.

Proposition 5.2. *Under the assumptions of Theorem 4.3 denote the objective function J by J^{upper} . We have the following representation:*

$$(5.2) \quad J^{\text{upper}} = C \left[j_1 \frac{1}{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right) - 1} + j_2 \frac{\rho_2 \exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right)}{(1-\rho_2) \left(\exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right) - 1\right)} \right] + c^{\text{upper}},$$

where

$$(5.3) \quad c^{\text{upper}} = \frac{2C}{(\tilde{\rho}_{1,2}-1)\mathbb{E}\zeta + \mathbb{E}\zeta^2} \cdot \frac{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right)}{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right) - 1} \\ \times \lim_{L \rightarrow \infty} \frac{1}{L} \widehat{C}_L \left(1 - \frac{2C}{(\tilde{\rho}_{1,2}\mathbb{E}\zeta + \mathbb{E}\zeta^2 - \mathbb{E}\zeta)L} \right),$$

and $\widehat{C}_L(z) = \sum_{j=0}^{L-1} c_{L-j} z^j$ is a backward generating cost function.

Proof. The representation for the term

$$C \left[j_1 \frac{1}{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right) - 1} + j_2 \frac{\rho_2 \exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right)}{(1-\rho_2) \left(\exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right) - 1\right)} \right]$$

of the right-hand side of (5.2) follows from (3.35) and (3.36) (Theorem 3.10). This term is similar to that (5.2) in [4]. The new term, which takes into account the water costs, is c^{upper} . Taking into account representation (4.7), for this term we obtain:

$$c^{\text{upper}} = \lim_{L \rightarrow \infty} \sum_{j=0}^{L-1} q_{L-j} c_{L-j} \\ = \lim_{L \rightarrow \infty} \sum_{j=0}^{L-1} c_{L-j} \cdot \frac{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right)}{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2}-1)\mathbb{E}\zeta + \mathbb{E}\zeta^2}\right) - 1} \\ \times \left(1 - \frac{2\delta L}{(\tilde{\rho}_{1,2}\mathbb{E}\zeta + \mathbb{E}\zeta^2 - \mathbb{E}\zeta)L} \right)^j \frac{2\delta L}{(\tilde{\rho}_{1,2}\mathbb{E}\zeta + \mathbb{E}\zeta^2 - \mathbb{E}\zeta)L},$$

and, because of $\lim_{L \rightarrow \infty} \delta L = C$, representation (5.3) follows. \square

5.3. **The case** $\rho = 1 - \delta$, $\delta > 0$. In the case $\rho = 1 - \delta$, $\delta > 0$ we have the following statement.

Proposition 5.3. *Under the assumptions of Theorem 4.4 denote the objective function J by J^{lower} . We have the following representation*

$$(5.4) \quad J^{\text{lower}} = C \left[j_1 \exp \left(\frac{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2}{2C} \right) + j_2 \frac{\rho_2}{1 - \rho_2} \left(\exp \left(\frac{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2}{2C} \right) - 1 \right) \right] + c^{\text{lower}},$$

where

$$(5.5) \quad c^{\text{lower}} = \frac{2C}{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2} \cdot \frac{1}{\exp \left(\frac{2C}{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2} \right) - 1} \times \lim_{L \rightarrow \infty} \frac{1}{L} \widehat{C}_L \left(1 + \frac{2C}{(\tilde{\rho}_{1,2}E\zeta + E\zeta^2 - E\zeta)L} \right),$$

and $\widehat{C}(z) = \sum_{j=0}^{L-1} c_{L-j} z^j$ is a backward generating cost function.

Proof. The representation for the term

$$C \left[j_1 \exp \left(\frac{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2}{2C} \right) + j_2 \frac{\rho_2}{1 - \rho_2} \left(\exp \left(\frac{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2}{2C} \right) - 1 \right) \right]$$

of the right-hand side of (5.4) follows from (3.45) and (3.46) (Theorem 3.12). This term is similar to that (5.3) in [4]. The new term, which takes into account the water costs, is c^{lower} . Taking into account representation (4.22), for this term we obtain:

$$\begin{aligned} c^{\text{lower}} &= \lim_{L \rightarrow \infty} \sum_{j=0}^{L-1} q_{L-j} c_{L-j} \\ &= \lim_{L \rightarrow \infty} \sum_{j=0}^{L-1} c_{L-j} \cdot \frac{1}{\exp \left(\frac{2C}{(\tilde{\rho}_{1,2} - 1)E\zeta + E\zeta^2} \right) - 1} \\ &\quad \times \left(1 + \frac{2\delta L}{(\tilde{\rho}_{1,2}E\zeta + E\zeta^2 - E\zeta)L} \right)^j \frac{2\delta L}{(\tilde{\rho}_{1,2}E\zeta + E\zeta^2 - E\zeta)L}, \end{aligned}$$

and, because of $\lim_{L \rightarrow \infty} \delta L = C$, representation (5.5) follows. \square

6. A SOLUTION TO THE CONTROL PROBLEM AND ITS PROPERTIES

In this section we discuss the solution to the control problem and study its properties. The functionals J^{upper} and J^{lower} are correspondingly given by (5.2) and (5.4), and the last terms in these functionals are correspondingly given by (5.3) and (5.5). For our further analysis we need in other representations for these last terms.

Denote

$$(6.1) \quad \psi(C) = \lim_{L \rightarrow \infty} \frac{\sum_{j=0}^{L-1} c_{L-j} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta})L}\right)^j}{\sum_{j=0}^{L-1} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta})L}\right)^j},$$

and

$$(6.2) \quad \eta(C) = \lim_{L \rightarrow \infty} \frac{\sum_{j=0}^{L-1} c_{L-j} \left(1 + \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta})L}\right)^j}{\sum_{j=0}^{L-1} \left(1 + \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta})L}\right)^j}.$$

Since $\{c_i\}$ is a bounded sequence, then the limits of (6.1) and (6.2) do exist.

The relations between c^{upper} and $\psi(C)$ and, respectively, between c^{lower} and $\eta(C)$ are given in the lemma below.

Lemma 6.1. *We have:*

$$(6.3) \quad c^{\text{upper}} = \psi(C),$$

and

$$(6.4) \quad c^{\text{lower}} = \eta(C).$$

Proof. From (6.1) and (6.2) we correspondingly have the representations:

$$(6.5) \quad \begin{aligned} & \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta})L}\right)^j \\ &= \psi(C) \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta})L}\right)^j, \end{aligned}$$

and

$$\begin{aligned}
(6.6) \quad & \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left(1 + \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta})L} \right)^j \\
& = \eta(C) \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 + \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta})L} \right)^j.
\end{aligned}$$

The desired results follow by direct transformations of the corresponding right-hand sides of (6.5) and (6.6).

Indeed, for the right-hand side of (6.5) we obtain:

$$\begin{aligned}
(6.7) \quad & \psi(C) \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta})L} \right)^j \\
& = \psi(C) \left[1 - \exp \left(-\frac{2C}{(\tilde{\rho}_{1,2} - 1)E_{\zeta} + E_{\zeta}^2} \right) \right] \frac{(\tilde{\rho}_{1,2} - 1)E_{\zeta} + E_{\zeta}^2}{2C}.
\end{aligned}$$

On the other hand, from (5.3) we have:

$$\begin{aligned}
(6.8) \quad & c^{\text{upper}} \left[1 - \exp \left(-\frac{2C}{(\tilde{\rho}_{1,2} - 1)E_{\zeta} + E_{\zeta}^2} \right) \right] \frac{(\tilde{\rho}_{1,2} - 1)E_{\zeta} + E_{\zeta}^2}{2C} \\
& = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta})L} \right)^j.
\end{aligned}$$

Hence, from (6.5), (6.7) and (6.8) we obtain (6.1). The proof of (6.4) is completely analogous and uses the representations of (5.5) and (6.6). \square

The next lemma establishes the main properties of functions $\psi(C)$ and $\eta(C)$.

Lemma 6.2. *The function $\psi(C)$ is a not increasing function, and its maximum is $\psi(0) = c^*$. The function $\eta(C)$ is a not decreasing function, and its minimum is $\eta(0) = c^*$. (Recall that $c^* = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{i=1}^L c_i$ is defined in Proposition 5.1.)*

Proof. Let us first prove that $\psi(0) = c^*$ is a maximum of $\psi(C)$. For this purpose we use the following well-known inequality (e.g. Hardy, Littlewood and Polya [10] or Marschall and Olkin [11]). Let $\{a_n\}$ and $\{b_n\}$ be arbitrary sequences, one of them is increasing and another decreasing. Then for any finite sum we have

$$(6.9) \quad \sum_{n=1}^l a_n b_n \leq \frac{1}{l} \sum_{n=1}^l a_n \sum_{n=1}^l b_n.$$

Applying inequality (6.9) to finite sums of the left-hand side of (6.5) and passing to limit as $L \rightarrow \infty$, we have

$$\begin{aligned}
(6.10) \quad & \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\
& \leq \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\
& = \psi(0) \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j.
\end{aligned}$$

Then, comparing (6.5) with (6.10) enables us to conclude,

$$\psi(0) = c^* \geq \psi(C),$$

i.e. $\psi(0) = c^*$ is the maximum value of $\psi(C)$.

Prove now, that $\psi(C)$ is a not increasing function, i.e. for any nonnegative $C_1 \leq C$ we have $\psi(C) \leq \psi(C_1)$.

To prove this note, that for small positive δ_1 and δ_2 we have $(1-\delta_1-\delta_2) = (1-\delta_1)(1-\delta_2) + O(\delta_1\delta_2)$. Using this idea, one can prove the monotonicity of $\psi(C)$ by replacing

$$\begin{aligned}
(6.11) \quad & 1 - \frac{2C}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \\
& = \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right) \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right) \\
& \quad + O\left(\frac{1}{L^2}\right), \quad C > C_1
\end{aligned}$$

in the above asymptotic relations for large L . Indeed, notice that

$$\begin{aligned}
(6.12) \quad & \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\
& = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\
& \quad \times \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j
\end{aligned}$$

Therefore, for any not decreasing sequence a_j

$$\begin{aligned}
(6.13) \quad & \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\
& \leq \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\
& \quad \times \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j.
\end{aligned}$$

Indeed, assume for contrary that

$$\begin{aligned}
(6.14) \quad & \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\
& > \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\
& \quad \times \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j.
\end{aligned}$$

Then, applying the inequality (6.9) to the right-hand side of (6.14), we obtain:

$$\begin{aligned}
(6.15) \quad & \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\
& \quad \times \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\
& \leq \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\
& \quad \times \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\
& = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j.
\end{aligned}$$

Since the left-hand side of (6.14) is

$$\lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left(1 - \frac{2C}{(\tilde{\rho}_{1,2}E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j$$

(see relation (6.11)), then comparison of the last obtained term in (6.15) with the left-hand side of (6.14) enables us to write:

$$\begin{aligned} & \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\ & > \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j. \end{aligned}$$

The contradiction with the basic inequality (6.9) proves (6.13).

Taking into account (6.12) and (6.13), the extended version of (6.10) after application (6.9) now looks

$$\begin{aligned} (6.16) \quad & \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\ & = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\ & \leq \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\ & \quad \times \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\ & = \psi(C_1) \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\ & \quad \times \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \end{aligned}$$

On the other hand, the right-hand side of (6.5) can be rewritten

$$\begin{aligned} (6.17) \quad & \psi(C) \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\ & = \psi(C) \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\ & = \psi(C) \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j \\ & \quad \times \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C - 2C_1}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L} \right)^j. \end{aligned}$$

The last equality in (6.17) is the application of (6.12). From (6.16) and (6.17) we finally obtain $\psi(C_1) \leq \psi(C)$ for any positive $C_1 \geq C$.

The first statement of Lemma 6.2 is proved. The proof of the second statement of this lemma is similar. \square

In the following we need in stronger result that is given by Lemma 6.2. Namely, we will prove the following lemma.

Lemma 6.3. *If the sequence $\{c_i\}$ contains at least two distinct values, then the function $\psi(C)$ is a strictly decreasing function, and the function $\eta(C)$ is a strictly increasing function.*

Proof. In order to prove this lemma it is sufficient to prove that if the sequence $\{c_i\}$ is nontrivial, that is there are at least two distinct values of this sequence, then for any distinct real numbers $C_1 \neq C_2$ the values of functions are also distinct, that is, $\psi(C_1) \neq \psi(C_2)$ and $\eta(C_1) \neq \eta(C_2)$. Let us prove the first inequality: $\psi(C_1) \neq \psi(C_2)$. Rewrite (6.1) as

$$(6.18) \quad \psi(C) = \lim_{L \rightarrow \infty} \frac{\frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L}\right)^j}{\frac{1}{L} \sum_{j=0}^{L-1} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_\zeta + E_\zeta^2 - E_\zeta)L}\right)^j}.$$

The limit of the denominator is equal to $\exp\left(-\frac{2C}{(\tilde{\rho}_{1,2}-1)E_\zeta+E_\zeta^2}\right)$. The limit of the numerator does exist and bounded, since the sequence $\{c_i\}$ is assumed to be bounded. As well, according to the other representation following from Lemma 6.1 and relation (5.3), it is an analytic function in C taking a nontrivial set of values.

The analytic function $\psi(C)$ is defined for all real $C \geq 0$ and it can be extended analytically for the whole complex plane. For example, for real negative values C we arrive at the function $\eta(C) = \psi(-C)$. According to the maximum principle for the module of an analytic function, if an analytic function takes the same values in two distinct points inside a domain, that the function must be the constant. If $c_i = c^*$ for all $i = 1, 2, \dots$, then this is just the case where $\psi(C) = c^*$ for all C . If there exist i_0 and i_1 such that $c_{i_0} \neq c_{i_1}$, then the function $\psi(C)$ cannot be a constant, because the analytic function is uniquely defined by the coefficients in the series expansion. So, the inequality $\psi(C_1) \neq \psi(C_2)$ is proved. The proof of the second inequality $\eta(C_1) \neq \eta(C_2)$ is similar. \square

We are ready now to formulate and prove a main theorem on optimal control of the dam model considered in the present paper.

Theorem 6.4. *Under the assumption that the costs c_i are not increasing, and under additional mild conditions of Theorems 4.3 and 4.4, there is a unique solution to the control problem. The solution to the control problem is defined by choice of the parameter ρ_1 as follows.*

Let \overline{C} be the minimum value of the functional J^{upper} defined in (5.2) and (5.3) and, respectively, let \underline{C} be the minimum value of the functional J^{lower} defined in (5.4) and (5.5). Then at least one of two parameters \overline{C} or \underline{C} must be equal to zero.

(1) In the case $\overline{C} = 0$ and $\underline{C} > 0$, the solution to the control problem is achieved for $\rho_1 = 1 - \delta$, where positive δ vanishes such that $\delta L \rightarrow \underline{C}$ as $L \rightarrow \infty$.

(2) In the case $\underline{C} = 0$ and $\overline{C} > 0$, the solution to the control problem is achieved for $\rho_1 = 1 + \delta$, where positive δ vanishes such that $\delta L \rightarrow \overline{C}$ as $L \rightarrow \infty$.

(3) In the case where both $\overline{C} = 0$ and $\underline{C} = 0$, the solution to the control problem is $\rho_1 = 1$.

Proof. Note first, that under the assumptions made there is a unique solution to the control problem considered in this paper. Indeed, a solution contains two terms one of them corresponds to the expression for $p_1 J_1 + p_2 J_2$ in (1.2) and another one corresponds to the term $\sum_{i=L^{\text{lower}}+1}^{L^{\text{upper}}} c_i q_i$ in (1.2). The first term of a solution is related to the models where the water costs are not taken into account, while the additional second term is related to the extended problem, where the water costs are taken into account.

In the case where the water costs are not taken into account, the existence of a unique solution to the control problem for the particular system in [4] follows from the main result of that paper. The same result holds true for a more general model with compound Poisson input flow but without water costs included. The last is supported by Theorems 3.10 - 3.13, which are similar to those Theorems 4.1 - 4.4 of [4].

In the case of the dam model, where the water costs are taken into account, the second term that is present in the solution is either c^{upper} or c^{lower} . According to Lemma 6.1 $c^{\text{upper}} = \psi(C)$ and $c^{\text{lower}} = \eta(C)$, and according to Lemmas 6.2 and 6.3 the function $\psi(C)$ is strictly decreasing in C , while the function $\eta(C)$ is strictly

increasing in C , and $\psi(0) = \eta(0) = c^*$. According to these properties, there is a unique solution to the control problem considered in the present paper as well, and it satisfies the following properties.

In the case where the both minima of J^{upper} and J^{lower} are achieved in $C = 0$, that is both $\overline{C} = 0$ and $\underline{C} = 0$, then $c^{\text{upper}} = c^{\text{lower}} = c^*$ and the term $p_1 J_1 + p_2 J_2$ of the objective function in (1.2) coincides with the term

$$j_1 \frac{(\tilde{\rho}_{1,2} - 1)E_\zeta + E_\zeta^2}{2} + j_2 \frac{\rho_2}{1 - \rho_2} \cdot \frac{(\tilde{\rho}_{1,2} - 1)E_\zeta + E_\zeta^2}{2}$$

in (5.1). That is both the minimum of J^{upper} and that of J^{lower} are the same, and they are equal to the right-hand side of (5.1). In this case the minimum of the objective function in (1.2) is achieved for $\rho_1 = 1$.

If the minimum of J^{lower} is achieved for $C = \underline{C} > 0$, then, since $\eta(C)$ is strictly increasing, we have $c^{\text{lower}} > c^*$, and hence the term $p_1 J_1 + p_2 J_2$ of the objective function in (1.2) satisfies the inequality:

$$p_1 J_1 + p_2 J_2 < j_1 \frac{(\tilde{\rho}_{1,2} - 1)E_\zeta + E_\zeta^2}{2} + j_2 \frac{\rho_2}{1 - \rho_2} \cdot \frac{(\tilde{\rho}_{1,2} - 1)E_\zeta + E_\zeta^2}{2}.$$

This implies that J^{lower} is less than the right-hand side of (5.1). On the other hand, in this case $c^{\text{upper}} < c^*$, and the minimum of J^{upper} must be achieved for $C = \overline{C} = 0$. In this case the minimum of the objective function in (1.2) is achieved for $\rho_1 = 1 - \delta$, where positive δ vanishes as $L \rightarrow \infty$, and $L\delta \rightarrow \underline{C}$.

In the opposite case, if the minimum of J^{upper} is achieved for $C = \overline{C} > 0$, then the arguments are similar to those above, and J^{upper} is not greater than the right-hand side of (5.1). The minimum of J^{lower} must be achieved for $C = \overline{C} = 0$. In this case the minimum of the objective function in (1.2) is achieved for $\rho_1 = 1 + \delta$, where positive δ vanishes as $L \rightarrow \infty$, and $L\delta \rightarrow \overline{C}$. \square

From Theorem 6.4 we have the following evident property of the optimal control.

Corollary 6.5. *The solution to the control problem can be $\rho_1 = 1$ only in the case $j_1 \leq j_2 \frac{\rho_2}{1 - \rho_2}$. Specifically, the equality is achieved only for $c_i \equiv c$, $i = 1, 2, \dots, L$, where c is an arbitrary positive constant.*

Although Corollary 6.5 provides a partial answer to the question 2 posed in the introduction, the answer is not useful, since it is an evident extension of the result

of [4]. A more constructive answer to question 2 of the introduction is obtained for the special case considered in the next section.

7. EXAMPLE OF LINEAR COSTS

In this section we study an example related to the case of linear costs.

Assume that c_1 and $c_L < c_1$ are given. Then the assumption that the costs are linear means, that

$$(7.1) \quad c_i = c_1 - \frac{i-1}{L-1}(c_1 - c_L), \quad i = 1, 2, \dots, L.$$

It is assumed that as L is changed, the costs are recalculated as follows. The first and last values of the cost c_1 and c_L remains the same. Other costs in the intermediate points are recalculated according to (7.1).

Therefore, to avoid confusing with the appearance of the index L for the fixed (unchangeable) values of cost c_1 and c_L , we use the other notation: $c_1 = \bar{c}$ and $c_L = \underline{c}$. Then (7.1) has the form

$$(7.2) \quad c_i = \bar{c} - \frac{i-1}{L-1}(\bar{c} - \underline{c}), \quad i = 1, 2, \dots, L.$$

In the following we shall also use the inverse form of (7.2). Namely,

$$(7.3) \quad c_{L-i} = \underline{c} + \frac{i}{L-1}(\bar{c} - \underline{c}), \quad i = 0, 1, \dots, L-1.$$

Apparently,

$$(7.4) \quad c^* = \frac{\bar{c} + \underline{c}}{2}.$$

For c^{upper} we have

$$\begin{aligned}
(7.5) \quad c^{\text{upper}} = \psi(C) &= \lim_{L \rightarrow \infty} \frac{\sum_{j=0}^{L-1} \left(\underline{c} + \frac{j}{L-1} (\bar{c} - \underline{c}) \right) \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta}) L} \right)^j}{\sum_{j=0}^{L-1} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta}) L} \right)^j} \\
&= \underline{c} + (\bar{c} - \underline{c}) \lim_{L \rightarrow \infty} \frac{1}{L-1} \cdot \frac{\sum_{j=0}^{L-1} j \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta}) L} \right)^j}{\sum_{j=0}^{L-1} \left(1 - \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta}) L} \right)^j} \\
&= \underline{c} + (\bar{c} - \underline{c}) \frac{(\tilde{\rho}_{1,2} - 1) E_{\zeta} + E_{\zeta}^2}{2C} \\
&\quad \times \frac{-\frac{2C}{(\tilde{\rho}_{1,2} - 1) E_{\zeta} + E_{\zeta}^2} + \exp\left(\frac{2C}{(\tilde{\rho}_{1,2} - 1) E_{\zeta} + E_{\zeta}^2}\right) - 1}{\exp\left(\frac{2C}{(\tilde{\rho}_{1,2} - 1) E_{\zeta} + E_{\zeta}^2}\right) - 1}.
\end{aligned}$$

For example, as C converges to zero in (7.5), then c^{upper} converges to $\underline{c} + \frac{1}{2}(\bar{c} - \underline{c}) = c^*$. This is in agreement with the statement of Proposition 5.1.

In turn, for c^{lower} we have

$$\begin{aligned}
(7.6) \quad c^{\text{lower}} = \eta(C) &= \lim_{L \rightarrow \infty} \frac{\sum_{j=0}^{L-1} \left(\underline{c} + \frac{j}{L-1} (\bar{c} - \underline{c}) \right) \left(1 + \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta}) L} \right)^j}{\sum_{j=0}^{L-1} \left(1 + \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta}) L} \right)^j} \\
&= \underline{c} + (\bar{c} - \underline{c}) \lim_{L \rightarrow \infty} \frac{1}{L-1} \cdot \frac{\sum_{j=0}^{L-1} j \left(1 + \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta}) L} \right)^j}{\sum_{j=0}^{L-1} \left(1 + \frac{2C}{(\tilde{\rho}_{1,2} E_{\zeta} + E_{\zeta}^2 - E_{\zeta}) L} \right)^j} \\
&= \underline{c} + (\bar{c} - \underline{c}) \frac{(\tilde{\rho}_{1,2} - 1) E_{\zeta} + E_{\zeta}^2}{2C} \\
&\quad \times \frac{\frac{2C}{(\tilde{\rho}_{1,2} - 1) E_{\zeta} + E_{\zeta}^2} - 1 + \exp\left(-\frac{2C}{(\tilde{\rho}_{1,2} - 1) E_{\zeta} + E_{\zeta}^2}\right)}{1 - \exp\left(-\frac{2C}{(\tilde{\rho}_{1,2} - 1) E_{\zeta} + E_{\zeta}^2}\right)}.
\end{aligned}$$

Again, as C converges to zero in (7.6), then c^{lower} converges to $\underline{c} + \frac{1}{2}(\bar{c} - \underline{c}) = c^*$.

So, we arrive at the agreement with the statement of Proposition 5.1.

Let us now discuss question 2 posed in the introduction. We cannot give the explicit solution because the calculations are very routine and cumbersome. However, we explain the way of the solution of this problem and find a numerical result. For simplicity, the input flow in the numerical example is assumed to be ordinary Poisson, that is we set $E_\zeta = 1$ and $E_\zeta^2 = 1$ in our calculations.

Following Corollary 6.5, take first $j_1 = j_2 \frac{\rho_2}{1-\rho_2}$. Clearly, that for these relation between parameters j_1 and j_2 the minimum of J^{lower} must be achieved for $C = 0$, while the minimum of J^{upper} must be achieved for a positive C . Now, keeping j_1 fixed assume that j_2 increases. Then, the problem is to find the value for parameter j_2 such that the value C corresponding to the minimization problem of J^{upper} reaches the point 0.

In our example we take $j_1 = 1$, $\rho_2 = \frac{1}{2}$, $\underline{c} = 1$, $\bar{c} = 2$, $\tilde{\rho}_{1,2} = 1$. In the table below we outline some values j_2 and the corresponding value C for optimal solution of functional J^{upper} . It is seen from the table that the optimal value is achieved in the case $j_2 \approx 1.34$. Therefore, in the present example $j_1 = 1$ and $j_2 \approx 1.34$ lead to the optimal solution $\rho_1 = 1$.

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Parameter	Argument of optimal value
j_2	C
1.06	0.200
1.07	0.190
1.08	0.182
1.09	0.174
1.10	0.165
1.11	0.156
1.12	0.149
1.13	0.140
1.14	0.134
1.15	0.126
1.16	0.120
1.17	0.112
1.18	0.104
1.19	0.096
1.20	0.090
1.25	0.055
1.30	0.022
1.33	0.010
1.34	0

TABLE 1. The values of parameter j_2 and corresponding arguments of optimal value C

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