

PREDICTION OF MISSING OBSERVATIONS BY A CONTROL METHOD

VYACHESLAV M. ABRAMOV and FIMA C. KLEBANER

School of Mathematical Sciences
Monash University
Building 28MClayton Campus, Clayton
VIC 3800, Australia
e-mail: fima.klebaner@sci.monash.edu.au

Abstract

Consider a time series with missing observations but a known final point. Using control theory ideas we estimate/predict these missing observations. We obtain recurrence equations which minimize sum of squares control sequence. An advantage of this method is in easily computable formulae and flexibility of its application to different structures of missing data.

1. Introduction

Analysis and forecasting missing data is a well-known area of statistics going back to earlier works of Bartlett [4], Tocher [28], Wilks [30], Yates [31] and many others (see review paper [1]). There is a large number of review papers and books related to this subject, [1], [2], [12], [19] and [20], to mention a few. There are various approaches to missing data, including Bayes methods [6], maximum likelihood, multiple imputations methods, methods of non-parametric regression and others, e.g., [2], [9], [29].

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In the present paper we suggest a new method of predicting a special class of missing observations in different time series including regression and auto-regression models. We suggest a simple recurrence procedure, and to the authors knowledge, it is new and simpler than the computational procedures that were known before.

Study autoregressive time series with missing observations, which we propose to predict using a control method. This method is developed for different types of autoregressive models including $AR(p)$ models in the case of scalar variables and $AR(1)$ in the case of vector-valued observations. Forecasting missing data in autoregressive time series has received a special attention in the literature: [12], [14], [15], [18], [20], [22], [24] and [27]. The typical approach for forecasting missing data in autoregressive models considered in most of papers is based on maximization of likelihood ratio, which can be computationally intensive. The approach of the present paper, referred to as a control method allows to obtain easily computable formulae for missing data.

It is known that a one-dimensional $AR(p)$ model can be transformed to the special case of a p -dimensional $AR(1)$ model (e.g., Anderson [3]). In the present paper we consider both one-dimensional $AR(p)$ models and multidimensional $AR(1)$ models nevertheless. The representations obtained in the case of one-dimensional $AR(p)$ models are simpler for computations than that for a multidimensional $AR(1)$ model. Whereas representations for a one-dimensional $AR(p)$ model is recurrence formulae and can be calculated directly, the computations for a multidimensional $AR(1)$ model requires two steps. In the first step we calculate the vector norm, and then the vector corresponding to a missing value. We assume that in the time series:

$$x_1, x_2, \dots, x_{n_0}, \tilde{x}_{n_0+1}, \tilde{x}_{n_0+2}, \dots, \tilde{x}_{N-1}, x_N = \bar{x} \quad (1.1)$$

the first n_0 observations are known/observed as well as the last value $x_N = \bar{x}$ is assumed to be given too. The values $\tilde{x}_{n_0+1}, \tilde{x}_{n_0+2}, \dots, \tilde{x}_{N-1}$ are missing.

This set up may have various applications, for example in economics and finance, where historical data indices are given, while the last value can be obtained from financial derivatives, or might be set externally. In

finance, for example, on basis of historical volatilities and a future value obtained from options one predicts the dynamics of the volatility.

Although the paper concerns with data structure (1.1), the results can be extended to different more complicated structures of missing values. Indeed, consider for instance the following data

$$\begin{aligned} x_1, \dots, x_{n_0}, \tilde{x}_{n_0+1}, \dots, \tilde{x}_{n_1}, \\ x_{n_1+1}, \dots, x_{n_2}, \tilde{x}_{n_2+1}, \dots, \tilde{x}_{N-1}, x_N = \bar{x}. \end{aligned} \quad (1.2)$$

Here in (1.2) the data indexed from 1 to n_0 and from $n_1 + 1$ to n_2 are known, the last point $x_N = \bar{x}$ is assigned, and the rest of data are missing. Then we have two groups of missing data, and standard decomposition arguments can be used to reduce analysis to that of a single group with missing data.

The paper is structured as follows. In Section 2 we discuss the known methods of forecasting missing observation as well as the method of the present paper with comparisons. In the following section we discuss forecasting missing data by a control method in order of increasing complexity. Specifically, in Section 3 we study the problem for the simplest AR(1) model of time series, and in Section 4 we extend the results for AR(p) models, $p \geq 1$. The multi-dimensional observations of AR(1) model are studied in Section 5. Then, in Section 6 the problem is solved for models of regression. The results of this section are easily understandable and simple. In Section 7 two numerical examples are considered in finance and archaeology.

2. Review of Methods for Missing Values

There is a large number of papers on estimation and forecasting of missing observation in autoregressive models.

Jones [12] provides the method for calculation of exact likelihood function of stationary ARMA time series. The method is based on Akake's Markovian representation and application of Kalman's recursions [5]. An advantage of Kalman's recursions is that the matrices and vectors being used in calculations have dimensions $\max\{p, q + 1\}$, where p is the order

of the auto-regression, and q is the order of moving average, rather than dimensions corresponding to the number of observations. A non-linear optimization program is then used to find the maximum likelihood estimates.

Kohn and Ansley [18] study interpolation missing data for nonstationary ARIMA models. The likelihood ratio for these models does not exist in the usual sense, and the authors define marginal likelihood ratio. They show, that marginal likelihood approach reduces in some cases to the usual likelihood approach forecasting missing observations in is based on a modified Kalman filter, which has been introduced in the earlier paper of these authors [17].

Shin and Pantula [26] discuss the testing problem for a unit root in an autoregressive model where data are available for each m -th period. The idea is to use characteristic polynomials and properties of their coefficients. Under special assumption [26] estimate parameters of ARMA($p, p - 1$) by fitting an ARMA(1, $p - 1$) model. By using a Monte Carlo simulation, the results were compared by those obtained in earlier papers of Pantula and Hall [21], Said and Dickey [23] and Shin and Fuller [25], who also studied the same testing problem.

Forecasting in autoregressive models has also been studied by Kharin and Huryn [15] and [16]. [15] investigate the case of unknown parameters of an autoregressive model based on the so-called *plug-in* approach. The *plug-in* approach consists of two steps: (i) estimation of the model parameters by some known approach and (ii) forecasting, based on estimation of the parameters in the first step. This method has lower computational complexity than other methods, such as straightforward joint maximum likelihood estimation of the parameters and future values of time series, or Expectation-Maximization algorithm (e.g., Little and Rubin [20], Jordan and Jacobs [13]). In [16] the mean-squared error of maximum likelihood forecasting in the case of missing values is obtained for many autoregressive time series.

The above-mentioned papers [15] and [16] all study a general scheme of missing data. Together with vector-valued time series they introduce a

binary vector characterizing a *missing pattern* but the solution for this general formulation is hard to implement in practice.

The aim of the present paper is prediction (interpolation) of missing observations whereas the aim of two above-mentioned papers is forecasting in the presence of missing observations, i.e., the forecasting procedure takes into account missing observations. Furthermore, the approach of the present paper deals with specific data structures (Section 1), and can be extended to more complicated structures of missing data. In the initial step we use least squares predictors for the preliminary extrapolation of missing values. Then, taking into account the last known observation we make corrections by formulating and solving a control problem. The control problem is formulated in terms of minimization of sums of squares of errors, which in itself is a classical approach. However, our method of is based on a novel application of the Cauchy-Schwartz inequality in a simple case, and then extended to other more complicated cases. The use of the Cauchy-Schwartz inequality is a known technique in optimization, e.g., [8] and [11], however, in the context of prediction of missing data this method seems to be new. In addition, this method yields easily computable recurrence formulae for missing values.

3. A Control Method for Missing Data

In this section we consider autoregressive time series of the following type:

$$x_1, x_2, \dots, x_{n_0}, \tilde{x}_{n_0+1}, \tilde{x}_{n_0+2}, \dots, \tilde{x}_{N-1}, x_N = \bar{x}. \quad (3.1)$$

The values x_1, x_2, \dots, x_{n_0} are assumed to be observed, while by $\tilde{x}_{n_0+1}, \tilde{x}_{n_0+2}, \dots, \tilde{x}_{N-1}$ we denote estimates of missing observations. The value $x_N = \bar{x}$ is also known. It is convenient to denote this value by tilde, i.e., $x_N = \tilde{x}_N = \bar{x}$.

Theorem 3.1. *Best predictors for the missing values are given by*

$$\tilde{x}_n = a\tilde{x}_{n-1} + b + \frac{\bar{x} - \hat{x}_N}{\sum_{i=n_0+1}^N a^{2(N-i)}} \cdot a^{N-n}, \quad n = n_0 + 1, \dots, N - 1, \quad (3.2)$$

where the coefficients a and b are the least squares solutions of the autoregressive equations

$$x_n = ax_{n-1} + b,$$

for the first n_0 observations, $n = 1, 2, \dots, n_0$.

Proof. Taking into consideration the first n_0 observed values one can build the linear least square predictor as

$$\hat{x}_n = a\hat{x}_{n-1} + b \quad (3.3)$$

for $n = n_0 + 1, n_0 + 2, \dots, N$, where parameters a and b are the regression coefficients. These a and b are then used for control problem, which is to find the unknown points, minimizing sum of squares of controls leading to the known final value. Namely, for $n = n_0 + 1, \dots, N$ ($\tilde{x}_{n_0} = \hat{x}_{n_0}$)

$$\tilde{x}_n = a\tilde{x}_{n-1} + b + u_n. \quad (3.4)$$

It can be seen as a *correction* of the initial linear equation for \tilde{x}_n with a control sequence u_n . The control problem is to minimize the sum of squares of controls under the condition that the auto-regression ends up at the specified point $\tilde{x}_N = \bar{x}$

$$\min_{u_n: \tilde{x}_N = \bar{x}} \sum_{n=n_0+1}^N u_n^2. \quad (3.5)$$

This minimization problem is solved as follows. By (3.3) and (3.4)

$$u_n^2 = (\hat{x}_n - \tilde{x}_n)^2, \quad (3.6)$$

and taking into account that

$$\tilde{x}_N = \tilde{x}_{n_0} a^{N-n_0} + \sum_{n=n_0+1}^N a^{N-n} (b + u_n)$$

and

$$\hat{x}_N = \hat{x}_{n_0} a^{N-n_0} + b \sum_{n=n_0+1}^N a^{N-n},$$

from (3.6) we obtain

$$u_N^2 = \left(\sum_{n=n_0+1}^N \alpha^{N-n} u_n \right)^2. \quad (3.7)$$

By the Cauchy-Schwartz inequality,

$$\left(\sum_{n=n_0+1}^N \alpha^{N-n} u_n \right)^2 \leq \sum_{n=n_0+1}^N \alpha^{2(N-n)} \cdot \sum_{n=n_0+1}^N u_n^2. \quad (3.8)$$

The equality in (3.8) is achieved if and only if $\alpha^{N-n} = cu_n$ for some constant c , and since the equality in (3.8) is associated with the minimum of the left-hand side of (3.8), the problem reduces to find an appropriate value $c = c^*$ such that

$$u_n = c^* \alpha^{N-n}.$$

Therefore,

$$\tilde{x}_N = \hat{x}_N + c^* \sum_{i=n_0+1}^N \alpha^{2(N-i)},$$

and then finally for c^* we have:

$$c^* = \frac{\tilde{x}_N - \hat{x}_N}{\sum_{i=n_0+1}^N \alpha^{2(N-i)}}. \quad (3.9)$$

Thus, the sequence u_n satisfying (3.5) is

$$u_n = \frac{\bar{x} - \hat{x}_N}{\sum_{i=n_0+1}^N \alpha^{2(N-i)}} \cdot \alpha^{N-n},$$

and its substitution for (3.4) yields the desired result (3.2).

4. Extension of the Result for AR(p) Model

Under the assumption that (3.1) is given, we first find the best linear predictor for AR(2) model as

$$\hat{x}_n = a_1 \hat{x}_{n-1} + a_2 \hat{x}_{n-2} + b, \quad (4.1)$$

($n = n_0 + 2, n_0 + 3 + \dots, N$), and then extend the result for the general case of AR(p) model.

Theorem 4.1. *For AR(2) model, the best predictor is given by*

$$\tilde{x}_n = a_1 \tilde{x}_{n-1} + a_2 \tilde{x}_{n-2} + b + \frac{\bar{x} - \hat{x}_N}{\sum_{i=n_0+2}^N \gamma_{N-i}^2} \cdot \gamma_{N-n}, \quad (4.2)$$

where $n = n_0 + 2, n_0 + 3, \dots, N - 1$, and the coefficients γ_n are as follows:

$$\gamma_n = \alpha_n + \beta_n,$$

$$\alpha_0 = 1,$$

$$\beta_0 = 0,$$

$$\alpha_n = a_1(\alpha_{n-1} + \beta_{n-1})(n \geq 1),$$

$$\beta_1 = 1,$$

$$\beta_n = a_2(\alpha_{n-2} + \beta_{n-2}) + 1(n \geq 2).$$

The coefficients a_1, a_2 and b for equation (4.2) are the minimum in the least-square sense of the autoregressive equation

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + b,$$

which are obtained by the first n_0 observations.

Proof. In the case of AR(2) model we have

$$\hat{x}_{n_0+2} = a_1 \hat{x}_{n_0+1} + a_2 \hat{x}_{n_0} + b, \quad (4.3)$$

and similarly to (3.4),

$$\tilde{x}_n = a_1 \tilde{x}_{n-1} + a_2 \tilde{x}_{n-2} + b + u_n. \quad (4.4)$$

($n = n_0 + 2, n_0 + 2, \dots, N$).

Let us now consider the difference $\tilde{x}_N - \hat{x}_N = u_N$. For this difference we have the following expansion

$$u_N = \sum_{n=n_0+2}^N \gamma_{N-n} u_n, \quad (4.5)$$

with some coefficients γ_{N-n} . Now, the main task is to determine these coefficients. Write $\gamma_n = \alpha_n + \beta_n$. Then, using induction we obtain

$$\begin{aligned} \alpha_0 &= 1, \\ \beta_0 &= 0, \\ \alpha_n &= a_1(\alpha_{n-1} + \beta_{n-1}) \quad (n \geq 1), \\ \beta_1 &= 1, \\ \beta_n &= a_2(\alpha_{n-2} + \beta_{n-2}) + 1 \quad (n \geq 2). \end{aligned} \quad (4.6)$$

Specifically, for the first step we have the following. Setting $N = n_0 + 2$ leads to the obvious identity $u_N = \gamma_0 u_N = (\alpha_0 + \beta_0) u_N$. In the case $N = n_0 + 3$ we have

$$\begin{aligned} u_N &= u_N + (\alpha_1 + 1)u_{N-1} \\ &= \gamma_0 u_N + [a_1(\alpha_0 + \beta_0) + 1]u_{N-1} \\ &= \gamma_0 u_N + \gamma_1 u_{N-1}. \end{aligned}$$

In the case $N = n_0 + 4$ we have

$$\begin{aligned} u_N &= u_N + a_1(\gamma_0 + 1)u_{N-1} + [a_1(a_1 + 1) + (a_2 + 1)]u_{N-2} \\ &= \gamma_0 u_N + \gamma_1 u_{N-1} + [a_1(\alpha_1 + \beta_1) + a_2\gamma_0 + 1]u_{N-2} \\ &= \gamma_0 u_N + \gamma_1 u_{N-1} + \gamma_2 u_{N-2}. \end{aligned}$$

The next steps follow by induction, and we have recurrence relation (4.5)-(4.6) above.

Therefore,

$$u_N^2 = \left(\sum_{n=n_0+2}^N \gamma_{N-n} u_n \right)^2, \quad (4.7)$$

and similarly to (3.8) by Cauchy-Schwartz inequality

$$\left(\sum_{n=n_0+1}^N \gamma_{N-n} u_n \right)^2 \leq \sum_{n=n_0+2}^N \gamma_{N-n}^2 \cdot \sum_{n=n_0+2}^N u_n^2 \quad (4.8)$$

The equality in (4.8) is achieved if and only if $\gamma_{N-n} = cu_n$ for some constant c , and since the equality in (4.8) is associated with the minimum of the left-hand side of (4.8), the problem reduces to find an appropriate value $c = c^*$ such that

$$u_n = c^* \gamma_{N-n}.$$

This finishes the proof.

The results above are easily extended to general AR(p) models.

Specifically, we have

$$\hat{x}_n = \alpha_1 \hat{x}_{n-1} + \alpha_2 \hat{x}_{n-2} + \cdots + \alpha_p \hat{x}_{n-p} + b, \quad (4.9)$$

and

$$\tilde{x}_n = \alpha_1 \tilde{x}_{n-1} + \alpha_2 \tilde{x}_{n-2} + \cdots + \alpha_p \tilde{x}_{n-p} + b + u_n, \quad (4.10)$$

($n = n_0 + p, n_0 + p + 1 + \dots, N$),

and

$$u_N = \tilde{x}_N - \hat{x}_N = \sum_{n=n_0+p}^N \gamma_{N-n} u_n, \quad (4.11)$$

where

$$\gamma_n = (\alpha_{n,1} + \alpha_{n,2} + \cdots + \alpha_{n,p}),$$

and

$$\alpha_{0,1} = 1,$$

$$\alpha_{0,k} = 0, \quad k = 2, 3, \dots, p,$$

$$\alpha_{n,1} = \alpha_1 (\alpha_{n-1,1} + \alpha_{n-1,2} + \cdots + \alpha_{n-1,p}) \quad (n \geq 1),$$

$$\alpha_{1,2} = 1,$$

$$\begin{aligned}
\alpha_{1,k} &= 0, \quad k = 3, 4, \dots, p, \\
\alpha_{n,2} &= a_2(\alpha_{n-2,1} + \alpha_{n-2,2} + \dots + \alpha_{n-2,p}) \quad (n \geq 2), \\
&\dots = \dots \\
\alpha_{k-1,k} &= 1, \quad k = 1, 2, \dots, p-1, \\
\alpha_{k-1,l} &= 0, \quad l = k+1, k+2, \dots, p, \\
\alpha_{n,k} &= a_k(\alpha_{n-k,1} + \alpha_{n-k,2} \dots + \alpha_{n-k,p}) \quad (n \geq k), \\
&\dots = \dots \\
\alpha_{n,p} &= a_p(\alpha_{n-p,1} + \alpha_{n-p,2} \dots + \alpha_{n-p,p}) + 1 \quad (n \geq p).
\end{aligned} \tag{4.12}$$

Thus, similarly to (4.2) we have the following formula

$$\begin{aligned}
\tilde{x}_n &= a_1 \tilde{x}_{n-1} + a_2 \tilde{x}_{n-2} + \dots + a_p \tilde{x}_{n-p} \\
&+ b + \frac{\bar{x} - \hat{x}_N}{\sum_{i=n_0+2}^N \gamma_{N-i}^2} \cdot \gamma_{N-n}
\end{aligned} \tag{4.13}$$

($n = n_0 + p, n_0 + p + 1, \dots, N - 1$), where γ_n are now defined according to (4.12).

5. Multi-Dimensional Autoregressive Model

In this section we study a multidimensional version of the problem for AR(1). Let

$$\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n_0}, \tilde{\mathbf{x}}_{n_0+1}, \tilde{\mathbf{x}}_{n_0+2}, \dots, \tilde{\mathbf{x}}_{N-1}, \mathbf{x}_N = \bar{\mathbf{x}}. \tag{5.1}$$

For this last value we shall also write $\mathbf{x}_N = \tilde{\mathbf{x}}_N$ (with tilde).

As above, the values $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n_0}$ are assumed to be observed values, while $\tilde{\mathbf{x}}_{n_0+1}, \tilde{\mathbf{x}}_{n_0+2}, \dots, \tilde{\mathbf{x}}_{N-1}$ are missing observations.

Taking into consideration only the first n_0 observed values one can

build the linear least square predictor as

$$\hat{\mathbf{x}}_n = A\hat{\mathbf{x}}_{n-1} + \mathbf{b} \quad (5.2)$$

for $n = n_0 + 1, n_0 + 2, \dots, N$. Here A is a square matrix, and \mathbf{b} is a vector.

For $n = n_0 + 1, \dots, N$ ($\tilde{\mathbf{x}}_{n_0} = \hat{\mathbf{x}}_{n_0}$) we find the unknown points by

$$\tilde{\mathbf{x}}_n = A\tilde{\mathbf{x}}_{n-1} + \mathbf{b} + \mathbf{u}_n. \quad (5.3)$$

The problem is to find the vectors \mathbf{u}_n , $n = n_0 + 1, \dots, N$ such that they minimize the sum of squares of their lengths subject to the constraint that the auto-regression attains the specified point $\tilde{\mathbf{x}}_N$

$$\min_{\mathbf{u}_n: \tilde{\mathbf{x}}_N = \tilde{\mathbf{x}}_N} \sum_{n=n_0+1}^N \|\mathbf{u}_n\|^2, \quad (5.4)$$

where

$$\|\mathbf{u}_n\| = \sqrt{u_{n,1}^2 + u_{n,2}^2 + \dots + u_{n,k}^2},$$

and $u_{n,j}$ denotes the j th component of the (k -dimensional) vector \mathbf{u}_n .

According to (5.2) and (5.3)

$$\|\mathbf{u}_n\|^2 = \|\hat{\mathbf{x}}_n - \tilde{\mathbf{x}}_n\|^2, \quad (5.5)$$

and for end point \mathbf{x}_N we have

$$\|\mathbf{u}_N\|^2 = \left\| \sum_{n=n_0+1}^N A^{N-n} \mathbf{u}_n \right\|^2. \quad (5.6)$$

Let $a_{i,j}^{(n)}$ denote element (i, j) of matrix A^n . We have the following.

The i th element of multiplication of A^{N-n} to vector \mathbf{u}_n can be written as

$$\sum_{j=1}^k a_{i,j}^{(N-n)} u_{n,j},$$

where $u_{n,j}$ is the j th element of the vector \mathbf{u}_n . Therefore (5.6) can be written as

$$\begin{aligned}
 \|\mathbf{u}_N\|^2 &= \left(\sum_{i=1}^k \sum_{n=n_0+1}^N \sum_{j=1}^k a_{i,j}^{(N-n)} u_{n,j} \right)^2 \\
 &= \left(\sum_{n=n_0+1}^N \left[\sum_{i=1}^k \sum_{j=1}^k a_{i,j}^{(N-n)} u_{n,j} \right] \right)^2 \\
 &= \left(\sum_{n=n_0+1}^N \sum_{j=1}^k \underbrace{\left\{ \sum_{i=1}^k a_{i,j}^{(N-n)} \right\}}_{\text{first term}} \underbrace{u_{n,j}}_{\text{second term}} \right)^2.
 \end{aligned} \tag{5.7}$$

Therefore, by Cauchy-Schwartz inequality, we have

$$\begin{aligned}
 \|\mathbf{u}_N\|^2 &\leq \left(\sum_{n=n_0+1}^N \sum_{j=1}^k \left[\sum_{i=1}^k a_{i,j}^{(N-n)} \right]^2 \right) \times \left(\sum_{n=n_0+1}^N \sum_{j=1}^k u_{n,j}^2 \right) \\
 &= \left(\sum_{n=n_0+1}^N \sum_{j=1}^k \left[\sum_{i=1}^k a_{i,j}^{(N-n)} \right]^2 \right) \times \left(\sum_{n=n_0+1}^N \|\mathbf{u}_n\|^2 \right).
 \end{aligned} \tag{5.8}$$

The equality in (5.8) is achieved if and only if for some constant c ,

$$\sum_{i=1}^k a_{i,j}^{(N-n)} = c u_{n,j} \tag{5.9}$$

and similarly to that of Section 3 the optimal value of this constant c^* is

$$c^* = \frac{\|\tilde{\mathbf{x}}_N - \hat{\mathbf{x}}_N\|}{\sum_{n=n_0+1}^N \sum_{j=1}^k \left[\sum_{i=1}^k a_{i,j}^{(N-n)} \right]^2}. \tag{5.10}$$

For the sequence \mathbf{u}_n we have

$$\|\mathbf{u}_n\| = \frac{\|\bar{\mathbf{x}} - \hat{\mathbf{x}}_N\|}{\sum_{l=n_0+1}^N \sum_{j=1}^k \left[\sum_{i=1}^k a_{i,j}^{(N-l)} \right]^2} \cdot \sum_{j=1}^k \sum_{i=1}^k a_{i,j}^{(N-n)}, \quad (5.11)$$

Let us now find the vectors \mathbf{u}_n , $n = n_0 + 1, n_0 + 2, \dots, N$. From (5.2) and (5.3) we have the following:

$$\mathbf{u}_N = \sum_{n=n_0+1}^N A^{N-n} \mathbf{u}_n. \quad (5.12)$$

Therefore for components of the vector \mathbf{u}_n we have equations

$$u_{N,i} = \sum_{n=n_0+1}^N A_i^{(N-n)} \mathbf{u}_n, \quad (5.13)$$

where $A_i^{(N-n)}$ denotes the i th row of the matrix A^{N-n} . Therefore, by Cauchy-Schwartz inequality

$$\begin{aligned} u_{N,i}^2 &= \left(\sum_{n=n_0+1}^N A_i^{(N-n)} \mathbf{u}_n \right)^2 \\ &= \left(\sum_{n=n_0+1}^N \sum_{j=1}^k a_{i,j}^{(N-n)} u_{n,j} \right)^2 \\ &\leq \left(\sum_{n=n_0+1}^N \sum_{j=1}^k [a_{i,j}^{(N-n)}]^2 \right) \left(\sum_{n=n_0+1}^N \sum_{j=1}^k u_{n,j}^2 \right), \end{aligned} \quad (5.14)$$

where the equality achieves in the case if for some c_i

$$\sqrt{\sum_{j=1}^k [a_{i,j}^{(N-n)}]^2} = c_i \|\mathbf{u}_n\|. \quad (5.15)$$

Therefore, substituting (5.15) for (5.13) we obtain:

$$c_i = \frac{u_{N,i}}{\sum_{n=n_0}^N A_i^{(N-n)} [A_i^{(N-n)}]^T}. \quad (5.16)$$

6. Models of Multi-Regression

Regression models with incomplete data has been studied intensively in the literature, and there are many approaches the solution of this problem. The theoretical aspect of the present approach seems to be new nevertheless.

1. Consider first the following data:

$$\begin{aligned} y_1, y_2, \dots, y_{n_0}, \tilde{y}_{n_0+1}, \dots, \tilde{y}_{N-1}, y_N = \bar{y}_N \\ \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n_0}, \mathbf{x}_{n_0+1}, \dots, \mathbf{x}_{N-1}, \mathbf{x}_N. \end{aligned} \quad (6.1)$$

As above we use the notation $\tilde{y}_N = y_N$.

We first find the vector \mathbf{a} and parameter b by linear least square predictor, so for $n = n_0 + 1, n_0 + 2, \dots, N$ we have

$$\hat{y}_n = \mathbf{a}^\top \mathbf{x}_n + b. \quad (6.2)$$

We have

$$\begin{aligned} \hat{y}_n &= \mathbf{a}^\top \mathbf{x}_n + b \\ &= \mathbf{a}^\top \mathbf{x}_{n-1} + b + \mathbf{a}^\top (\mathbf{x}_n - \mathbf{x}_{n-1}) \\ &= \hat{y}_{n-1} + b_n, \end{aligned}$$

where $b_n = \mathbf{a}^\top (\mathbf{x}_n - \mathbf{x}_{n-1})$.

Therefore considering

$$\tilde{y}_n = \tilde{y}_{n-1} + b_n + u_n, \quad (6.3)$$

where $\tilde{y}_{n_0} = \hat{y}_{n_0}$, and $\tilde{y}_N = \bar{y}_N$ and the same problem to

$$\text{minimize } \sum_{n=n_0+1}^N u_n^2,$$

we arrive at

$$\tilde{y}_n = \tilde{y}_{n-1} + b_n + \frac{\bar{y}_N - \hat{y}_N}{N - n_0}. \quad (6.4)$$

2. Let us consider a more extended problem

$$\begin{aligned} \mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{n_0}, \tilde{\mathbf{y}}_{n_0+1}, \dots, \tilde{\mathbf{y}}_{N-1}, \mathbf{y}_N = \bar{\mathbf{y}}_N \\ \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n_0}, \mathbf{x}_{n_0+1}, \dots, \mathbf{x}_{N-1}, \mathbf{x}_N, \end{aligned} \quad (6.5)$$

where the vectors \mathbf{y} of the first row all of dimension m .

By the linear least square predictor we have

$$\hat{\mathbf{y}}_n = A\mathbf{x}_n + \mathbf{b}. \quad (6.6)$$

Here the vectors $\hat{\mathbf{y}}_n$ are of dimension m , the matrix A is of $m \times k$ and the vector \mathbf{b} is of m . We have

$$\begin{aligned} \hat{\mathbf{y}}_n &= A\mathbf{x}_n + \mathbf{b} \\ &= A\mathbf{x}_{n-1} + \mathbf{b} + A(\mathbf{x}_n - \mathbf{x}_{n-1}) \\ &= \hat{\mathbf{y}}_{n-1} + \mathbf{b}_n, \end{aligned} \quad (6.7)$$

where

$$\mathbf{b}_n = A(\mathbf{x}_n - \mathbf{x}_{n-1}).$$

Let us now consider the equation

$$\tilde{\mathbf{y}}_n = \tilde{\mathbf{y}}_{n-1} + \mathbf{b}_n + \mathbf{u}_n. \quad (6.8)$$

In this specific case we have

$$\mathbf{u}_N = \sum_{n=n_0+1}^N \mathbf{u}_n. \quad (6.9)$$

By the same calculations as earlier (see (5.11)) we have:

$$\|\mathbf{u}_n\| = \frac{\|\bar{\mathbf{y}} - \hat{\mathbf{y}}_N\|}{N - n}, \quad (6.10)$$

and all the constants c_i defined in Section 5 are the same. Therefore

$$u_{n,l}^2 = \frac{\|u_n\|^2}{m}.$$

We finally have

$$\tilde{\mathbf{y}}_n = \tilde{\mathbf{y}}_{n-1} + \mathbf{b}_n + \frac{\bar{\mathbf{y}}_N - \tilde{\mathbf{y}}_N}{N - n_0},$$

7. Numerical Work

Numerical work of this paper consists of two different parts. The first part is related to the case of interpolating Numerical work of this paper consists of two different parts. The first part is related to the case of interpolating missing data in autoregressive models. The two numerical results of this part are reflected in Figure 1a and Figure 1b. The second part of numerical work is related to two- dimensional autoregressive model. The data for this model are related to archaeological field and taken from the paper of Cavanagh, Buck and Litton [7]. missing data in autoregressive models. The two numerical results of this part are reflected in Figure 1a and Figure 1b. The second part of numerical work is related to two- dimensional autoregressive model. The data for this model are related to archaeological field and taken from the paper of Cavanagh, Buck and Litton [7].

7.1. Part 1

The real data of volatility dynamic of IBM company calculated on the base of the stock information by the method of [10] have been used for Figure 1a. We removed some data from the middle and the end of this dynamic and then forecasted missing data by AR(1) model for the construction of missing data described by (1.2). The value n_0 is equal to 418, and the corresponding number of missing data is 85. Then the value n_1 is equal to 1058 and the corresponding number of missing data is 106.

In the second example (Figure 1b) we use the volatility dynamic of exchange rates of USD and New Israel Shekel. The historical period, n_0 , is 1319, and the total length, N , is 1466. Assuming that volatility dynamic is AR(1) model, $v_{n+1} = av_n + b$, then by calculation of parameters by linear least square predictor we have $a \approx 0.999576$ and $b \approx 2.652 \cdot 10^{-7}$. Assuming that volatility dynamics satisfies AR(2) model, $v_{n+2} = a_1v_{n+1} + a_2v_n + b$, we correspondingly obtain $a_1 \approx 2.4$,

$a_2 \approx -1.4$ and $b \approx 2.8 \cdot 10^{-7}$. As we can see, although the difference between these predicted models is small, both curves are visible in the graph nevertheless.

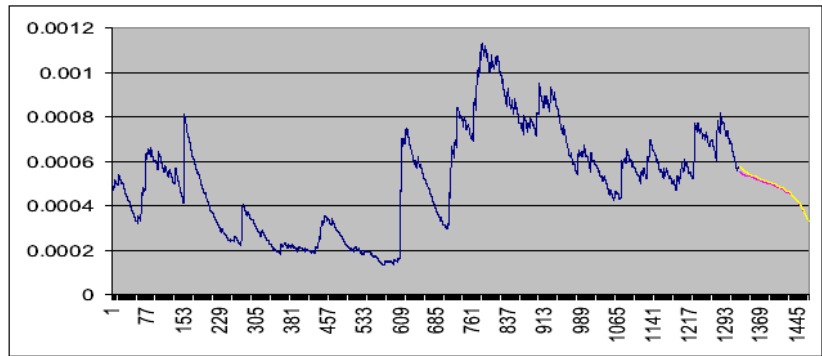
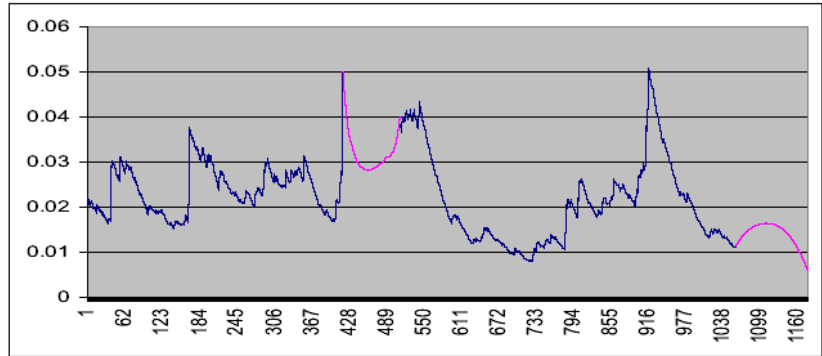


Figure 1(a). Forecasting missing data of volatility for IBM Co.: Blue line- known values, purple line- predicted values by AR(1) model.

Figure 1(b). Forecasting missing data of volatility for USD-New Israel Shekel exchange: Blue line- known values, purple line- predicted values by AR(1) model and yellow line- predicted values AR(2) model).

Table 1. Phosphate concentration (the fragment of data from [7])

59	57	80	71	19	80	60	60	60	62	+	+	166	77	+	68
60	68	75	85	57	44	30	62	38	91	+	+	68	77	+	59

7.2. Part 2

In this part we use data from [7]. This is data on Phosphate concentration reflected in Figure 1 (p. 94). There are missing data in the fifth and sixth row of these data, and these two rows are the rows of Table 1 corresponding to two-dimensional vector \mathbf{x} with missing data, where the missing data there are indicated by ‘+’.

We use a first order autoregressive model in order to predict the missing data. We do not provide all intermediate calculations, only meaningful results are shown here.

The filling of these missing data is carried out by two steps. According to our notation $n_0 = 10$ and $n_1 = 12$. We have

$$\hat{\mathbf{x}}_{11} = \begin{pmatrix} 60.43 \\ 28.57 \end{pmatrix}, \hat{\mathbf{x}}_{12} = \begin{pmatrix} 61.10 \\ 47.98 \end{pmatrix}.$$

Next, taking into account the value $\hat{\mathbf{x}}_{13} = \begin{pmatrix} 166 \\ 68 \end{pmatrix}$, we obtain the following

values for $\hat{\mathbf{x}}_{11}$ and $\hat{\mathbf{x}}_{12}$

$$\hat{\mathbf{x}}_{11} = \begin{pmatrix} 95.10 \\ 55.33 \end{pmatrix} \text{ and } \hat{\mathbf{x}}_{12} = \begin{pmatrix} 130.43 \\ 56.67 \end{pmatrix}.$$

Next, according to the accepted notation, $N = 16$. Similarly, we first find

$$\hat{\mathbf{x}}_{15} = \begin{pmatrix} 76.90 \\ 62.93 \end{pmatrix}. \text{ Then } \tilde{\mathbf{x}}_{15} = \begin{pmatrix} 71.45 \\ 61.15 \end{pmatrix}.$$

The finally modified table after calculation of missing data is now Table 2.

Table 2. Phosphate concentration (the finally modified table)

59	57	80	71	19	80	60	60	60	62	95.10	130.43	166	77	71.45	68
60	68	75	85	57	44	30	62	38	91	55.33	56.67	68	77	61.15	59

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