On Solutions of First Order Stochastic Partial Differential Equations

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Abstract

This note is concerned with an important for modelling question of existence of solutions of stochastic partial differential equations as proper stochastic processes, rather than processes in the generalized sense. We consider a first order stochastic partial differential equations of the form $\frac{\partial U}{\partial t} = DW$, and $\frac{\partial U}{\partial t} - \frac{\partial U}{\partial x} = DW$, where D is a differential operator and $W(t,x)$ is a continuous but non-differentiable function (field). We give a necessary and sufficient condition for stochastic equations to have solutions as functions. The result is then applied to the equation for a yield curve. Proofs are based on probability arguments.

1 Introduction

Stochastic differential equations are often obtained from ordinary ones by introduction of noise, which is taken to be a generalized function (Schwartz distribution). Following Rozanov 1995, the noise in SPDE is defined as follows. Let $W(t,x)$ be a continuous function (field) and D a differential operator. If W is not differentiable, then DW does not exist in the usual sense of a function, but can be considered as a generalized function. It is defined by the following action on any test function $\varphi(t,x) \in C_K^\infty((0,\infty)\times(0,\infty))$, the space of infinitely differentiable compactly supported functions,

$$\langle DW, \varphi\rangle = \langle W, D^\star \varphi\rangle = \iint W(t,x) D^\star \varphi(t,x)\, dt dx,$$

where $D^\star$ is the adjoint operator of D. The adjoint operator is defined by the identity $\langle Df, \varphi\rangle = \langle f, D^\star\varphi\rangle$ that holds for all smooth functions f and test functions φ. For details see Rozanov 1995 p. 99-103, or Lang 1993.

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In applications the noise is typically random. White noise, for example, is a generalized derivative of Brownian motion $\dot{B}(t) = \frac{dB(t)}{dt}$, and the noise considered in second order partial differential equations is the second derivative of Brownian sheet $B(t,x)$, $\dot{B}(t,x) = \frac{\partial^2 B}{\partial t \partial x}(t,x)$, (Walsh 1984, Carmona and Nualart 1988, Freidlin 1988.) First we recall a definition of Brownian sheet $B(t,x)$ on positive quadrant of the plane R^2_+, which is the basic model for White noise field with its various modifications. In our applications t is time and is non-negative, as well as the state variable x. This is not really a restriction, if needed the Brownian sheet can be taken on the whole plane R^2. A Gaussian random measure $\mathcal{B}$ on R^2 is defined by the following properties. For any Borel set $A \subset R^2_+$, $\mathcal{B}(A)$ is a Normal random variable with zero mean and variance given by the area of A. For non overlapping A_1, A_2, $\mathcal{B}(A_1)$ and $\mathcal{B}(A_2)$ are independent and $\mathcal{B}(A_1 \cup A_2) = \mathcal{B}(A_1) + \mathcal{B}(A_2)$. Put $B(t,x) = \mathcal{B}([0,t] \times ([0,x])$. It is known that for almost all realizations, $B(t,x)$ is a continuous but nowhere differentiable function of t, x. For details see e.g. Walsh 1984.

In the next section we give results for solutions of first order partial differential equations to be functions when the noise is a generalized function, and then use probabilistic arguments to derive conditions for the case when noise is obtained from a stochastic process, such as a white noise. In Section 3 we apply the result to the equation of a yield curve. Section 4 contains the proofs. Since our focus is on modelling with SPDE's it is of prime concern that solutions will be proper functions. The classical existence and uniqueness results apply in the space of generalized functions. Perhaps a classical approach is to use existence and uniqueness in the space of generalized functions and then apply some regularization results. Our approach is more direct, and utilizes the probabilistic nature of noise in the equations.

2 First Order SPDE's

Let D be a first order differential operator

$$D\varphi(t,x) = a(t,x)\frac{\partial \varphi}{\partial t}(t,x) + b(t,x)\frac{\partial \varphi}{\partial x}(t,x) + c(t,x)\varphi(t,x), \tag{1}$$

where a and b are smooth functions from C^1, and c is continuous. The adjoint operator $D^\star$ is easily found to be

$$D^\star \varphi = -\frac{\partial(a\varphi)}{\partial t} - \frac{\partial(b\varphi)}{\partial x} + c\varphi.$$

It turns out that only a particular form of the differential operator D yields solutions as functions of the SPDE (2).

Theorem 1 *Let D is given by (1) and U be a solution to the pde*

$$\frac{\partial U}{\partial t} = DW, \tag{2}$$

in the sense that $-\left\langle U, \frac{\partial \varphi}{\partial t} \right\rangle = \langle W, D^\star \varphi \rangle$ *. for any test function* φ*. Then it holds*

$$\begin{aligned}
&\int_0^x [U(t,y) - U(0,y)]dy \\
&\quad = \int_0^t [b(s,x)W(s,x) - b(s,0)W(s,0)]\, ds + \int_0^x [a(t,y)W(t,y) - a(0,y)W(0,y)]\, dy \\
&\quad + \int_0^x \left[\int_0^t \left(\frac{\partial a}{\partial t} + \frac{\partial b}{\partial x} - c \right) (s,y) W(s,y) ds \right] dy. \qquad (3)
\end{aligned}$$

Thus a solution of (2) as a function exists if and only if $\int_0^t b(s,x)W(s,x)ds$ *is differentiable in* x*. In particular, when* b *is identically zero, a solution as a function exists.*

Consider now the equation

$$\frac{\partial r}{\partial t} - \frac{\partial r}{\partial x} = DW. \qquad (4)$$

We say that $r(t,x)$ is a solution of (4) if (5) holds for any test function φ

$$\iint r(t,x) \left(\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial x} \right) (t,x) dt dx = \iint W(t,x) \left(\frac{\partial (a\varphi)}{\partial t} + \frac{\partial (b\varphi)}{\partial x} - c\varphi \right) (t,x) dt dx. \qquad (5)$$

Theorem 2 *Suppose that the noise process is such that for any* $x > 0$*,* $W(0,x) = 0$*. Then the equation*

$$\frac{\partial r}{\partial t}(t,x) - \frac{\partial r}{\partial x}(t,x) = a(t,x) \left(\frac{\partial W}{\partial t}(t,x) - \frac{\partial W}{\partial x}(t,x) \right) + c(t,x)W(t,x) \qquad (6)$$

(this is equation (4) with $b(t,x) = -a(t,x)$*) with the initial condition* $r(0,x) = r_0(x)$ *and* $r(t,x)$ *is continuous at* $t = 0$*, for all* x*, has for unique solution the function*

$$r(t,x) = a(t,x)W(t,x) + \int_0^t W(s,x) \left[\frac{\partial a}{\partial x}(s,x) - \frac{\partial a}{\partial t}(s,x) + c(s,x) \right] ds + r_0(t+x). \qquad (7)$$

Next (motivated by applications) we take the noise $W(t,x)$ as the Brownian sheet

$$W(t,x) = B(t, t+x) := B^x(t). \qquad (8)$$

Theorem 3 *A solution to equation (4), where D is given by (1) and $W(t,x)$ by (8), as a function exists if and only if*

$$a(t,x) = -b(t,x). \tag{9}$$

Moreover, in this case, the general solution (7) can be written in the form

$$r(t,x) = \int_0^t a(s,x)dB^x(s) + \int_0^t B^x(s)\left[\frac{\partial a}{\partial x}(s,x) + c(s,x)\right]ds + r_0(t+x),$$

where the integral $\int_0^t a(s,x)dB^x(s)$ is the usual Wiener-Ito integral.

3 Application to Yield Curves

Let $r(0,x) = r_0(x)$ be a given yield curve at time $t = 0$, that is the interest on the investment maturing at x. Let $r(t,x)$, $t, x \geq 0$ denote similar curve at time t, that is the interest rate at time t on the investment maturing at $t+x$. In the absence of noise it should hold that $r(t,x) = r(0,t+x) = r_0(t+x)$, otherwise one can make a riskless profit. Assuming that $r_0(x)$ is smooth, the evolution of the yield curve therefore is described by pde

$$\frac{\partial r}{\partial t} - \frac{\partial r}{\partial x} = 0, \tag{10}$$

with the initial condition $r(0,x) = r_0(x)$, see Musiela and Sondermann 1994. Consider now a stochastic analogue of this pde given by (4) $\frac{\partial r}{\partial t} - \frac{\partial r}{\partial x} = DW$, where $W(t,x) = B(t,t+x)$. The reason we take $B(t,t+x)$ rather than $B(t,x)$ is because we model the yield at the point $t+x$, and it would be natural to write $r(t,t+x)$ instead of $r(t,x)$ and add to it the basic noise at the same point $(t,t+x)$. The process $W(t,x) = B(t,t+x) = B^x(t)$ for any fixed $x > 0$, is not a Brownian motion, however it is a continuous martingale with independent but non-stationary increments. Condition $W(0,x) = 0$ for any $x > 0$ in Theorem 2 means here that at time zero there is no uncertainty in the yield curve.

Musiela and Sondermann 1994 introduced the noise by considering for all $x > 0$ the stochastic differential equations

$$dr(t,x) = \alpha(t,x)dt + \sigma(t,x)dW(t), \tag{11}$$

which hold for all $x \geq 0$. In comparison with this equation, the general solution (7) for any fixed $x > 0$ satisfies

$$dr(t,x) = a(t,x)dB^x(t) + \left[B^x(t)\left(\frac{\partial a}{\partial x}(t,x) + c(t,x)\right) + r_0'(t+x)\right]dt.$$

Hence in the spde solution the "drift" term includes random noise, moreover the equations are driven by martingales $B^x(t)$ dependent on x.

4 Proofs

The proof of Theorem 1 requires the following basic lemma.

Lemma 4 *Let g be an integrable function on $[0,+\infty)\times[0,+\infty)$, and assume that for any test function, on $(0,+\infty)\times(0,+\infty)$, φ,*

$$\iint g(t,x)\frac{\partial^2\varphi}{\partial t\partial x}(t,x)dx = 0. \tag{12}$$

Then, $g(t,x) = g(t,0) + g(0,x) + \text{const}$.

Proof of Theorem 1

For any test function φ, $-\left\langle U, \frac{\partial\varphi}{\partial t}\right\rangle =< W, D^\star\varphi >$, or equivalently

$$\begin{aligned}\iint U(t,x)\frac{\partial\varphi}{\partial t}(t,x)dtdx &= \iint W(t,x)\left[\frac{\partial(a\varphi)}{\partial t}+\frac{\partial(b\varphi)}{\partial x}-c\varphi\right](t,x)dtdx\\ &= \iint W(t,x)\left[a\frac{\partial\varphi}{\partial t}+b\frac{\partial\varphi}{\partial x}+\left(\frac{\partial a}{\partial t}+\frac{\partial b}{\partial x}-c\right)\varphi\right](t,x)dtdx.\end{aligned}$$

Integrating by parts, we get

$$\iint\left\{\int_0^x U(t,y)dy-\int_0^x a(t,y)W(t,y)dy-\int_0^t b(s,x)W(s,x)ds+\right.$$

$$\left.\int_0^x\left[\int_0^t\left(\frac{\partial a}{\partial t}+\frac{\partial b}{\partial x}-c\right)(s,y)W(s,y)ds\right]dy\right\}\frac{\partial^2\varphi}{\partial t\partial x}(t,x)dtdx = 0.$$

(3) now follows from Lemma 4. □

Proof of Theorem 2

Make a change of variables $\tau = t$, $\xi = t+x$, $\rho(\tau,\xi) = r(t,x)$, $\alpha(\tau,\xi) = a(t,x)$, $\beta(\tau,\xi) = b(t,x)$, $\gamma(\tau,\xi) = c(t,x)$, $\psi(\tau,\xi) = \varphi(t,x)$ and $V(\tau,\xi) = W(t,x)$. Perform the above change of variables in (5) and use the relation $b(t,x) = -a(t,x)$ to obtain

$$\left\langle\rho,\frac{\partial\psi}{\partial\tau}\right\rangle = \left\langle V,\alpha\frac{\partial\psi}{\partial\tau}+\left(\frac{\partial\alpha}{\partial\tau}-\gamma\right)\psi\right\rangle. \tag{13}$$

It follows from Theorem 1 that

$$\int_0^\xi[\rho(\tau,\eta)-\rho(0,\eta)]d\eta = \int_0^\xi\alpha(\tau,\eta)V(\tau,\eta)d\eta+\int_0^\xi\int_0^\tau\left(\frac{\partial\alpha}{\partial t}-c\right)(\theta,\eta)V(\theta,\eta)d\theta d\eta.$$

Differentiating with respect to ξ, we get

$$\rho(\tau,\xi) = \alpha(\tau,\xi)V(\tau,\xi) - \int_0^\tau V(\theta,\xi)\left(\frac{\partial\alpha}{\partial\tau} - \gamma\right)(\theta,\xi)d\theta + \rho(0,\xi). \tag{14}$$

Going back to the original variables and taking into account the initial condition we obtain (7). Notice that a formal derivation of (14) is obtained by integrating with respect to τ the equality $\frac{\partial\rho}{\partial\tau} = \alpha\frac{\partial V}{\partial\tau} + \gamma V$ (to be understood in the sense of generalized functions). □

Proof of Theorem 3

The proof of Theorem 3 is based on the following lemmas.

Lemma 5 *Let $\mathcal{X}$ be a Gaussian measure on a measurable space $(E,\mathcal{E})$ with intensity μ, R and S be two functions in $L^2(E,\mathcal{E},\mu)$, F and G be two measurable sets, and $(F_k^n)_k$ and $\{G_k^n)_k$ be measurable partitions of F and G respectively. Assume that*

1. *$\mu(F) < +\infty$ and $\mu(G) < +\infty$;*
2. *$\lim_{n\to\infty}\sup_{1\le k\le n}\mu(F_k^n) = \lim_{n\to\infty}\sup_{1\le k\le n}\mu(G_k^n) = 0$.*

If $(F_k^n \cap G_k^n)_k$ is a partition of $F \cap G$, then

$$\lim_{n\to\infty}\sum_{k=1}^n R_k^n S_k^n \mathcal{X}(F_k^n)\mathcal{X}(G_k^n) = \int_{F\cap G} RSd\mu$$

in the L^2-sense, where (R_k^n) and (S_k^n) are approximating sequences on $(F_k^n \cap G_k^n)_k$ of R and S respectively.
If, on the other hand, for all k and l, one or both of $F_k^n \cap G_l^n$ and $F_l^n \cap G_k^n$ are empty (in particular $F_k^n \cap G_k^n = \emptyset$), then

$$\lim_{n\to\infty}\sum_{k=1}^n R_k^n S_k^n \mathcal{X}(F_k^n)\mathcal{X}(G_k^n) = 0$$

in the L^2-sense.

Proof of Lemma 5:

Let us first note that $\mathrm{var}(\mathcal{X}(F_k^n)\mathcal{X}(G_k^n)) = \mu(F_k^n)\mu(G_k^n) + \mu(F_k^n \cap G_k^n)^2$, and that, for $k \neq l$, $\mathrm{cov}(\mathcal{X}(F_k^n)\mathcal{X}(G_k^n), \mathcal{X}(F_l^n)\mathcal{X}(G_l^n)) = \mu(F_k^n \cap G_l^n)\mu(F_l^n \cap G_k^n)$. It follows that, if $(F_k^n \cap G_k^n)_k$

is a partition of $F \cap G$, then the random variables $\mathcal{X}(F_k^n)\mathcal{X}(G_k^n)$ and $\mathcal{X}(F_l^n)\mathcal{X}(G_l^n)$, for $k \neq l$, are independent.

$$\left\| \sum_{k=1}^{n} R_k^n S_k^n \mathcal{X}(F_k^n)\mathcal{X}(G_k^n) - \int_{F\cap G} RSd\mu \right\|_2 \leq$$
$$\left\| \sum_{k=1}^{n} R_k^n S_k^n \mathcal{X}(F_k^n)\mathcal{X}(G_k^n) - \sum_{k=1}^{n} R_k^n S_k^n \mu(F_k^n \cap G_k^n) \right\|_2 + \left| \sum_{k=1}^{n} R_k^n S_k^n \mu(F_k^n \cap G_k^n) - \int_{F\cap G} RSd\mu \right|.$$

The second term of the right hand side goes to zero. By the independence of the random variables $\mathcal{X}(F_k^n)\mathcal{X}(G_k^n)$,the first term becomes

$$\begin{aligned}
\text{var}\left(\sum_{k=1}^{n} R_k^n S_k^n \mathcal{X}(F_k^n)\mathcal{X}(G_k^n)\right) &= \sum_{k=1}^{n}(R_k^n S_k^n)^2 \text{var}\left(\mathcal{X}(F_k^n)\mathcal{X}(G_k^n)\right) \\
&= \sum_{k=1}^{n}(R_k^n S_k^n)^2 [\mu(F_k^n)\mu(G_k^n) + \mu(F_k^n \cap G_k^n)^2] \\
&\leq 2\sum_{k=1}^{n}(R_k^n S_k^n)^2 \mu(F_k^n \cup G_k^n)^2 \\
&\leq 2 \sup_{1\leq k\leq n} \mu(F_k^n \cup G_k^n) \sum_{k=1}^{n}(R_k^n S_k^n)^2 \mu(F_k^n \cup G_k^n),
\end{aligned}$$

which completes the proof of the first statement. The second statement follows in the same way. □

Lemma 6 *Let $\mathcal{X}$ be a Gaussian measure on a measurable space $(E, \mathcal{E})$ with intensity μ, F be a measurable set, and $(F_k^n)_k$ be a measurable partition of F. If there exists $\kappa > 0$ such that*

$$\lim_{n\to\infty} n^{\kappa} \sup_{1\leq k\leq n} \mu(F_k^n) = 0,$$

then, with probability one,

$$\lim_{n\to\infty} \sup_{1\leq k\leq n} |\mathcal{X}(F_k^n)| = 0.$$

Proof of Lemma 6: Fix $\varepsilon > 0$.

$$\begin{aligned}
\mathrm{P}\left[\sup_{1\leq k\leq n} |\mathcal{X}(F_k^n)| > \varepsilon\right] &= 1 - \mathrm{P}\left[\sup_{1\leq k\leq n} |\mathcal{X}(F_k^n)| \leq \varepsilon\right] \\
&= 1 - \prod_{k=1}^{n}\left[2\Phi\left(\frac{\varepsilon}{\sqrt{\mu(F_k^n)}}\right) - 1\right]
\end{aligned}$$

$$\leq \; 1-\prod_{k=1}^{n}\left[1-\exp\left\{-\sqrt{\frac{2}{\pi}}\frac{\varepsilon}{\sqrt{\mu(F_k^n)}}\right\}\right]$$
$$\leq \; 1-\left[1-\exp\left\{-\sqrt{\frac{2}{\pi}}\frac{\varepsilon}{\sqrt{\mu(F^n)}}\right\}\right]^n,$$

where $\mu(F^n)=\sup_{1\leq k\leq n}\mu(F_k^n)$. The proof is ended by the fact that

$$\sum_n\left(1-\left[1-\exp\left\{-\sqrt{\frac{2}{\pi}}\frac{\varepsilon}{\sqrt{\mu(F^n)}}\right\}\right]^n\right)<+\infty.$$

□

Proof of Theorem 3: The " if " part is given in Theorem 2. We now prove the " only if " part. Using the notations of the proof of Theorem 2, we obtain, instead of (13) and without the assumption $b(t,x)=-a(t,x)$,

$$\left\langle \rho,\frac{\partial\psi}{\partial\tau}\right\rangle=\left\langle V,\alpha\frac{\partial\psi}{\partial\tau}+(\alpha+\beta)\frac{\partial\psi}{\partial\xi}+\left(\frac{\partial\alpha}{\partial\tau}+\frac{\partial(\alpha+\beta)}{\partial\xi}-\gamma\right)\psi\right\rangle.$$

It now follows from Theorem 1 that for a solution to exist in the space of functions,

$$Z(t,x):=\int_0^t A(s,x)B(s,s+x)ds=\int_0^t A(s,x)W(s,x)ds=\int_0^\tau(\alpha+\beta)(\theta,\xi)V(\theta,\xi)d\theta,$$

where $A=a+b$, must be differentiable in x. Let $Y(t,x)=A(t,x)B(t,t+x)$, $x<y$, $\delta^n=(y-x)/n$ and $\delta_k^n=k\delta^n$. We have,

$$\begin{aligned}&\sum_{k=1}^{n}\Big(Z(t,x+\delta_k^n)-Z(t,x+\delta_{k-1}^n\Big)^2\\ &=\;\sum_{k=1}^{n}\left(\int_0^t\Big(Y(s,x+\delta_k^n)-Y(s,x+\delta_{k-1}^n)\Big)\,ds\right)^2\\ &=\;\int_0^t\int_0^t\sum_{k=1}^{n}\Big[\Big(Y(r,x+\delta_k^n)-Y(r,x+\delta_{k-1}^n)\Big)\Big(Y(s,x+\delta_k^n)-Y(s,x+\delta_{k-1}^n)\Big)\Big]\,drds.\end{aligned}$$

Now $\Delta Y_k^n(s):=Y(s,x+\delta_k^n)-Y(s,x+\delta_{k-1}^n)=A_k^n(s)\mathcal{B}(F_k^n(s))+B_{k-1}^n(s)\mathcal{A}(G_k^n(s))$, with $A_k^n(s)=A(s,x+\delta_k^n)$, $B_{k-1}^n(s)=B(s,s+x+\delta_k^n)$, $F_k^n(s)=[0,s]\times[s+x+\delta_{k-1}^n,s+x+\delta_k^n]$, $G_k^n(s)=[0,s]\times[x+\delta_{k-1}^n,x+\delta_k^n]$, $\mathcal{B}$ is the Gaussian measure associated to the Brownian sheet B, and $\mathcal{A}$ the measure associated to the C^1 function A. Thus

$$\sum_{k=1}^{n}\Delta Y_k^n(r)\Delta Y_k^n(s)$$

$$= \sum_{k=1}^{n} A_k^n(r)A_k^n(s)\mathcal{B}(F_k^n(r))\mathcal{B}(F_k^n(s)) + \sum_{k=1}^{n} A_k^n(r)B_{k-1}^n(s)\mathcal{B}(F_k^n(r))\mathcal{A}(G_k^n(s))$$
$$+ \sum_{k=1}^{n} B_{k-1}^n(r)A_k^n(s)\mathcal{A}(G_k^n(r))\mathcal{B}(F_k^n(s)) + \sum_{k=1}^{n} B_{k-1}^n(r)B_{k-1}^n(s)\mathcal{A}(G_k^n(r))\mathcal{A}(G_k^n(s)).$$

Now each one of the last three terms converges, with probability one, to 0. Indeed

$$\left|\sum_{k=1}^{n} A_k^n(r)B_{k-1}^n(s)\mathcal{B}(F_k^n(r))\mathcal{A}(G_k^n(s))\right| \le \sup_{1\le k\le n} |\mathcal{B}(F_k^n(r))| \left|\sum_{k=1}^{n} A_k^n(r)B_{k-1}^n(s)\mathcal{A}(G_k^n(s))\right|,$$

which, according to Lemma 6, goes to 0. Recall that, if μ is the intensity of the Gaussian measure $\mathcal{B}$, then

$$\sup_{1\le k\le n} \mu(F_k^n(r)) = r\frac{y-x}{n}.$$

The same goes for the third term. The convergence to 0 of the fourth term follows from

$$\lim_{n\to\infty} \sup_{1\le k\le n} |\mathcal{A}(F_k^n(r))| = 0.$$

To investigate the asymptotic behavior of the first term, we first make the following observation. For $r \ne s$, there is n_0 ($n_0 \ge (y-x)/|r-s|$) such that for any $n \ge n_0$, and any k and l, $F_k^n(r) \cap F_l^n(s) = \emptyset$ or $F_l^n(r) \cap F_k^n(s) = \emptyset$. Applying Lemma 5, we see that $\sum_{k=1}^n A_k^n(r)A_k^n(s)\mathcal{B}(F_k^n(r))\mathcal{B}(F_k^n(s))$ converges in L^2 to 0. On the other hand, if $r = s$, then, by Lemma 5,

$$\lim_{n\to\infty} \sum_{k=1}^{n} A_k^n(s)^2 \mathcal{B}(F_k^n(s))^2 = \int_x^y A(s,z)^2 s dz$$

in the L^2 sense. It follows that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta Y_k^n(r)\Delta Y_k^n(s) = \begin{cases} \int_x^y A(s,z)^2 s dz & \text{if} \quad r = s \\ 0 & \text{if} \quad r \ne s \end{cases}$$

in probability, and that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Big(Z(t, x+\delta_k^n) - Z(t, x+\delta_{k-1}^n\Big)^2 = \int_0^t \left(\int_x^y A(s,z)^2 s dz\right) ds \tag{15}$$

in probability. Now assume that $Z(t,x)$, as a process in x, is locally Hölder of order $\varepsilon > 1/2$, then

$$\Big|Z(t, x+\delta_k^n) - Z(t, x+\delta_{k-1}^n)\Big| \le \text{const } (\delta^n)^\varepsilon,$$

and

$$\sum_{k=1}^{n} (Z(t, x+k\delta_n) - Z(t, x+(k-1)\delta_n))^2 \le \text{const}^2 \ (y-x)(\delta^n)^{2\varepsilon-1} \longrightarrow 0, \quad \text{as } n \to \infty.$$

This combined with (15) shows that $A(s,x)$ must be nil. □

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