CRAMER'S THEOREM FOR NONNEGATIVE MULTIVARIATE POINT PROCESSES WITH INDEPENDENT INCREMENTS

KLEBANER F. AND R. LIPTSER

Abstract. We consider a continuous time version of Cramer's theorem with nonnegative summands $S_t = \frac{1}{t}\sum_{i:\tau_i\le t}\xi_i,\ t\to\infty$, where $(\tau_i,\xi_i)_{i\ge 1}$ is a sequence of random variables such that tS_t is a random process with independent increments.

1. **Introduction and main result**

The following version of the Cramer theorem [1] can be extracted from Dembo and Zeitouni, [2].

Theorem 1. *Let $(\xi_i)_{i\ge 1}$ be a sequence of nonnegative identically distributed and independent random variables with ξ_1 admitting the Laplace transform*:

$$\mathscr{L}(\lambda)=\mathsf{E}e^{\lambda\xi_1},\ \lambda\in(-\infty,\Lambda),\quad \exists\ \Lambda=\inf\{\lambda>0:\mathscr{L}(\lambda)=\infty\}.$$

Then, the family

$$S_n=\frac{1}{n}\sum_{i=1}^{n}\xi_i,\quad n\to\infty$$

obeys the Large Deviation principle (*LDP*) *in the metric space* $(\mathbb{R}_+,\varrho)$ (*for the Euclidean metric* ϱ) *with the rate* $\frac{1}{n}$ *and the rate function*

$$I(u)=\begin{cases}\sup\limits_{\lambda\in(-\infty,\Lambda)}\big[\lambda u-g(\lambda)\big], & u>0\\ -\log\mathsf{P}(\xi_1=0), & u=0,\end{cases}$$

where $g(\lambda)$ *is the log moment generation function,*

$$g(\lambda)=\log\mathsf{E}e^{\lambda\xi_1},\ \lambda<\Lambda.$$

In this paper, we study a "continuous time version" of this theorem. For $t\ge 0$, set

$$S_t=\frac{1}{t}\sum_{i:\tau_i\le t}\xi_i,$$

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where $(\xi_i, \tau_i)_{i\ge 1}$ is a sequence of random pairs, where ξ_i's and τ_i's are random variables:

$$\xi_i \ge 0 \quad \text{and} \quad \tau_0 = 0 < \tau_1 < \tau_2 < \ldots \tau_i < \ldots$$

defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $(\mathscr{G}_n)_{n\ge 0}$ be the filtration with $\mathscr{G}_0 = (\varnothing, \Omega)$ and $\mathscr{G}_n := \sigma\{(\tau_i, \xi_i)_{i\le n}\}$. Random variables ξ_i are assumed to be identically distributed and independent of $\mathscr{G}_{i-1}$ with the distribution function

$$G(x) = \mathsf{P}(\xi_1 \le x), \ x \ge 0.$$

The conditional distribution of τ_i given $\mathscr{G}_{i-1}$ is exponential:

$$\mathsf{P}\big(\tau_i \le t|\mathscr{G}_{i-1}\big) = \big(1 - e^{-r(t-\tau_{i-1})}\big), \ t \ge \tau_{i-1},$$

where r is a positive number. Moreover, we assume that

$$\mathsf{P}\big(\xi_i \le x, \tau_i \le t|\mathscr{G}_{i-1}\big) = G(x)\big(1 - e^{-r(t-\tau_{i-1})}\big), \ i \ge 1.$$

The following theorem is an analogue of Theorem 1.

Theorem 2. *The family*

$$S_t = \frac{1}{t}\sum_{i:\tau_i\le t} \xi_i, \quad t \to \infty$$

obeys the LDP in the metric space $(\mathbb{R}_+, \varrho)$ with the rate $\frac{1}{t}$ an the rate function

$$I(u) = \begin{cases} \sup\limits_{\lambda\in(-\infty,\Lambda)} \big[\lambda u - r\int_0^\infty (e^{\lambda z} - 1)dG(z)\big], & u > 0 \\ r[1 - G(0+)], & u = 0. \end{cases}$$

We give two examples illustrating compatibility with Theorems 1 and 2. For both discrete and continuous time cases, let

$$\mathsf{P}(\xi_1 \le x) = 1 - e^{-x}, \ x \ge 0,$$

so that, ξ_1 has the Laplace transform with $\Lambda = 1$. The log moment generating function is

$$g(\lambda) = -\log(1 - \lambda), \ \lambda < 1.$$

For $G(x) = 1 - e^{-x}$, $x \ge 0$.

$$\int_0^\infty r(e^{\lambda z} - 1)dG(z) = \frac{r\lambda}{1 - \lambda}, \ \lambda < 1.$$

In both cases, rate functions are explicitly computable (see also figures (1) and (2)),

$$I^d(u) = \begin{cases} u - 1 - \log(u), & u > 0 \\ \infty, & u = 0 \end{cases} \quad \text{(discrete time case)}$$

$$I^c(u) = \begin{cases} \big(\sqrt{r} - \sqrt{u}\big)^2, & u > 0 \\ r, & u = 0. \end{cases} \quad \text{(continuous time case)}$$

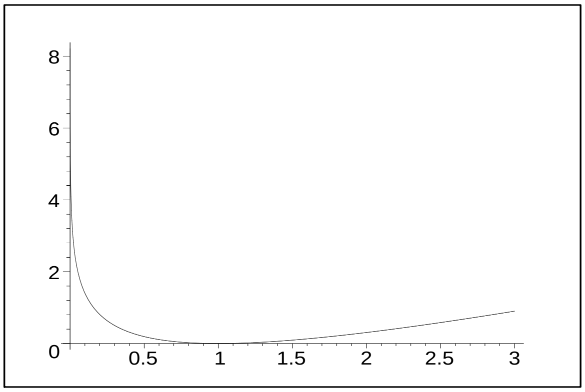

FIGURE 1. The rate function $I^d(u)$

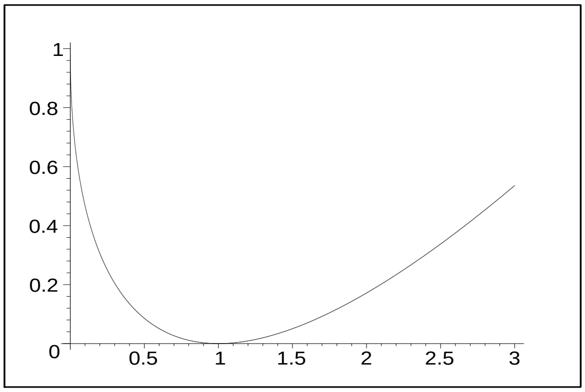

FIGURE 2. The rate function $I^c(u)$ for $r = 1$

Remark 1. Related topics to Theorem 2 can be found e.g. in Georgii and Zessin, [3], serving a class of marked point random fields. Probably, the proof of Theorem 2 can be adapted with arguments from proofs in [3] provided that many details not related to our setting have to be omitted and other ones concerning to the boundary effect have to be added.

We prefer to give a complete and direct proof of Theorem 2.

2. **Counting random measure, its compensator, Laplace transform**

We consider $(\xi_i, \tau_i)_{i\ge 1}$ as a multivariate (marked) point process (see, e.g. [4], [5]) with the counting measure

$$\mu(dt, dy) = \sum_{i\ge 1} \mathbf{I}_{\{\tau_i<\infty\}} \delta_{\{\tau_i,\xi_i\}}(t, y) dt dy,$$

where $\delta_{\{\tau_i,\xi_i\}}$ is the Dirac delta-function on $\mathbb{R}_+ \times \mathbb{R}_+$. Parallel to $(\mathscr{G}_n)_{n\ge 0}$, we introduce one more filtration $(\mathscr{G}_t)_{t\ge 0}$ related to $(\xi_i, \tau_i)_{i\ge 1}$:

$$\mathscr{G}_t := \sigma(\mu([0, t'] \times \mathit{\Gamma}) : t' \le t, \mathit{\Gamma} \in \mathscr{B}(\mathbb{R}_+)),$$

where $\mathscr{B}(\mathbb{R}_+)$ is the Borel σ-algebra on $\mathbb{R}_+$, and assume that $\mathscr{G}_0$ is augmented by P-zero sets from $\mathcal{F}$ (notice that, then, $(\mathscr{G}_t)_{t\ge 0}$ satisfies the general conditions). With the help of the counting measure μ, one can present tS_t in a form of a stochastic integral with respect to μ:

$$tS_t = \int_0^t \int_{x>0} x\mu(ds, dx). \tag{2.1}$$

Then, the Levy measure $\nu(ds, dx)$, related to μ, is explicitly computed (see, e.g. Theorem III.1.33, [5])

$$\nu(ds, dx) = \sum_{i\ge 1} I_{\rrbracket \tau_{i-1}, \tau_i \rrbracket}(t) \frac{dG(x) de^{-r(s-\tau_{i-1})}}{e^{-r(s-\tau_{i-1})}} = rdsdG(x) \tag{2.2}$$

and is not random. It is well known (see, e.g. Corollary to Theorem 1 in §4, Ch.4, [6] that under the deterministic Levy process tS_t is the random process with independent increments. We recall a useful property: for any nonnegative and $(\mathscr{G}_t)$-predictable function $f(\omega, x, t)$,

$$\mathsf{E}\int_0^t \int_{x>0} f(\omega, x, s)\mu(ds, dx) = \mathsf{E}\int_0^t \int_{x>0} f(\omega, x, s)\nu(ds, dx)$$

with "$\infty = \infty$".

Lemma 1. [Laplace transform] *For any $\lambda < \Lambda$ and $t > 0$,*

$$\mathsf{E}e^{\lambda t S_t} = e^{rt\int_{x>0}[e^{\lambda z}-1]dG(z)}.$$

Proof. Though the direct computation of Laplace's transform is permissible, we prefer to apply the stochastic calculus. The process $U_t = e^{\lambda t S_t}$ has right-continuous piece-wise constant paths with jumps

$$\triangle U_s = (U_s - U_{s-}) = U_{s-}\int_{x>0} [e^{\lambda x} - 1]\mu(\{s\}, dx),$$

so that, for any $t > 0$,

$$U_t = 1 + \int_0^t \int_{x>0} U_{s-}[e^{\lambda x} - 1]\mu(ds, dx).$$

Since the function $f(\omega, x, s) := U_{s-}[e^{\lambda x} + 1]$ is nonnegative and predictable, the following equality with $\lambda < \Lambda$ holds true:

$$E\int_0^t \int_{x>0} U_{s-}[e^{\lambda x}+1]\mu(ds, dx) = E\int_0^t \int_{x>0} U_{s-}[e^{\lambda x}+1]\nu(ds, dx)\ (< \infty).$$

Then, we also have

$$E\int_0^t \int_{x>0} U_{s-}[e^{\lambda x}-1]\mu(ds, dx) = E\int_0^t \int_{x>0} U_{s-}[e^{\lambda x}-1]\nu(ds, dx)\ (\in \mathbb{R}).$$

Since $\nu(ds, dx) = rdsdG(x)$, the later provides

$$(\mathsf{E}U_t) = 1 + \int_0^t \int_{x>0} (EU_s)[e^{\lambda x} - 1]dG(x)rds.$$

This can be written in an equivalent form of differential equation

$$\frac{d(\mathsf{E}U_t)}{dt}=(\mathsf{E}U_t)\int_{x>0}[e^{\lambda x}-1]dG(x)r$$

subject to $(\mathsf{E}U_0)=1$.

Thus, the desired result holds. □

3. The proof of Theorem 2

We verify the necessary and sufficient conditions for the LDP to hold (for more details, see Puhalskii, [7]):

1) exponential tightness,

$$\lim_{j\to\infty}\overline{\lim_{t\to\infty}}\frac{1}{t}\log\mathsf{P}\big(S_t\in\mathbb{R}_+\setminus\mathcal{K}_j\big)=-\infty,$$

where $\mathcal{K}_j$'s are compacts increasing to $\mathbb{R}_+$;

2) local LDP, defining the rate function $I(u)$, $u\in\mathbb{R}_+$

$$\lim_{\delta\to 0}\lim_{t\to\infty}\frac{1}{t}\log\mathsf{P}\big(|S_t-u|\le\delta\big).=-I(u).$$

3.1. The exponential tightness.

By choosing

$$K_j=\{x\in\mathbb{R}_+:x\in[0,j]\}$$

and applying Chernoff's inequality with parameter 0.5Λ, we find that

$$\mathsf{P}(S_t>j)\le e^{-0.5\Lambda j+\log\mathsf{E}e^{0.5\Lambda tS_t}}.$$

By Lemma 1,

$$\mathsf{E}e^{0.5\Lambda tS_t}=e^{rt\int_{x>0}[e^{0.5\Lambda x}-1]dG(z)}$$

and, therefore,

$$\frac{1}{t}\log\mathsf{P}\big(S_t>j\big)\le-0.5\Lambda j+r\int_{x>0}[e^{0.5\Lambda x}-1]dG(x)\xrightarrow[j\to\infty]{}-\infty$$

and 1) is done.

3.2. The local LDP.

We begin with computation of $I(0)$ and prove

$$\begin{aligned}\underline{\lim}_{\delta\to 0}\underline{\lim}_{t\to\infty}\frac{1}{t}\log\mathsf{P}(S_t\le\delta)&\ge-r[1-G(0+)]\\ \overline{\lim}_{\delta\to 0}\overline{\lim}_{t\to\infty}\frac{1}{t}\log\mathsf{P}(S_t\le\delta)&\le-r[1-G(0+)]\end{aligned}\tag{3.1}$$

By (2.2), $\{tS_t=0\}=\{\mu((0,t]\times\{x>0\})=0$. Consequently for any $t>0$,

$$\mathsf{P}(S_t\le\delta)\ge\mathsf{P}(S_t=0)=\mathsf{P}(tS_t=0)=\mathsf{P}\big(\mu\big((0,t],\{x>0\}\big)=0\big).$$

The counting process $\pi_t:=\mu\big((0,t],\{x>0\}\big)$ has independent increments and the rate

$$\mathsf{E}\mu\big((0,t],\{x>0\}\big)=\nu\big((0,t],\{x>0\}\big)=r[1-G(0+)]t.$$

It is a counting process with the compensator $\nu\big((0,t],\{x>0\}\big)=r[1-G(0+)]t$. Therefore, by the Watanabe theorem, [8], π_t is a Poisson process with parameter $r[1-G(0+)]$. Hence, due to well known property of the Poisson process

$$\mathsf{P}(\pi_t=0)=e^{-tr[1-G(0+)]}.$$

We find that

$$\frac{1}{t}\log\mathsf{P}(S_t\le\delta)\ge\frac{1}{t}\log\mathsf{P}(\pi_t=0)=-r[1-G(+)]$$

implying the lower bond from (3.1).

The upper bound from (3.1) is derived with the help of Laplace's transform with $0<\lambda<\Lambda$. To this end, we use identity

$$1=\mathsf{E}\exp\Big(\lambda tS_t-tr\int_{x>0}[e^{\lambda x}-1]dG(x)\Big)$$

implying the inequality

$$1\ge\mathsf{E}I_{\{S_t\le\delta\}}\exp\Big(t\Big[\lambda\delta-r\int_{x>0}[e^{\lambda x}-1]dG(x)\Big]\Big)$$

being equivalent to

$$\frac{1}{t}\log\mathsf{P}(S_t\le\delta)\le-\lambda\delta+r\int_{x>0}[e^{-\lambda x}-1]dG(x).$$

Now, passing $t\to\infty$, we obtain the following upper bound depending on δ and λ:

$$\varlimsup_{t\to\infty}\frac{1}{t}\log\mathsf{P}(S_t\le\delta)\le-\lambda\delta+r\int_{x>0}[e^{\lambda x}-1]dG(x).$$

Now, passing $\delta\to0$ and λ to $-\infty$ we find that

$$\varlimsup_{\delta\to0}\varlimsup_{t\to\infty}\frac{1}{t}\log\mathsf{P}(S_t\le\delta)\le-r\int_{x>0}dG(x)=-r[1-G(0+)].$$

We continue the proof by checking the formula for $I(u)$ when $u>0$, i.e.

$$\varlimsup_{\delta\to0}\varlimsup_{t\to\infty}\frac{1}{t}\log\mathsf{P}(|S_t-u|\le\delta)\le-I(u)$$
$$\varliminf_{\delta\to0}\varliminf_{t\to\infty}\frac{1}{t}\log\mathsf{P}(|S_t-u|\le\delta)\ge-I(u),$$

with

$$I(u)=\sup_{\lambda\in(-\infty,\Lambda)}\big[\lambda u-r\int_0^\infty(e^{\lambda x}-1)dG(x)\big].$$

The Laplace transform

$$1=\mathsf{E}\exp\Big(\lambda tS_t-tr\int_{x>0}[e^{\lambda x}-1]dG(x)\Big),\ \lambda<\Lambda,$$

implies the inequality

$$1 \geq \mathsf{E} I_{\{|S_t - u| \leq \delta\}} \exp\Big(-t\delta u + \lambda t u - tr \int_{x>0} [e^{\lambda x} - 1] dG(x)\Big)$$

prviding the following upper bound depending on λ:

$$\overline{\lim_{\delta\to 0}}\, \overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathsf{P}(|S_t - u| \leq \delta) \leq -\Big(\lambda u - r \int_{x>0} [e^{\lambda x} - 1] dG(x)\Big).$$

A further minimization of the right hand side of the above inequality in λ over $(-\infty, \Lambda)$ gives the desired result.

The lower bound proof uses a standard approach of changing "probability measure". Denote by $\lambda^* = \operatorname{argmax}_{\lambda<\Lambda}\big(\lambda u - r\int_{x>0}[e^{\lambda x}-1]dG(x)\big)$. Since λ^* solves the equation (with $u > 0$)

$$u - r \int_{x>0} x e^{\lambda x} dG(x)\big) = 0, \tag{3.2}$$

λ^* is a proper number strictly less than Λ. Set

$$\mathfrak{L}_t(\lambda^*) = \exp\left(\lambda^* t S_t - \int_0^t \int_{x>0} r[e^{\lambda^* x} - 1] dG(x) ds\right). \tag{3.3}$$

First of all we notice that the Laplace transform for tS_t with λ^* guarantees $\mathsf{E}\mathfrak{L}_t(\lambda^*) = 1$. Moreover, taking into account (2.1) and applying the Itô formula to $\mathfrak{L}_t(\lambda^*)$ one can see that $(\mathfrak{L}_t(\lambda^*), \mathscr{G}_t)_{t\geq 0}$ is a positive local martingale with paths from the Skorokhod space $\mathbb{D}_{[0,\infty)}$. Then a measure $\widetilde{\mathsf{P}}_t$, defined by $d\widetilde{\mathsf{P}}_t = \mathfrak{L}_t(\lambda^*) d\mathsf{P}_t$, where P_t is a restriction of P on $\mathscr{G}_t$, is the probability measure. We introduce the probability space $(\varOmega, \mathcal{F}, \widetilde{\mathsf{P}}_t)$. Since $\mathfrak{L}_t(\lambda^*) > 0$, P-a.s., not only $\widetilde{\mathsf{P}}_t \ll \mathsf{P}_t$ but also $\widetilde{\mathsf{P}}_t \ll \mathsf{P}_t$ with $d\mathsf{P}_t = \mathfrak{L}_t^{-1}(\lambda^*)\widetilde{\mathsf{P}}_t$. This property and (3.3) provide a lower bound

$$\begin{aligned}\mathsf{P}\big(|S_t - u| \leq \delta\big) = \mathsf{P}_t\big(|S_t - u| \leq \delta\big) &= \int_{\{|S_t - u| \leq \delta\}} \mathfrak{L}_t^{-1}(\lambda^*) d\widetilde{\mathsf{P}}_t \\ &\geq e^{-\lambda^* t\delta - tI(u)} \widetilde{\mathsf{P}}_t\big(|S_t - u| \leq \delta\big).\end{aligned}$$

Therefore, we find that

$$\underline{\lim}_{t\to\infty} \frac{1}{t} \log \mathsf{P}\big(|S_t - u| \leq \delta\big) \geq -I(u) - \lambda^*\delta + \underline{\lim}_{t\to\infty} \frac{1}{t} \log \widetilde{\mathsf{P}}_t\big(|S_t - u| \leq \delta\big).$$

Obviously, the desired lower bound to obtain it is left to prove that

$$\underline{\lim}_{t\to\infty} \widetilde{\mathsf{P}}_t\big(|S_t - u| \leq \delta\big) = 1$$

or, equivalently,

$$\lim_{t\to\infty} \widetilde{\mathsf{P}}_t\big(|S_t - u| > \delta\big) = 0. \tag{3.4}$$

Thus the last step of the proof deal with (3.4). To this end, we show that ($\widetilde{\mathsf{E}}_t$ denotes the expectation relative to $\widetilde{\mathsf{P}}_t$)

$$\lim_{t\to\infty} \widetilde{\mathsf{E}}_t |S_t - u|^2 = 0.$$

Since $\mathsf{E}\mathfrak{L}_t(\lambda) = 1$, it holds

$$0 = \frac{\partial^2 \mathsf{E}\mathfrak{L}_t(\lambda)}{\partial \lambda^2}\Big|_{\lambda=\lambda^*}$$

$$= t^2 \mathsf{E}\Big(S_t - r\int_{\{x>0\}} x e^{\lambda^* x} dG(x)\Big)^2 \mathfrak{L}_t(\lambda^*) - t\,\underbrace{r\int_{\{x>0\}} x^2 e^{\lambda^* x} dG(x)}_{=u \ \text{(see (3.2))}}$$

$$= t^2 \widetilde{\mathsf{E}}\Big(S_t - r\int_{\{x>0\}} x e^{\lambda^* x} dG(x)\Big)^2 - tu.$$

Hence,

$$\widetilde{\mathsf{E}}\big(S_t - u\big)^2 = \frac{1}{t} r \mathsf{E}\int_{\{x>0\}} x^2 e^{\lambda^* x} dG(x) \xrightarrow[t\to\infty]{} 0.$$

□

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School of Mathematical Sciences, Building 28M, Monash University, Clayton Campus, Victoria 3800, Australia.
E-mail address: `fima.klebaner@sci.monash.edu.au`

Department of Electrical Engineering Systems, Tel Aviv University, 69978 Tel Aviv, Israel
E-mail address: `liptser@eng.tau.ac.il`