ASYMPTOTIC BEHAVIOR OF BRANCHING POPULATIONS BEFORE EXTINCTION

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Abstract. Under the assumption that the initial population size of a Galton-Watson branching process increases to infinity, the paper studies asymptotic behavior of the population size before extinction. More specifically, we establish asymptotic properties of the conditional moments (which are exactly defined in the paper).

1. Introduction and the main result

We consider a Galton-Watson branching process $\{X_n\}_{n\geq 0}$,

$$X_{n+1} = \sum_{j=1}^{X_n} \xi_{n,j}, \tag{1.1}$$

where X_n denotes the number of offspring in the nth generation for a population starting from K offsprings, $X_0 = K$, and throughout the paper the initial size of population K is assumed to be a large value. Such a type of branching process can be a model of real population of animals, insects etc., and the main results of our study can have applications to analysis of real populations arising in biology (e.g. Jagers [11], Haccou, Jagers and Vatutin [10], Jagers and Klebaner [13]). For other study of branching processes with a large initial population size see also Borovkov [1], Klebaner [17], Klebaner and Liptser [18].

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The study of branching populations before extinction has been initiated by Jagers [12] and then resulted in papers of Jagers, Klebaner and Sagitov [14] and [15]. The approach of these papers [14] and [15] is based on analytic techniques for studying paths to extinction with the following analysis of asymptotic behavior of these paths.

The present paper presents an alternative way to study asymptotic behavior of large populations before extinction, and the approach of the present paper is based on diffusion approximations of the original branching process with large initial population as well as a series of auxiliary processes. Those diffusion approximations are then used to study asymptotic behavior of conditional moments of a population size before extinction as it is explained below. The approach of our paper (including diffusion approximations, asymptotic expansions and sample path techniques) remains correct for much wider classes of branching processes than that traditional branching process and includes for instance bisexual Galton-Watson branching processes [4], [5] and different type of controlled ϕ-branching Galton-Watson processes (which need not be subcritical, as it is assumed in the paper later). The ϕ-branching processes have been introduced by Sevastyanov and Zubkov [21], and intensively studied in many papers (e.g. Bruss [2], [3], González, Molina and Del Puerto [7], [8], [9], Yanev [22], Zubkov [23]).

The main results of the present paper are presented in Theorem 1.1 below.

Assume that $\xi_{n,j}$, (n=1,2,...; j=1,2,...) have the same distribution for all n and j and are mutually independent, and there exists the second moment $\mathrm{E}\xi_{n,j}^2 < \infty$. Denoting $\mathfrak{m} = \mathrm{E}\xi_{n,j}$ and $\mathfrak{S}^2 = \mathrm{var}(\xi_{n,j})$, assume that $\mathfrak{m} < 1$. Under this last assumption the extinction time of the branching process always exists with probability 1. Let $\tau = \tau_K$ be that moment of extinction. The random variable τ_K is a stopping time associated with the sequence $\{X_n\}_{n\geq 0}$. We assume that the family of all stopping times $\{\tau_K\}$ (for different values K) is defined on a filtered probability space $\{\Omega, \mathscr{F}_0, \mathbf{F}=(\mathscr{F}_{0,K}), \mathrm{P}\}$, $\mathscr{F}_{0,K}\subset\mathscr{F}_{0,K+1}\subset\ldots\subset\mathscr{F}_0$. (The meaning of the index 0 will be clear later.)

The paper studies asymptotic behavior of the branching population before extinction as K increases to infinity, and the main result of our study, formulated below, as well as the analysis of the paper, use the notation $\asymp$ for asymptotic equivalence between two main parts of expansion. The notation is used in order to reduce irrelevant background explanations and to avoid multiple using of lim in different senses or expansions with remainder, where it is not significant. For example, relations (1.2) and (1.3) (see formulation of the theorem below) should be read as follows: *For any sufficiently small positive ϵ and δ there exists a large integer K such that*

$$\begin{aligned}&\mathrm{P}\Big\{(1-\delta)X^l_{\lfloor u_2\tau_K\rfloor}\mathrm{E}\mathfrak{m}^{l(\lfloor u_1\tau_K\rfloor-\lfloor u_2\tau_K\rfloor)}\\&\quad\leq\mathrm{E}(X^l_{\lfloor u_1\tau_K\rfloor}\mid X_{\lfloor u_2\tau_K\rfloor})\leq(1+\delta)X^l_{\lfloor u_2\tau_K\rfloor}\mathrm{E}\mathfrak{m}^{l(\lfloor u_1\tau_K\rfloor-\lfloor u_2\tau_K\rfloor)}\Big\}>1-\epsilon,\\&\mathrm{P}\Big\{(1-\delta)X^l_{\lfloor u_1\tau_K\rfloor}\mathrm{E}\mathfrak{m}^{l(\lfloor u_2\tau_K\rfloor-\lfloor u_1\tau_K\rfloor)}\\&\quad\leq\mathrm{E}(X^l_{\lfloor u_2\tau_K\rfloor}\mid X_{\lfloor u_1\tau_K\rfloor})\leq(1+\delta)X^l_{\lfloor u_1\tau_K\rfloor}\mathrm{E}\mathfrak{m}^{l(\lfloor u_2\tau_K\rfloor-\lfloor u_1\tau_K\rfloor)}\Big\}>1-\epsilon,\end{aligned}$$

and

$$\begin{aligned}&\mathrm{P}\Big\{(1-\delta)K^l\mathfrak{m}^{l\lfloor u_1\tau_K\rfloor}\\&\quad\leq\mathrm{E}(X^l_{\lfloor u_1\tau_K\rfloor}\mid\tau_K)\leq(1+\delta)K^l\mathfrak{m}^{l\lfloor u_1\tau_K\rfloor}\Big\}>1-\epsilon.\end{aligned}$$

In the places where it is required and looks more profitable (e.g. Section 6), the explicit form of asymptotic expansion with remainder is used nevertheless.

Theorem 1.1. *Let $0 < u_1 < u_2 < 1$ be two real numbers. Then, as $K \to \infty$,*

$$\begin{aligned} &\mathrm{E}\{X^l_{\lfloor u_1\tau_K\rfloor} \mid X_{\lfloor u_2\tau_K\rfloor}\} \asymp X^l_{\lfloor u_2\tau_K\rfloor}\mathrm{E}\mathfrak{m}^{l(\lfloor u_1\tau_K\rfloor - \lfloor u_2\tau_K\rfloor)},\\ &\mathrm{E}\{X^l_{\lfloor u_2\tau_K\rfloor} \mid X_{\lfloor u_1\tau_K\rfloor}\} \asymp X^l_{\lfloor u_1\tau_K\rfloor}\mathrm{E}\mathfrak{m}^{l(\lfloor u_2\tau_K\rfloor - \lfloor u_1\tau_K\rfloor)}, \end{aligned} \tag{1.2}$$

and

$$\mathrm{E}\{X^l_{\lfloor u_1\tau_K\rfloor} \mid \tau_K\} \asymp K^l\mathfrak{m}^{l\lfloor u_1\tau_K\rfloor}, \tag{1.3}$$

where $\lfloor z\rfloor$ is the notation for the integer part of z. As $K \to \infty$, $\frac{\tau_K}{\log K}$ converges in probability to the constant $c = -\frac{1}{\log \mathfrak{m}}$.

The proof of the main result is based on the following lemma.

Lemma 1.2. *For any finite-dimensional vector $\{X_{i_1}, X_{i_2}, \ldots, X_{i_n}\}$, $1 \leq i_1 < i_2 < \ldots < i_n < \infty$,*

$$\lim_{K\to\infty} \mathrm{P}\left\{\frac{X_{i_1} - K\mathfrak{m}^{i_1}}{\mathfrak{S}\sqrt{K}} \leq x_1, \frac{X_{i_2} - K\mathfrak{m}^{i_2}}{\mathfrak{S}\sqrt{K}} \leq x_2, \ldots, \frac{X_{i_n} - K\mathfrak{m}^{i_n}}{\mathfrak{S}\sqrt{K}} \leq x_n\right\}$$
$$= \mathrm{P}\{\theta_{i_1} \leq x_1, \theta_{i_2} \leq x_2, \ldots, \theta_{i_n} \leq x_n\},$$

where $\{\theta_1, \theta_2, \ldots\}$ is a Gaussian sequence with $\mathrm{E}\theta_j = 0$ and $\mathrm{cov}(\theta_j, \theta_{j+n}) = \mathfrak{m}^n \,\mathrm{var}(\theta_j) + n\mathfrak{m}^{j+n-1}$, $\mathrm{var}(\theta_{j+1}) = \mathfrak{m}^2 \,\mathrm{var}(\theta_j) + \mathfrak{m}^j$, $\mathrm{var}(\theta_1) = 1$.

Lemma 1.2 is known from the literature, and its proof can be found in Klebaner and Nerman [19]. For the purpose of the present paper we, however, need in an alternative proof of this lemma, which follows from the asymptotic expansions presented here. Furthermore, the proof of Theorem 1.1 requires the intermediate asymptotic expansions obtained in the proof of Lemma 1.2 rather than the statement of Lemma 1.2 itself.

In this paper, simple asymptotic representations for all conditional moments before extinction are obtained. The most significant consequence of this analysis is a so-called *invariance* property of the conditional expectations. This property is discussed in Section 6.

The main idea of the method is as follows. The random sequence $\{X_n\}_{n\geq 0}$ is approximated by appropriate random sequences $\{Y_n^{(a)}\}_{n\geq 0}$ $(a\leq 1)$, as a tends to zero. For each fixed a we define stopping times $\tau_{a,K}$ (for different values K) associated with the process $Y_n^{(a)}$. $\tau_{a,K}$ is assumed to be measurable with respect to the σ-field $\mathscr{F}_{a,K}\subset\mathscr{F}_a$, where $\mathscr{F}_a=\cup_{K\geq 1}\mathscr{F}_{a,K}$, and $\mathscr{F}_a\subset\mathscr{F}_0$. For that fixed a the sequence $\tau_{a,K}$ converges (in definite sense) to $\ell(a)$ as $K\to\infty$ (the details are given in the paper). Then knowledge of the behavior of $Y_{u\tau_{a,K}}^{(a)}$, $0<u<1$, for which we have the corresponding relationship, enables us to study the behavior of its limit as a tends to zero. This limit is just $X_{u\tau_K}$, $0<u<1$. Other assumptions associated with definition of X_n and that of the associated processes $X_n^{(a)}$, $Y_n^{(a)}$ and other processes are given in the next section.

The rest of the paper is organized as follows. In Section 2 we introduce the auxiliary stochastic sequences $X_n^{(a)}$ and $Y_n^{(a)}$ and the stopping times associated with these sequences. The elementary properties of these random objects are studied. In Section 3 we continue to study the properties of the sequences $X_n^{(a)}$ and $Y_n^{(a)}$. Specifically, it is shown that these sequences are upper and lower bounds for the branching process X_n, and these bounds are tight as $a\to 0$. These properties are then used in order to prove the convergence results in the next sections. In Section 4 we derive asymptotic expansions and prove the convergence lemma to the Gaussian process, the parameters of which are explicitly defined in the formulation of Lemma 1.2. In Section 5 we prove Theorem 1.1. Last Section 6 discusses application of the main results of this study and establishes the invariance property.

2. **Stopping times and auxiliary processes associated with the Galton-Watson process**

In this section we approach the stopping time τ_K, the extinction moment, by introducing a parametric family of stopping times $\{\tau_{a,K}\}$, depending on the two parameters a and K. Specifically, for any real a, $0 \leq a < 1$ and integer K

$$\tau_{a,K} = \inf\{l : X_l \leq \lfloor aK \rfloor\}, \tag{2.1}$$

where $\lfloor aK \rfloor$ is the integer part of aK. The stopping time $\tau_{a,K}$ as well as the associated with these parameters a and K other corresponding random variables defined below are assumed to be measurable with respect $\mathscr{F}_{a,K} \subset \mathcal{F}_K$, and for two different values a_1 and a_2, $0 \leq a_2 < a_1 < 1$, we have $\mathscr{F}_{a_1,K} \subset \mathscr{F}_{a_2,K}$. If $a < 1$ is fixed and K_1, K_2 are distinct, $K_1 < K_2 < \infty$, then we have $\mathscr{F}_{a,K_1} \subset \mathscr{F}_{a,K_2}$. Then the two-parametric family of σ-fields $\{\mathscr{F}_{a,K}\}$ is increasing in the following sense. For any $0 \leq a_2 \leq a_1 < 1$ and integer $K_1 \leq K_2 < \infty$ we have $\mathscr{F}_{a_1,K_1} \subseteq \mathscr{F}_{a_2,K_2}$.

In accordance with this family of stopping times (2.1), consider a family of processes $X_{j,K}^{(a)} = X_j^{(a)}$ satisfying the recurrence relation (for notational convenience the additional index K is not provided):

$$X_{n+1}^{(a)} = \max\left\{\lfloor aK \rfloor, \sum_{j=1}^{X_n^{(a)}} \xi_{n,j}\right\}, \quad X_0^{(a)} = K. \tag{2.2}$$

The processes $X_{j,K}^{(a)}$ are assumed to be adapted with respect to the σ-fields $\mathscr{F}_{a,K}$. In addition, the processes $X_{j,K}^{(a)}$ are assumed to be measurable with respect to the the wider σ-field $\mathscr{F}_0$. Specifically, if there are two processes $X_{j,K}^{(a_1)}$ and $X_{j,K}^{(a_2)}$ with different a_1 and a_2, say $0 \leq a_2 \leq a_1 < 1$, then both of these processes $X_{j,K}^{(a_1)}$ and $X_{j,K}^{(a_2)}$ are measurable with respect to the σ-field $\mathscr{F}_{0,K}$, and, of course, with respect to the σ-field $\mathscr{F}_{a_2,K}$. All of these processes with different a are defined due to representation (2.2). This means that the processes $X_{j,K}^{(a)}$ are actually defined after their stopping times as well. For different a_1 and a_2 ($0 \leq a_2 \leq a_1 < 1$) the processes $X_{j,K}^{(a_1)}$ and $X_{j,K}^{(a_2)}$ are 'coupled' until the stopping time $\tau_{a_1,K}$, i.e. until that time instant their sample paths coincide, but after the time instant $\tau_{a_1,K}$ these processes are decoupled i.e. their paths become different. But the coupling arguments can be used nevertheless: after the time instant $\tau_{a_1,K}$ with the aid of Kalmykov's theorem [16] we have $X_{j,K}^{(a_1)} \geq_{st} X_{j,K}^{(a_2)}$, $j \geq \tau_{a_1,K}$ (see the next section for details).

Some mathematical details about these processes can be found in the next section. The similar coupling arguments hold for the processes $Y_{j,K}^{(a)}$ defined later, which are derivative from the processes $X_{j,K}^{(a)}$ (the further details can be found in the next section).

Let us transform (2.2) by adding and subtracting the term $\lfloor aK \rfloor$. To this end we use the following elementary property of numbers: $\max\{a,b\} - a = \max\{0, b-a\}$. Also there is used the fact that

$X_n^{(a)} \geq \lfloor aK \rfloor$ for any n. Then, we have

$$\begin{aligned}
&\left(X_{n+1}^{(a)} - \lfloor aK \rfloor\right) + \lfloor aK \rfloor \\
&= \left(\max\left\{\lfloor aK \rfloor, \sum_{j=1}^{X_n^{(a)}} \xi_{n,j}\right\} - \lfloor aK \rfloor\right) + \lfloor aK \rfloor \\
&= \max\left\{0, \sum_{j=1}^{X_n^{(a)}} \xi_{n,j} - \sum_{j=1}^{\lfloor aK \rfloor}\left[\xi_{n,j} + \left(1-\xi_{n,j}\right)\right]\right\} + \lfloor aK \rfloor \qquad (2.3)\\
&= \max\left\{0, \sum_{j=\lfloor aK \rfloor+1}^{X_n^{(a)}} \xi_{n,j} - \sum_{j=1}^{\lfloor aK \rfloor}\left(1-\xi_{n,j}\right)\right\} + \lfloor aK \rfloor.
\end{aligned}$$

Hence, denoting $Y_n^{(a)} = X_n^{(a)} - \lfloor aK \rfloor$ from (2.3) we obtain

$$\begin{aligned}
Y_{n+1}^{(a)} &= \max\left\{0, \sum_{j=\lfloor aK \rfloor+1}^{X_n^{(a)}} \xi_{n,j} - \sum_{j=1}^{\lfloor aK \rfloor}\left(1-\xi_{n,j}\right)\right\} \\
&= \max\left\{0, \sum_{j=1}^{Y_n^{(a)}} \xi'_{n,j} - \sum_{j=1}^{\lfloor aK \rfloor}\left(1-\xi_{n,j}\right)\right\} \\
&= \left(\sum_{j=1}^{Y_n^{(a)}} \xi'_{n,j} - \sum_{j=1}^{\lfloor aK \rfloor}\left(1-\xi_{n,j}\right)\right) \qquad (2.4)\\
&\quad \times \mathrm{I}\left(\sum_{j=1}^{Y_n^{(a)}} \xi'_{n,j} > \sum_{j=1}^{\lfloor aK \rfloor}\left(1-\xi_{n,j}\right)\right) \\
&= \left(\sum_{j=1}^{Y_n^{(a)}} \xi'_{n,j} - \sum_{j=1}^{\lfloor aK \rfloor}\left(1-\xi_{n,j}\right)\right) I_n^{(a)},
\end{aligned}$$

where $\xi'_{n,j} = \xi_{n,j+\lfloor aK \rfloor}$ ($\xi'_{n,1}$, $\xi'_{n,2}$,…are independent and identically distributed random variables having the same distribution as $\xi_{n,j}$), and $I_n^{(a)} = I_n^{(a)}(K) = \mathrm{I}\{\sum_{j=1}^{Y_n^{(a)}} \xi'_{n,j} > \sum_{j=1}^{\lfloor aK \rfloor}(1-\xi_{n,j})\}$ is the notation used in (2.4).

Thus we have the new family of processes $Y_n^{(a)}$, which is assumed, as mentioned before, to be measurable with respect to $\mathscr{F}_0$ and given on the same probability space $\{\Omega,\ \mathscr{F}_0,\ \mathrm{P}\}$. Recall that a stopping time $\tau_{a,K}$ and the sequence

$$\left\{Y_0^{(a)}, Y_1^{(a)}, \dots\right\} \tag{2.5}$$

are assumed to be adapted with respect to the σ-field $\mathscr{F}_{a,K}$, and the family of these σ-fields $\{\mathscr{F}_{a,K}\}$ is increasing when a decreases and K increases.

It is known that as $K \to \infty$, $\frac{X_n}{K}$ converges to $\mathfrak{m}^n$ in probability (see Klebaner and Nerman [19]). Using this result it is not difficult to prove that, as $K \to \infty$,

(*i*) $$\frac{Y_n^{(a)}}{K} \text{ converges to } \max\{0, \mathfrak{m}^n - a\} \text{ in probability,}$$

(*ii*) $$\frac{\mathrm{E}Y_n^{(a)}}{K} \text{ converges to } \max\{0, \mathfrak{m}^n - a\},$$

(*iii*) $$I_n^{(a)}(K) \text{ converges to } \chi_{n+1} = \chi_{n+1}(\mathfrak{m}, a) = \begin{cases} 1, & \text{if } \mathfrak{m}^{n+1} > a, \\ 0, & \text{otherwise} \end{cases}$$

in probability,

as well as,

(*iv*) $$\tau_{a,K} \text{ converges in probability to } \ell(a) = \min\{l : \mathfrak{m}^l \le a\}.$$

The proof of (i) is postponed to the end of Section 3. The proofs of (ii) – (iv) are similar to the proof of (i).

3. Properties of the sequences $X_n^{(a)}$ and $Y_n^{(a)}$

The study of this section we start from the properties of the random vectors (2.5). Let a_1, a_2 be two numbers, and $0 \leq a_2 \leq a_1 < 1$ and K is fixed. Then, in the suitable probability space for all events $\omega \in \Omega$ and $n \geq 0$

$$Y_n^{(a_1)}(\omega) \leq Y_n^{(a_2)}(\omega). \tag{3.1}$$

Indeed, consider two random vectors

$$\left\{Y_0^{(a_1)}, Y_1^{(a_1)}, \ldots\right\} \tag{3.2}$$

and

$$\left\{Y_0^{(a_2)}, Y_1^{(a_2)}, \ldots\right\} \tag{3.3}$$

Consider the stopping times $\{\tau_{a_1,K}, \mathscr{F}_{a_1,K}\}$ and $\{\tau_{a_2,K}, \mathscr{F}_{a_2,K}\}$ associated with the sequences (3.2) and (3.3). Since for fixed K, $\mathscr{F}_{a_1,K} \subseteq \mathscr{F}_{a_2,K}$, then $\tau_{a_1,K}(\omega) \leq \tau_{a_2,K}(\omega)$.

According to the definition of the sequence $X_n^{(a)}$ (see (2.2)), on the suitable probability space containing $\mathscr{F}_{a_2,K}$ we have the correspondence

$$X_i^{(a_1)}(\omega) = X_i^{(a_2)}(\omega), \quad i = 1, 2, \ldots, \tau_{a_1,K} - 1, \tag{3.4}$$

and at this stopping time $\tau_{a_1,K}$ we have $X_{\tau_{a_1,K}}^{(a_1)}(\omega) \geq X_{\tau_{a_1,K}}^{(a_2)}(\omega)$, and therefore according to Kalmykov's theorem [16]:

$$X_i^{(a_1)} \geq_{st} X_i^{(a_2)}, \quad i = \tau_{a_1,K}, \tau_{a_1,K} + 1, \ldots, \tau_{a_2,K}, \ldots, \tau_{0,K}.$$

Therefore, in a suitable probability space

$$X_i^{(a_1)}(\omega) \geq X_i^{(a_2)}(\omega), \quad i = \tau_{a_1,K}, \tau_{a_1,K} + 1, \ldots, \tau_{a_2,K}, \ldots, \tau_{0,K}. \tag{3.5}$$

Thus, we showed

$$X_i^{(a_1)}(\omega) \geq X_i^{(a_2)}(\omega), \quad i = 1, 2, \ldots, \tau_{a_1,K}, \tau_{a_1,K} + 1, \ldots, \tau_{a_2,K}, \ldots, \tau_{0,K}.$$

From this correspondence (3.4) and (3.5) according to the definition of the sequence $Y_n^{(a)}$ (see (2.4)) on the same probability space we have

$$Y_i^{(a_1)}(\omega) + \lfloor a_1 K \rfloor = Y_i^{(a_2)}(\omega) + \lfloor a_2 K \rfloor, \quad i = 1, \ldots, \tau_{a_1,K} - 1, \tag{3.6}$$

and therefore up to time $\tau_{a_1,K} - 1$ the inequality $Y_i^{(a_1)}(\omega) \leq Y_i^{(a_2)}(\omega)$ is obvious. At time instant $\tau_{a_1,K}$ we have $Y_{\tau_{a_1,K}}^{(a_1)}(\omega) = 0$, while $Y_{\tau_{a_1,K}}^{(a_2)}(\omega)$ is nonnegative in general. The further behavior of the processes $Y_i^{(a_1)}(\omega)$ and $Y_i^{(a_2)}(\omega)$ after time $\tau_{a_1,K}$ is specified by coupling arguments, where the initial inequality $Y_i^{(a_1)}(\omega) \leq Y_i^{(a_2)}(\omega)$ before the stopping time $\tau_{a_1,K}$ remains true after this stopping time as well. If for some $i = i_0$, $Y_{i_0}^{(a_1)}(\omega) = Y_{i_0}^{(a_2)}(\omega)$ $(= 0)$, then the both processes are coupled until $i_1 \geq i_0$. If after time i_1, $Y_{i_1+1}^{(a_2)}(\omega)$ becomes positive, then we again arrive at the inequality $Y_{i_1+1}^{(a_1)}(\omega) \leq Y_{i_1+1}^{(a_2)}(\omega)$, and so on.

Taking into account that according to the definition $X_n^{(0)}$ coincides with X_n, we obtain the inequality

$$Y_n^{(a)}(\omega) \leq X_n(\omega) \leq X_n^{(a)}(\omega), \tag{3.7}$$

being correct for all $\omega \in \Omega$ and all $n \geq 0$ as well as for any initial population K and any a. This inequality is also tight as $a \to 0$, because according to the definition of the above sequences, $Y_n^{(0)}(\omega) = X_n^{(0)}(\omega)$ for all n.

Let us now prove the above properties $(i) - (iv)$. Find the limit in probability of $\frac{Y_n^{(a)}}{K}$ as $K \to \infty$. Notice first, that according to (2.2) $\frac{X_1^{(a)}}{K}$ converges to $\max\{a, \mathfrak{m}\}$ in probability, and according to Wald's equation [6], p.384, $\frac{\mathrm{E}X_1^{(a)}}{K}$ converge to the same limit $\max\{a, \mathfrak{m}\}$. Therefore, $\frac{Y_1^{(a)}}{K}$ converges to $\max\{a, \mathfrak{m}\} - a = \max\{0, \mathfrak{m} - a\}$ in probability, and $\frac{\mathrm{E}Y_1^{(a)}}{K}$ converges to $\max\{0, \mathfrak{m} - a\}$. Now, assuming that for some k it is already proved that $\frac{Y_k^{(a)}}{K}$ converges to $\max\{0, \mathfrak{m}^k - a\}$ in probability and $\frac{\mathrm{E}Y_k^{(a)}}{K}$ converges to $\max\{0,\mathfrak{m}^k-a\}$, by induction we have as follows. If $\mathfrak{m}^k\leq a$ then $\frac{Y_k^{(a)}}{K}$ converges to 0 in probability and $\frac{\mathrm{E}Y_k^{(a)}}{K}$ converges to 0, and consequently,

$$\mathrm{E}I_k^{(a)}=\mathrm{P}\left\{\frac{1}{K}\sum_{j=1}^{Y_k^{(a)}}\xi'_{k,j}>\frac{1}{K}\sum_{j=1}^{\lfloor aK\rfloor}\Big(1-\xi_{k,j}\Big)\right\}\to 0.$$

The last is true because

$$\frac{1}{K}\mathrm{E}\sum_{j=1}^{Y_k^{(a)}}\xi'_{k,j}=\frac{1}{K}\mathrm{E}\sum_{j=1}^{Y_k^{(a)}}\xi_{k,j}=\frac{\mathfrak{m}\mathrm{E}Y_k^{(a)}}{K}\to 0.$$

Therefore, according to (2.4) $\frac{\mathrm{E}Y_{k+1}^{(a)}}{K}$ vanishes, and $\frac{Y_{k+1}^{(a)}}{K}$ vanishes in probability. Therefore, the assumption $\mathfrak{m}^k\leq a$ is not the case. Hence, assuming that $\frac{Y_k^{(a)}}{K}$ converges to $\mathfrak{m}^k-a$ in probability, where $\mathfrak{m}^k>a$, we have the following:

$$\begin{aligned}\lim_{K\to\infty}\frac{\mathrm{E}Y_{k+1}^{(a)}}{K}&=\max\{0,(\mathfrak{m}^k-a)\mathfrak{m}-a(1-\mathfrak{m})\}\\&=\max\{0,\mathfrak{m}^{k+1}-a\}.\end{aligned}$$

Thus, as $K\to\infty$, $\frac{Y_n^{(a)}}{K}$ converges to $\max\{0,\mathfrak{m}^n-a\}$ in probability, and (i) is proved. Notice, that (ii), (iii) and (iv) follow together with (i). All these claims are closely related, and their proof is similar.

Notice also, that the convergence of $\frac{Y_n^{(a)}}{K}$ to $\max\{0,\mathfrak{m}^n-a\}$ in probability means that in a suitable probability space, the sequence $\frac{Y_n^{(a)}(\omega)}{K}$ converges almost surely to $\max\{0,\mathfrak{m}^n-a\}$.

4. **Asymptotic expansions and the proof of Lemma 1.2**

Pathwise inequalities (3.7) and $\frac{Y_n^{(a)}}{K}\leq\frac{X_n}{K}$ hold for any initial size K and any a. Therefore the appropriate normalized sequences $\frac{Y_n^{(a)}}{K}$ and $\frac{X_n^{(a)}}{K}$ converge to the same limit in probability as $K\to\infty$. If there exists the limit in distribution of $\frac{X_n - \mathrm{E}X_n}{\sqrt{K}}$ as $K \to \infty$, then because of the equality $\frac{Y_n^{(a)} - \mathrm{E}Y_n^{(a)}}{\sqrt{K}} = \frac{X_n^{(a)} - \mathrm{E}X_n^{(a)}}{\sqrt{K}}$, and the inequality $Y_n^{(a)}(\omega) \leq X_n(\omega)$ for all $a \geq 0$ and all $\omega \in \Omega$ (see ref. (3.7)), there are also the limits in distribution of $\frac{Y_n^{(a)} - \mathrm{E}Y_n^{(a)}}{\sqrt{K}}$ and $\frac{X_n^{(a)} - \mathrm{E}X_n^{(a)}}{\sqrt{K}}$ as $K \to \infty$ and $a \to 0$ independently. That is, one can let $K \to \infty$ before $a \to 0$, or converse. Notice, that the limiting distribution of $\frac{X_n - \mathrm{E}X_n}{\sqrt{K}}$ has been obtained in [19], and it also follows from asymptotic expansions obtained in this section.

It follows from the results of Section 3 that, as $K \to \infty$, $\tau_{a,K}$ converges in probability to

$$\ell(a) = \min\{l : \mathfrak{m}^l \leq a\}.$$

and hence, in the case where K increases to infinity first, $\ell(a) = \text{P-}\lim_{K \to \infty} \tau_{a,K}$ (P-lim denotes a limit in probability). It is known (see e.g. Pakes [20]), that $\frac{\tau_K}{\log K}$ converges to the constant $c = -\frac{1}{\log \mathfrak{m}}$ in probability. This result of Pakes [20] can be proved by different ways. The advantage of the proof given below is that it remains true for more general models than the usual Galton-Watson branching process, resulting in the justice of the results of the paper for general models as well. For instance, one can reckon that a bisexual Galton-Watson branching process starting with K mating units is considered, where $\mathfrak{m}$ now has the meaning of the average reproduction mean per mating unit (see Bruss [4]). For the relevant result related to the ϕ-branching processes see Bruss [3], Theorem 1.

For large $X_0 = K$ we have as follows:

$$\begin{aligned}\tau_K :&= \inf\{t \in \mathbb{N} : X_t = 0\}\\ &= \inf\{t \in \mathbb{N} : X_t < 1\}\\ &= \inf\left\{t \in \mathbb{N} : \frac{X_t}{X_0} < \frac{1}{K}\right\}\\ &= \inf\left\{t \in \mathbb{N} : \prod_{n=1}^{t} \frac{X_n}{X_{n-1}} < \frac{1}{K}\right\}.\end{aligned} \tag{4.1}$$

Now note that, as $K \to \infty$, each fraction $\frac{X_n}{X_{n-1}}$ converge to $\mathfrak{m}$ in probability. Indeed,

$$\frac{X_n}{X_{n-1}} = \frac{X_n}{K} \cdot \frac{K}{X_{n-1}}. \tag{4.2}$$

According to [19], $\frac{X_n}{K} \to \mathfrak{m}^n$ in probability as $K \to \infty$. Therefore, the fraction (4.2) converges to $\frac{\mathfrak{m}^n}{\mathfrak{m}^{n-1}} = \mathfrak{m}$ in probability for any n.

On the other hand, by virtue of Wald's identity [6], p.384 we obtain:

$$\frac{\mathrm{E}X_n}{\mathrm{E}X_{n-1}} = \frac{\mathrm{E}\sum_{j=1}^{X_{n-1}} \xi_{n-1,j}}{\mathrm{E}X_{n-1}} = \frac{\mathfrak{m}\mathrm{E}X_{n-1}}{\mathrm{E}X_{n-1}} = \mathfrak{m}. \tag{4.3}$$

So, according to (4.2) and (4.3), the limit in probability of the fraction $\frac{X_n}{X_{n-1}}$ as $K \to \infty$ and the fraction of the corresponding expectations $\frac{\mathrm{E}X_n}{\mathrm{E}X_{n-1}}$ are the same.

From (4.3) we therefore obtain:

$$\begin{aligned}\mathrm{E}\left(\frac{X_t}{X_0}\right) = \frac{1}{K}\mathrm{E}X_t &= \prod_{n=1}^{t} \frac{\mathrm{E}X_n}{\mathrm{E}X_{n-1}}\\ &= \mathfrak{m}^t.\end{aligned} \tag{4.4}$$

So,

$$\lim_{K\to\infty} K\mathrm{E}\mathfrak{m}^{\tau_K} = 1.$$

Similarly to (4.4), we also have that $\left(\frac{X_t}{X_0}\right)$ converges to $\mathfrak{m}^t$ in probability as $K \to \infty$ for any integer t. So, from (4.1) and (4.4) we have the similar limit as in the case of expectations, i.e. for any positive ϵ and δ there exists integer K large enough such that $\mathrm{P}\{|K\mathfrak{m}^{\tau_K}-1|>\delta\}<\epsilon$, i.e. $K\mathfrak{m}^{\tau_K}\to 1$ in probability as $K\to\infty$, and thus $\frac{\tau_K}{\log K}$ converges to $-\frac{1}{\log\mathfrak{m}}$ in probability.

From (2.4) for $Y_{j+1}^{(a)}$, $j=0,1,\ldots$, we obtain the following equations:

(4.5)
$$Y_{j+1}^{(a)}-\mathfrak{m}I_j^{(a)}Y_j^{(a)}=\mathfrak{S}\sqrt{I_j^{(a)}K}\sum_{i=1}^{Y_j^{(a)}}\frac{\xi_{j,i}-\mathfrak{m}}{\mathfrak{S}\sqrt{I_j^{(a)}K}}-I_j^{(a)}\sum_{i=1}^{\lfloor aK\rfloor}(1-\xi_{j,i})$$

where $0\cdot\infty$ is assumed to be 0.

Assuming that K increases to infinity, and dividing both sides of (4.5) by large parameter $\mathfrak{S}\sqrt{K}$ we have the following expansions

(4.6)
$$\begin{aligned}&\frac{Y_{j+1}^{(a)}-\mathfrak{m}\chi_{j+1}Y_j^{(a)}}{\mathfrak{S}\sqrt{K}}\\ &\asymp\chi_{j+1}\sum_{i=1}^{Y_j^{(a)}}\frac{\xi_{j,i}-\mathfrak{m}}{\mathfrak{S}\sqrt{\chi_{j+1}K}}-\chi_{j+1}\frac{\sqrt{K}}{\mathfrak{S}K}\sum_{i=1}^{\lfloor aK\rfloor}(1-\xi_{j,i})\end{aligned}$$

or

(4.7)
$$\begin{aligned}&\chi_{j+1}\cdot\frac{Y_{j+1}^{(a)}-\mathfrak{m}Y_j^{(a)}}{\mathfrak{S}\sqrt{K}}\\ &\asymp\chi_{j+1}\left(\sum_{i=1}^{Y_j^{(a)}}\frac{\xi_{j,i}-\mathfrak{m}}{\mathfrak{S}\sqrt{K}}-\frac{a\sqrt{K}}{\mathfrak{S}}\Big(1-\mathfrak{m}\Big)\right).\end{aligned}$$

For $j=0,1,\ldots,\ell(a)-2$, $\ell(a)\geq 2$, one can remove the term χ_{j+1} from the both sides of (4.7).

Therefore, for $j=0,1,\ldots,\ell(a)-2$, the left-hand side of (4.7) can be transformed as follows:

$$\begin{aligned}
&\frac{Y_{j+1}^{(a)}-\mathfrak{m}Y_j^{(a)}}{\mathfrak{S}\sqrt{K}}\\
&=\frac{Y_{j+1}^{(a)}-\mathfrak{m}\mathrm{E}Y_j^{(a)}+\mathfrak{m}\mathrm{E}Y_j^{(a)}-\mathfrak{m}Y_j^{(a)}}{\mathfrak{S}\sqrt{K}}\\
&=\frac{Y_{j+1}^{(a)}-\mathrm{E}Y_{j+1}^{(a)}}{\mathfrak{S}\sqrt{K}}-\mathfrak{m}\frac{Y_j^{(a)}-\mathrm{E}Y_j^{(a)}}{\mathfrak{S}\sqrt{K}}-\frac{a\sqrt{K}}{\mathfrak{S}}\Big(1-\mathfrak{m}\Big)\\
&\asymp\theta_{j+1}^{(a)}-\mathfrak{m}\theta_j^{(a)}-\frac{a\sqrt{K}}{\mathfrak{S}}\Big(1-\mathfrak{m}\Big),
\end{aligned}\tag{4.8}$$

where $\{\theta_j^{(a)}\}$ is a Gaussian sequence. (The values of the parameters of this Gaussian sequence are not discussed here.)

In turn, for $j=0,1,\ldots,\ell(a)-2$, $\ell(a)\geq 2$, the right-hand side of (4.7) is transformed as

$$\begin{aligned}
&\sum_{i=1}^{Y_j^{(a)}}\frac{\xi_{j,i}-\mathfrak{m}}{\mathfrak{S}\sqrt{K}}-\frac{\sqrt{K}}{\mathfrak{S}K}\sum_{i=1}^{\lfloor aK\rfloor}(1-\xi_{j,i})\\
&\qquad=\sqrt{\frac{Y_j^{(a)}}{K}}\frac{\xi_{j,i}-\mathfrak{m}}{\mathfrak{S}\sqrt{Y_j^{(a)}}}-\frac{a\sqrt{K}}{\mathfrak{S}}\Big(1-\mathfrak{m}\Big)\\
&\qquad=\zeta_j\sqrt{\frac{Y_j^{(a)}}{K}}-\frac{a\sqrt{K}}{\mathfrak{S}}\Big(1-\mathfrak{m}\Big),
\end{aligned}\tag{4.9}$$

where $\{\zeta_j\}$ is a sequence of independent standard normally distributed random variables.

Therefore for $j=0,1,\ldots,\ell(a)-2$ from (4.8) and (4.9) we have:

$$\theta_{j+1}^{(a)}-\mathfrak{m}\theta_j^{(a)}\asymp\zeta_j\sqrt{y_j^{(a)}},\tag{4.10}$$

where $y_j^{(a)}=\frac{Y_j^{(a)}}{K}$.

The analysis of (4.10) is standard. According to the definition $\mathrm{E}\theta_j^{(a)} = 0$. Therefore, rewriting (4.10) as

$$\theta_{j+1}^{(a)} \asymp \mathfrak{m}\theta_j^{(a)} + \zeta_j\sqrt{y_j^{(a)}}, \tag{4.11}$$

we obtain:

$$\mathrm{E}\left(\theta_{j+1}^{(a)}\right)^2 = \lim_{K\to\infty} \mathrm{E}\left(\mathfrak{m}\theta_j^{(a)} + \zeta_j\sqrt{\frac{Y_j^{(a)}}{K}}\right)^2. \tag{4.12}$$

Taking into account that $\mathrm{E}\left(\zeta_j\sqrt{y_j^{(a)}}\right)^2 = \mathrm{E}y_j^{(a)} \to \mathfrak{m}^j - a$, as $K \to \infty$, we obtain

$$\mathrm{var}(\theta_{j+1}^{(a)}) = \mathfrak{m}^2\,\mathrm{var}(\theta_j^{(a)}) + \mathfrak{m}^j - a. \tag{4.13}$$

Next, from (4.10) we have:

$$\begin{aligned}
\mathrm{cov}(\theta_j^{(a)}, \theta_{j+1}^{(a)}) &= \mathrm{E}\theta_j^{(a)}\theta_{j+1}^{(a)}\\
&= \lim_{K\to\infty} \mathrm{E}\left(\mathfrak{m}\theta_j^{(a)}\theta_j^{(a)} + \zeta_j\theta_j^{(a)}\sqrt{\frac{Y_j^{(a)}}{K}}\right)\\
&= \mathfrak{m}\,\mathrm{var}(\theta_j^{(a)}) + \mathfrak{m}^j - a,
\end{aligned}$$

and it is easy to show by induction

$$\begin{aligned}
&\mathrm{cov}(\theta_j^{(a)}, \theta_{j+n}^{(a)}) = \mathfrak{m}^n\,\mathrm{var}(\theta_j^{(a)}) + \sum_{i=1}^{n} \mathfrak{m}^{i-1}\mathrm{E}y_{j+n-i}^{(a)},\\
&\mathrm{E}y_{j+n-i}^{(a)} = \mathfrak{m}^{j+n-i} - a,
\end{aligned}$$

where $j + n \le \ell(a) - 1$. Assuming now that $a \to 0$, we obtain the convergence of the sequence

$$\left\{\frac{X_1 - K\mathfrak{m}}{\mathfrak{S}\sqrt{K}}, \frac{X_2 - K\mathfrak{m}^2}{\mathfrak{S}\sqrt{K}}, \ldots\right\}$$

to the Gaussian process $\{\theta_1, \theta_2, \ldots\}$ with mean 0 and covariance function

$$\begin{aligned} &\operatorname{cov}(\theta_j, \theta_{j+n}) = \mathfrak{m}^n \operatorname{var}(\theta_j) + n\mathfrak{m}^{j+n-1}, \\ &\operatorname{var}(\theta_{j+1}) = \mathfrak{m}^2 \operatorname{var}(\theta_j) + \mathfrak{m}^j, \ \operatorname{var}(\theta_1) = 1. \end{aligned} \tag{4.14}$$

This implies the statement of Lemma 1.2.

5. **Proof of Theorem 1.1**

Let us now study equation (4.10) more carefully. Let u_1 and u_2 be two real numbers, $0 < u_1 < u_2 < 1$. Assume that K is so large that the probability $\mathrm{P}\{|\tau_{a,K} - \ell(a,K)| > \epsilon\}$ is negligible ($\epsilon > 0$ is an arbitrary fixed value, K is large enough), where $\ell(a,K)$ is a (not random) integer number. Such a number does always exist for any given a since, as $K \to \infty$, $\tau_{a,K}$ converges to $\ell(a)$ in probability.

For large K we have the following two expansions:

$$\theta^{(a)}_{\lfloor u_1\tau_{a,K}\rfloor+1} - \mathfrak{m}\theta^{(a)}_{\lfloor u_1\tau_{a,K}\rfloor} \asymp \zeta_{\lfloor u_1\tau_{a,K}\rfloor}\sqrt{\frac{Y^{(a)}_{\lfloor u_1\tau_{a,K}\rfloor}}{K}}, \tag{5.1}$$

$$\theta^{(a)}_{\lfloor u_2\tau_{a,K}\rfloor+1} - \mathfrak{m}\theta^{(a)}_{\lfloor u_2\tau_{a,K}\rfloor} \asymp \zeta_{\lfloor u_2\tau_{a,K}\rfloor}\sqrt{\frac{Y^{(a)}_{\lfloor u_2\tau_{a,K}\rfloor}}{K}}, \tag{5.2}$$

where $y^{(a)}_{\lfloor u_i\tau_{a,K}\rfloor}$ in the right-hand side of equations (5.1) and (5.2), $i = 1, 2$, are correspondingly replaced by $\frac{Y^{(a)}_{\lfloor u_i\tau_{a,K}\rfloor}}{K}$. It is worth noting as follows. Relations (5.1) and (5.2) are written in the form of an asymptotic expansion. The left-hand sides of these expansions are Gaussian martingale-differences, while the right-hand sides are the expressions with large parameter K. Since the probability $\mathrm{P}\{|\tau_{a,K} - \ell(a,K)| > \epsilon\}$ is negligible ($\epsilon > 0$ is an arbitrary fixed value, K is large enough), the expansion with the given right-hand side is correct. From (5.1) and (5.2) we obtain as follows:

$$\begin{aligned}&Y^{(a)}_{\lfloor u_1\tau_{a,K}\rfloor}\left(\theta_{\lfloor u_2\tau_{a,K}\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_2\tau_{a,K}\rfloor}\right)^2\zeta^2_{\lfloor u_1\tau_{a,K}\rfloor}\\&\asymp Y^{(a)}_{\lfloor u_2\tau_{a,K}\rfloor}\left(\theta_{\lfloor u_1\tau_{a,K}\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_1\tau_{a,K}\rfloor}\right)^2\zeta^2_{\lfloor u_2\tau_{a,K}\rfloor}\end{aligned}$$

and for any continuous function $f(\bullet)$

$$\begin{aligned}&f\left[Y^{(a)}_{\lfloor u_1\tau_{a,K}\rfloor}\left(\theta_{\lfloor u_2\tau_{a,K}\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_2\tau_{a,K}\rfloor}\right)^2\zeta^2_{\lfloor u_1\tau_{a,K}\rfloor}\right]\\&\asymp f\left[Y^{(a)}_{\lfloor u_2\tau_{a,K}\rfloor}\left(\theta_{\lfloor u_1\tau_{a,K}\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_1\tau_{a,K}\rfloor}\right)^2\zeta^2_{\lfloor u_2\tau_{a,K}\rfloor}\right]\end{aligned}\tag{5.3}$$

For example, from (5.3) we obtain:

$$\begin{aligned}&\left[Y^{(a)}_{\lfloor u_1\tau_{a,K}\rfloor}\left(\theta_{\lfloor u_2\tau_{a,K}\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_2\tau_{a,K}\rfloor}\right)^2\zeta^2_{\lfloor u_1\tau_{a,K}\rfloor}\right]^l\\&\asymp \left[Y^{(a)}_{\lfloor u_2\tau_{a,K}\rfloor}\left(\theta_{\lfloor u_1\tau_{a,K}\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_1\tau_{a,K}\rfloor}\right)^2\zeta^2_{\lfloor u_2\tau_{a,K}\rfloor}\right]^l\end{aligned}\tag{5.4}$$

Now estimate the conditional expectation $\mathrm{E}\left\{\left(Y^{(a)}_{\lfloor u_1\tau_{a,K}\rfloor}\right)^l \;\middle|\; Y^{(a)}_{\lfloor u_2\tau_{a,K}\rfloor}\right\}$. For brevity let us introduce a random vector

$$\mathbf{Z}_{u_1,u_2,\tau_{a,K}}=\left\{\theta_{\lfloor u_1\tau_{a,K}\rfloor},\ \theta_{\lfloor u_2\tau_{a,K}\rfloor},\ \zeta_{\lfloor u_1\tau_{a,K}\rfloor},\ \zeta_{\lfloor u_2\tau_{a,K}\rfloor}\right\}.$$

We have

$$\begin{aligned}&\mathrm{E}\left\{Y^{(a)}_{\lfloor u_i\tau_{a,K}\rfloor}\;\middle|\;\mathbf{Z}_{u_1,u_2,\tau_{a,K}}\right\}\\&=\mathrm{E}\left\{\mathrm{E}\left(Y^{(a)}_{\lfloor u_i\tau_{a,K}\rfloor}\;\middle|\;\mathbf{Z}_{u_1,u_2,\tau_{a,K}},\tau_{a,K}\right)\;\middle|\;\tau_{a,K}\right\}\\&=\mathrm{E}\left\{\mathrm{E}\left(Y^{(a)}_{\lfloor u_i\tau_{a,K}\rfloor}\;\middle|\;\mathbf{Z}_{u_1,u_2,\tau_{a,K}}\right)\;\middle|\;\tau_{a,K}\right\}\\&=\mathrm{E}Y^{(a)}_{\lfloor u_i\tau_{a,K}\rfloor},\quad i=1,2.\end{aligned}\tag{5.5}$$

The last equality of the right-hand side of (5.5) is a consequence of conditional independence of $Y^{(a)}_{\lfloor u_i\tau_{a,K}\rfloor}$ and $\mathbf{Z}_{u_1,u_2,\tau_{a,K}}$, that is for any given event $\{\tau_{a,K}=k\}$, the random variable $Y^{(a)}_{\lfloor u_ik\rfloor}$ and random vector $\mathbf{Z}_{u_1,u_2,k}$ are independent. (5.5) holds true also in the case of $a=0$ that will be discussed later.

Next, using the notation $\tau_K = \tau_{0,K}$ let us prove that

$$\mathrm{cov}\left(\theta_{\lfloor u_1\tau_K\rfloor+1} - \mathfrak{m}\theta_{\lfloor u_1\tau_K\rfloor}, \theta_{\lfloor u_2\tau_K\rfloor+1} - \mathfrak{m}\theta_{\lfloor u_2\tau_K\rfloor}\right) \to 0 \tag{5.6}$$

as $K \to \infty$.

Notice, first (see relation (4.14)) that $\mathrm{cov}(\theta_j, \theta_{j+n})$ vanishes as $n \to \infty$. Consequently, by the total expectation formula,

$$\mathrm{cov}(\theta_{\lfloor u_1\tau_K\rfloor}, \theta_{\lfloor u_1\tau_K\rfloor+n}) = \mathrm{E}\left(\mathrm{cov}(\theta_{\lfloor u_1\tau_K\rfloor}, \theta_{\lfloor u_1\tau_K\rfloor+n} \mid \tau_K)\right) \tag{5.7}$$

vanishes as $n \to \infty$, where here in relation (5.7) and later the notation for $\mathrm{cov}(\theta_{\lfloor u_1\tau_K\rfloor}, \theta_{\lfloor u_1\tau_K\rfloor+n} \mid \tau_K)$ or another similar notation means the conditional covariance. Taking into account that, as $K \to \infty$, τ_K increases to infinity in probability and $u_2 - u_1 > 0$, the difference $\lfloor u_2\tau_K\rfloor - \lfloor u_1\tau_K\rfloor$ increases to infinity in probability too. Hence, by virtue of (5.7) one can conclude that $\mathrm{cov}(\theta_{\lfloor u_1\tau_K\rfloor}, \theta_{\lfloor u_2\tau_K\rfloor})$ vanishes as $K \to \infty$. Therefore, as $K \to \infty$, $\mathrm{E}\theta_{\lfloor u_1\tau_K\rfloor}\theta_{\lfloor u_2\tau_K\rfloor}$ is asymptotically equal to $\mathrm{E}\theta_{\lfloor u_1\tau_K\rfloor}\mathrm{E}\theta_{\lfloor u_2\tau_K\rfloor}$, and (5.6) follows. In addition to (5.5) and (5.6) we have also the following. Since the sequence $\{\zeta_j\}$ consists of independent standard normally distributed random variables, then as $K \to \infty$

$$\mathrm{cov}\left(\zeta_{\lfloor u_1\tau_K\rfloor}, \zeta_{\lfloor u_2\tau_K\rfloor}\right) \to 0. \tag{5.8}$$

This is because $\mathrm{cov}\left(\zeta_{\lfloor u_1\tau_K\rfloor}, \zeta_{\lfloor u_2\tau_K\rfloor} \mid \tau_K\right) = \mathrm{I}\{\lfloor u_1\tau_K\rfloor = \lfloor u_2\tau_K\rfloor\}$, and the last vanishes in probability as $K \to \infty$.

Assuming that a vanishes we need a stronger assumption than above. Specifically, we assume that K is so large that the probability

$$\mathrm{P}\left\{\left|\frac{\tau_{a,K} - \ell(a,K)}{\frac{\log K}{\log \mathfrak{m}}}\right| > \epsilon\right\}$$

is negligible for all $0 \leq a < a_0$ ($\epsilon > 0$ is an arbitrary fixed value, K is large enough), where $a_0 < 1$ is some fixed small number. Such a large number K does always exist, since as $K \to \infty$ and a vanishing, $\frac{\tau_{a,K}}{\log K}$ converges to $-\frac{1}{\log \mathfrak{m}}$ in probability. Then, letting $a \to 0$ in (5.4) in view of pathwise inequalities (3.7) and $\frac{Y_n^{(a)}}{K} \le \frac{X_n}{K}$ we have

$$
\begin{aligned}
&X^l_{\lfloor u_1\tau_K\rfloor}\left(\theta_{\lfloor u_2\tau_K\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_2\tau_K\rfloor}\right)^{2l}\zeta^{2l}_{\lfloor u_1\tau_K\rfloor}\\
&\asymp X^l_{\lfloor u_2\tau_K\rfloor}\left(\theta_{\lfloor u_1\tau_K\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_1\tau_K\rfloor}\right)^{2l}\zeta^{2l}_{\lfloor u_2\tau_K\rfloor}.
\end{aligned}
\tag{5.9}
$$

Taking into account (5.5), (5.6) and (5.8) and conditional independency of $X_{\lfloor u_1\tau_K\rfloor}$, $\left(\theta_{\lfloor u_1\tau_K\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_1\tau_K\rfloor}\right)$ and $\zeta_{\lfloor u_2\tau_K\rfloor}$, and passing to the appropriate conditional expectations, from (5.9) we obtain:

$$
\begin{aligned}
\mathrm{E}\left\{X^l_{\lfloor u_1\tau_K\rfloor}\;\middle|\; X_{\lfloor u_2\tau_K\rfloor}\right\} &= \mathrm{E}\left[\mathrm{E}\left\{X^l_{\lfloor u_1\tau_K\rfloor}\;\middle|\; X_{\lfloor u_2\tau_K\rfloor},\tau_K\right\}\;\middle|\;\tau_K\right]\\
&\asymp X^l_{\lfloor u_2\tau_K\rfloor}\frac{\mathrm{E}\left(\theta_{\lfloor u_1\tau_K\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_1\tau_K\rfloor}\right)^{2l}}{\mathrm{E}\left(\theta_{\lfloor u_2\tau_K\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_2\tau_K\rfloor}\right)^{2l}}.
\end{aligned}
\tag{5.10}
$$

Thus, to this end our task is to determine the asymptotic of

$$
\mathrm{E}\left(\theta_{\lfloor u_i\tau_K\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_i\tau_K\rfloor}\right)^{2l},\quad i=1,2,
$$

for large K. Returning to basic equations (5.1) and (5.2), we have

$$
\begin{aligned}
\mathrm{E}\left(\theta^{(a)}_{\lfloor u_i\tau_{a,K}\rfloor+1}-\mathfrak{m}\theta^{(a)}_{\lfloor u_i\tau_{a,K}\rfloor}\right)^{2l} &\asymp \mathrm{E}\left(\zeta_{\lfloor u_i\tau_{a,K}\rfloor}\sqrt{\frac{Y^{(a)}_{\lfloor u_i\tau_{a,K}\rfloor}}{K}}\right)^{2l}\\
&= \mathrm{E}\left\{\mathrm{E}\left[\left(\zeta_{\lfloor u_i\tau_{a,K}\rfloor}\sqrt{\frac{Y^{(a)}_{\lfloor u_i\tau_{a,K}\rfloor}}{K}}\right)^{2l}\;\middle|\;\tau_{a,K}\right]\right\}\\
&= \mathrm{E}\left(y^{(a)}_{\lfloor u_i\tau_{a,K}\rfloor}\right)^{l}\mathrm{E}(\zeta_{\lfloor u_i\tau_{a,K}\rfloor})^{2l},
\end{aligned}
\tag{5.11}
$$

where $\zeta_{\lfloor u_i\tau_{a,K}\rfloor}$, $i=1,2$, are standard normally distributed random variables. As a vanishes, from (5.11) we obtain

$$
\mathrm{E}\left(\theta_{\lfloor u_i\tau_K\rfloor+1}-\mathfrak{m}\theta_{\lfloor u_i\tau_K\rfloor}\right)^{2l} \asymp \mathrm{E}\mathfrak{m}^{l\lfloor u_i\tau_K\rfloor}\mathrm{E}(\zeta_{\lfloor u_i\tau_K\rfloor})^{2l}.
\tag{5.12}
$$

Therefore, (5.10) can be rewritten

$$\mathrm{E}\left\{X_{\lfloor u_1\tau_K\rfloor}^l \;\Big|\; X_{\lfloor u_2\tau_K\rfloor}\right\} \asymp X_{\lfloor u_2\tau_K\rfloor}^l \mathrm{E}\mathfrak{m}^{l(\lfloor u_1\tau_K\rfloor-\lfloor u_2\tau_K\rfloor)}.$$

This proves the first equation of (1.2). The proof of the second equation of (1.2) is similar.

Consider basic equation (5.1) again, rewriting it as follows:

$$Y_{\lfloor u_1\tau_{a,K}\rfloor}^{(a)}\zeta_{\lfloor u_1\tau_{a,K}\rfloor}^2 \asymp K\left(\theta_{\lfloor u_1\tau_{a,K}\rfloor+1}^{(a)} - \mathfrak{m}\theta_{\lfloor u_1\tau_{a,K}\rfloor}^{(a)}\right)^2. \tag{5.13}$$

Assuming that as a vanishes we have:

$$X_{\lfloor u_1\tau_K\rfloor}\zeta_{\lfloor u_1\tau_K\rfloor}^2 \asymp K\left(\theta_{\lfloor u_1\tau_K\rfloor+1} - \mathfrak{m}\theta_{\lfloor u_1\tau_K\rfloor}\right)^2. \tag{5.14}$$

Therefore, taking into account that $X_{\lfloor u_1\tau_K\rfloor}$ and $\zeta_{\lfloor u_1\tau_K\rfloor}$ are conditionally independent, from (5.14) we obtain:

$$\begin{aligned}\mathrm{E}\left\{X_{\lfloor u_1\tau_K\rfloor}^l \;\Big|\; \tau_K\right\} &\asymp K^l\frac{\mathrm{E}\left\{\left(\theta_{\lfloor u_1\tau_K\rfloor+1} - \mathfrak{m}\theta_{\lfloor u_1\tau_K\rfloor}\right)^{2l} \mid \tau_K\right\}}{\mathrm{E}(\zeta_{\lfloor u_1\tau_{a,K}\rfloor} \mid \tau_K)^{2l}}\\ &\asymp K^l\mathfrak{m}^{l\lfloor u_1\tau_K\rfloor}.\end{aligned}$$

(1.3) is proved.

6. **Discussion**

The aim of this section is to present the main results in convenient form for application to analysis of real populations. In this section we also establish a so-called invariance property.

Let, when K is large, ϵ be a relatively small (positive or negative) parameter having the following meaning. The population size at time $\lfloor u_2\tau_K\rfloor$ is assumed to be equal to $\lfloor(1+\epsilon)K\mathfrak{m}^{u_2 t_K}\rfloor$, $t_K = -\frac{\log K}{\log \mathfrak{m}}$.

The meaning of this value is the following. The factor $K\mathfrak{m}^{\lfloor u_2 t_K\rfloor}$ is the expected size of the population at time $\lfloor u_2\tau_K\rfloor$, and the factor $1+\epsilon$ represents a parameter of relative deviation from the expected population at that time moment. Then, from Theorem 1.1 we obtain, that for large K

$$\begin{aligned}
&\log \mathrm{E}\left(X^l_{\lfloor u_1\tau_K\rfloor} \mid X_{\lfloor u_2\tau_K\rfloor} = \lfloor (1+\epsilon)K\mathfrak{m}^{u_2 t_K}\rfloor\right)\\
&= l\log(K+K\epsilon) + lu_1 t_K \log\mathfrak{m} + o(1) \qquad (6.1)\\
&= l\log(K+K\epsilon) - lu_1\log K + o(1).
\end{aligned}$$

In real computations the term $\log(K+K\epsilon)$ can be replaced by $\epsilon+\log K$ if ϵ is sufficiently small.

The result similar to (6.1) can be obtained for the conditional expectation of (1.3). Specifically, for large K write

$$\tau_K = -\left\lfloor (1+\epsilon)\frac{\log K}{\log\mathfrak{m}}\right\rfloor. \qquad (6.2)$$

(6.2) has the following meaning. As $K\to\infty$, the fraction $\frac{\tau_K}{\log K}$ converges in probability to $-\frac{1}{\log\mathfrak{m}}$, and therefore, as K is large, the factor $1+\epsilon$ is a parameter for relative deviation from the expected value of extinction time. Then,

$$\begin{aligned}
&\log\mathrm{E}\left\{X^l_{\lfloor u_1\tau_K\rfloor}\;\middle|\;\tau_K = -\left\lfloor(1+\epsilon)\frac{\log K}{\log\mathfrak{m}}\right\rfloor\right\} \qquad (6.3)\\
&= l\log(K+K\epsilon) - lu_1\log K + o(1).
\end{aligned}$$

As we can see the right-hand sides of (6.1) and (6.3) coincide. That is for any given relative deviation $1+\epsilon$ the asymptotic conditional expectations are invariant.

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