LIKELY PATH TO EXTINCTION IN SIMPLE BRANCHING MODELS WITH LARGE INITIAL POPULATION

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Abstract. We give explicit formulae for most likely paths to extinction in simple branching models when initial population is large. In discrete time we study the Galton-Watson process and in continuous time the Branching diffusion. The most likely paths are found with the help of the Large Deviation Principle (LDP). We also find asymptotics for the extinction probability, which gives a new expression in continuous time and recovers the known formula in discrete time. Due to the non-negativity of the processes, the proof of LDP at the point of extinction uses a nonstandard argument of independent interest.

1. Introduction and main results

In population genetics it is often important to look back at the development of populations. In this paper we consider the question of how extinctions occur, and in particular, what path a population takes on the road to extinction. Using asymptotic analysis when initial population values are large, we are able to find most likely path to extinction as well as the extinction probability in two simple branching models in discrete and continuous time. In both examples we use the large deviation principle (LDP) which is non-standard since random processes are nonnegative, and we use trajectories ending up at zero.

One of the contributions of this paper is in rigorous proofs of the LDP for processes on half space. It may appear to the reader that the LDP follows from known results in Markov chains and diffusions. This is only partly correct. In proofs of local LDP one needs to prove a lower bound. The standard proof relies on the change of measure. This requires a certain point (the point where maximum in the Fenchel-Legendre transform is achieved) to be finite. In our case this point is at infinity, and a standard approach for the proof of lower bound brake down. We therefore give complete proofs of LDP in Sections 4 and 5 following the scheme of Puhalskii [19]. His approach states that the LDP is equivalent to exponential tightness plus local LDP, and is based on method of stochastic exponential (instead of the Laplace transform). Although we follow the scheme of Puhalskii [19] we do not use idempotent probability and give direct proofs. Since these proofs are more technical, we placed them at the end, after results on extinction. Once the LDP is established, the problem of finding most likely path to extinction is in effect the problem of minimization of the rate function. This is typically a difficult problem due to nonlinearity. We are able to solve it by setting up the Bellman equation in discrete case, Section 2, and a dynamical control problem in continuous case, Section 3.

1.1. Galton-Watson process.

A prototype of branching model in discrete time is the Galton-Watson process, described as follows.

Let X_n denote the population size at time n, and ξ^j_{n+1} the number of offspring of the jth individual. For each $n = 1, 2, \dots$, $\{(\xi^j_n)_{j\ge1}\}$ is the sequence of independent

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identically distributed integer-valued random variables with the probability distribution function $\mathsf{P}(\xi_n^j = \ell) = p_\ell,\ \ell = 0, 1, \dots$. The population size at time $n+1$ is given by

$$X_{n+1} = \sum_{j=1}^{X_n} \xi_{n+1}^j,$$

where $X_0 = K > 0$. The state $\{0\}$ is absorbing, and the branching process $(X_n)_{n\ge 0}$ might be absorbed in $\{0\}$ at the extinction time

$$\tau = \inf\big\{n : X_n = 0\big\}.$$

If $p_0 = 0$, the population does not become extinct. However if $p_0 > 0$, it is well known (e.g. [7], [1]) that the extinction time τ is finite with probability one if and only if the offspring mean $\mathfrak{m} = \sum_{\ell\ge 1} \ell p_\ell$ does not exceed one ($\mathfrak{m} \le 1$). Moreover, for any $\mathfrak{m}$, the distribution function of τ is computed using the offspring probability generating function $\mathfrak{f}(s) = \sum_{\ell\ge 0} p_\ell s^\ell$, $0 \le s \le 1$: for any $N \ge 1$,

$$\mathsf{P}(\tau \le N) = (\mathfrak{f}_N(0))^K, \tag{1.1}$$

where $\mathfrak{f}_n(s)$ is the n-th iterate of $\mathfrak{f}(s)$, i.e. $\mathfrak{f}_n(s) = \mathfrak{f}(\mathfrak{f}_{n-1}(s))$ with $\mathfrak{f}_1(0) = \mathfrak{f}(0) = p_0$.

A natural question is how to find the "path to extinction" given that extinction occurred at time N, $\tau = N$. The conditional distribution of the chain conditioned on extinction: for $n = 1, \dots, N-1$,

$$\pi_{n|N}(i) := \mathsf{P}(X_n = i | \tau = N),\ i = 1, 2, \dots$$

gives the complete description. It can be used to find the conditional median or the traditional optimal estimate $\widehat{X}_n = \sum_{i=1}^\infty i\pi_{n|N}(i)$. Unfortunately such computations are involved, even using the Markov property of (X_n). However, for large values of $X_0 = K$, one path has an overwhelmingly large probability compared to the rest. Consider the normed branching process

$$x_n^K = \frac{X_n}{K}.$$

The limit in probability $\mathsf{P}\text{-}\lim_{K\to\infty} x_n^K = \hat{x}_n$ exists (see [10], [11]) and satisfies $\hat{x}_{n+1} = \mathfrak{m}\hat{x}_n, \quad \hat{x}_0 = 1$. The process $\hat{x}_n$ is always positive, irrespective of the value of $\mathfrak{m}$, so that, the approximation $\hat{x}_n$ is inadequate for study of extinction, the fact is already mentioned in [3]. In the approach we take, $(x_n^K)_{n\le N}$ is approximated on the set $\{\tau \le N\}$ by a deterministic sequence $u^*_\cdot := (u^*_n)_{n\le N}$ with $u^*_0 = 1$, positive u^*_n's and $u^*_N = 0$, such that for small $\delta > 0$ and large K,

$$\mathsf{P}\Big(\sum_{n=1}^N |x_n^K - u_n^*| \le \delta\Big) \approx \mathsf{P}\Big(\tau \le N\Big).$$

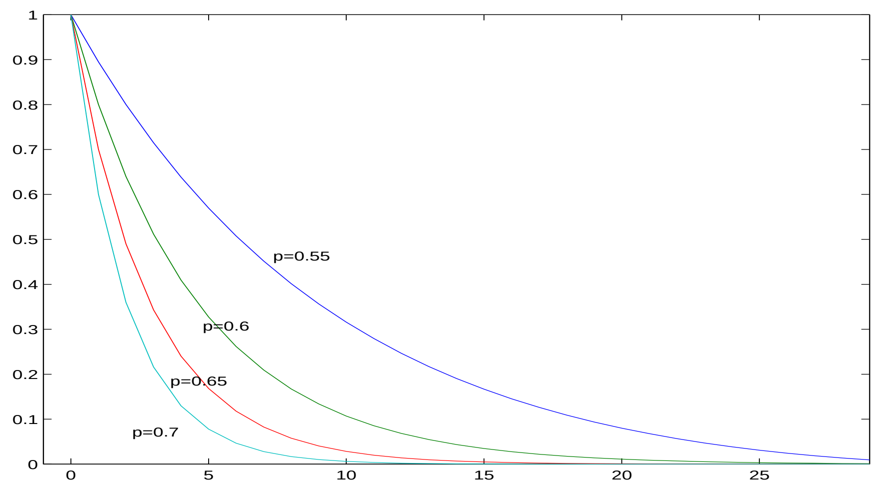
This choice of $u^*_\cdot$ might be warranted by the following argument. Since $\mathfrak{f}_n(0)$ increases in n, for large K, $(\mathfrak{f}_N(0))^K$ is considerably larger than any of $(\mathfrak{f}_n(0))^K$ for $n < N$. Then, by (1.1), $\mathsf{P}(\tau \le N) = \mathsf{P}(\tau = N) + \mathsf{P}(\tau \le N-1) \approx \mathsf{P}(\tau = N)$. Consequently, for any $u_\cdot = (u_n)_{n\le N}$ with $u_0 = 1$ and $u_n \ge 0$,

$$\mathsf{P}\Big(\sum_{n=1}^N |x_n^K - u_n| \le \delta\Big) \lesssim \mathsf{P}\Big(\tau \le N\Big).$$

For large K, extinction for the process x_n^K is a rare event, since the limit process $\hat{x}_n$ is positive. Therefore, as in [14], we approach the problem of extinction using the large deviations theory, obtaining a new result as well as recover an asymptotic version of the well-known result (1.1) by using this theory. According to LDP, Theorem 4.1 and by analogy with the maximal likelihood estimator, the path $(u_n^*)_{n\le N}$ is said to be the most likely path to extinction of the normed population x_n^K.

Clearly, τ is the extinction time for both processes X_n and x_n^K, so that, Ku_n^* (with large K) sets the pattern for the extinction path in the original branching process.

Figure below demonstrates likely paths to extinction for a binary splitting model with different parameters, $p = p_0$, illustrating the general result.



For formulating the main result, we use the log moment generating function, assuming its existence up to some $t_0 > 0$,

$$\mathfrak{g}(t) = \log \sum_{\ell \geq 0} e^{t\ell} p_\ell, \ t \in (-\infty, t_0). \tag{1.2}$$

It is related to the moment generating function by (Lemma 2.1):

$$\log \mathfrak{f}_n(0) \equiv \mathfrak{g}_n(-\infty).$$

Theorem 1.1. *Assume $p_0 > 0$ and (1.2). Then, for any $N \geq 1$,*

(i)

$$(u_n^*)_{n \leq N} = \underset{\substack{u_0=1, u_N=0 \\ u_n>0, n \leq N-1}}{\operatorname{argmax}} \lim_{\delta \to 0} \lim_{K \to \infty} \frac{1}{K} \log \mathsf{P}\Big(\sum_{n=1}^{N} |x_n^K - u_n| \leq \delta \Big)$$

with

$$u_n^* = \prod_{1 \leq i \leq n} \mathfrak{g}'(\mathfrak{g}_{N-i}(-\infty)), \ n \leq N, \tag{1.3}$$

where $\mathfrak{g}_i(t)$ is i-th iterate of $\mathfrak{g}(t)$, $\mathfrak{g}_0(t) = t$.

(ii)

$$\lim_{\delta \to 0} \lim_{K \to \infty} \frac{1}{K} \log \mathsf{P}\Big(\sum_{n=1}^{N} |x_n^K - u_n^*| \leq \delta \Big) = \lim_{K \to \infty} \frac{1}{K} \log \mathsf{P}(\tau \leq n)\big).$$

1.2. **Branching diffusion.** In continuous time, we consider the model of a branching diffusion X_t defined by the Itô equation

$$dX_t = \alpha X_t dt + \sigma \sqrt{X_t} dB_t \tag{1.4}$$

with a positive initial condition $X_0 = K$, where B_t is a Brownian motion, $\sigma^2 > 0$, and $\alpha \in \mathbb{R}$. Stochastic equation (1.4) possesses a strong nonnegative solution.

Since diffusion degenerates, one way to see this is to construct the solution from the following approximating sequence $(X^i_t)_{l\ge i}$:

$$X_t := X^1_t I_{\{t\le\tau_1\}} + \sum_{i\ge 1} X^i_{\tau_i} I_{\{\tau_i<t\le\tau_{i+1}\}},$$

where $dX^i_t = \alpha X^i_t dt + \sigma\sqrt{|X^i_t|\vee i^{-1}}dB_t$, $X^i_0 = K$, and $\tau_i = \inf\{X^i_t \le i^{-1}\}$ the increasing sequence of stopping times $(\tau_i)_{i\ge1}$ relative to the filtration generated by Brownian motion (B_t) (see also Theorem 13.1, [13]). The strong uniqueness of (1.4) follows from Yamada-Watanabe's theorem (see, e.g. Rogers and Williams, p. 265 [21]) since its drift and diffusion parameters are Lipschitz and Hölder (with coefficient 1/2) continuous respectively.

Obviously,

$$\tau = \inf\{t : X_t = 0\} = \lim_{i\to\infty} \tau_i.$$

We analyze the normed process $x^K_t = \frac{X_t}{K}$. Due to (1.4), x^K_t solves the Itô equation

$$dx^K_t = \alpha x^K_t dt + \frac{\sigma}{\sqrt{K}}\sqrt{x^K_t}dB_t, \tag{1.5}$$

with $x^K_0 = 1$. It can be readily shown that $\mathsf{P}\text{-}\lim_{K\to\infty} x^K_t = \hat{x}_t$ exists and solves $\dfrac{d\hat{x}_t}{dt} = \alpha\hat{x}_t,\ \hat{x}_0 = 1$. However, $\hat{x}_t$ is always positive and is far from to be estimated path to extinction. As in the discrete time, in order to evaluate path to extinction for $(x^K_t)_{t\le T}$ for fixed $T > 0$, we approximate $(x^K_t)_{t\le T}$ on the set $\{\tau \le T\}$ by a deterministic function $(u^*_t)_{t\le T}$ with $u^*_0 = 1, u^*_T = 0$ and $u^*_t > 0$, such that for a small $\delta > 0$ and large K,

$$\mathsf{P}\Big(\sup_{t\le T}|x^K_t - u^*_t| \le \delta\Big) \ge \mathsf{P}\Big(\sup_{t\le T}|x^K_t - u_t| \le \delta\Big)$$

for any $(u_t)_{t\le T}$ from the set $\{u_0 = 1, (u_t > 0)_{t<T}, u_T = 0\}$.

Unfortunately, the helpful formula of type (1.1) is not known to us in this case. Here we obtain its asymptotic version as $K \to \infty$, see (ii) below.

Theorem 1.2. *For any $T > 0$,*

(i)

$$(u^*_t)_{t\le T} = \operatorname*{argmax}_{\substack{u_0=1,u_T=0\\ u_t>0,t<T}} \lim_{\delta\to0}\lim_{K\to\infty}\frac{1}{K}\log\mathsf{P}\Big(\sup_{t\le T}|x^K_t - u_t| \le \delta\Big)$$

is given by

$$u^*_t = \begin{cases} e^{-\alpha t}\Big(\frac{1-e^{-\alpha(T-t)}}{1-e^{-\alpha T}}\Big)^2, & \alpha\ne0\\ \big(1-\frac{t}{T}\big)^2, & \alpha = 0.\end{cases} \tag{1.6}$$

(ii)

$$\begin{aligned}\lim_{K\to\infty}\frac{1}{K}\log\mathsf{P}(\tau\le T) &= \lim_{\delta\to0}\lim_{K\to\infty}\frac{1}{K}\log\mathsf{P}\Big(\sup_{t\le T}|x^K_t - u^*_t| \le \delta\Big)\\ &= -\frac{1}{2\sigma^2}\frac{\alpha}{1-e^{-\alpha T}}.\end{aligned}$$

Corollary 1. (1) *$u^*_\cdot$ has the remarkable property: it is the same for subcritical and supercritical case: $u^*_t(\alpha) \equiv u^*_t(-\alpha)$.*

(2) *For large K, the probability of extinction in $[0, T]$ is given by*

$$\mathsf{P}(\tau\le T) \approx \exp\Big(-\frac{K}{2\sigma^2}\frac{\alpha}{1-e^{-\alpha T}}\Big).$$

In particular, for $\alpha = 0$, $\mathsf{P}(\tau \le T) \approx e^{-\frac{K}{2\sigma^2 T}}$.

2. Proof of Theorem 1.1

We begin with

Lemma 2.1. *For any* $n\ge 1$, $\mathfrak{g}_n(-\infty)=\log\mathfrak{f}_n(0)$.

Proof. The result follows by induction from the identity $\mathfrak{g}_n(\log t)\equiv\log\mathfrak{f}_n(t)$ for $t\in(0,t_0)$. Write

$$\mathfrak{g}(\log(t))=\log\sum_{\ell\ge 0}e^{\ell\log(t)}p_\ell=\log\sum_{\ell\ge 0}e^{\log(t^\ell)}p_\ell=\log\sum_{\ell\ge 0}t^\ell p_\ell=\log\mathfrak{f}(t).$$

If $\mathfrak{g}_{n-1}(\log t)\equiv\log\mathfrak{f}_{n-1}(t)$, then

$$\mathfrak{g}_n(\log t)=\mathfrak{g}(\mathfrak{g}_{n-1}(t)=\mathfrak{g}(\log(\mathfrak{f}_{n-1}(t))=\log\mathfrak{f}(\mathfrak{f}_{n-1}(t))=\log(\mathfrak{f}_n(t)).$$

$\square$

The proof of Theorem 1.1 is done in a number of steps.

(1) Recall that $\mathfrak{g}(t)$ is convex function with $\mathfrak{g}(0)=0$, $\mathfrak{g}(-\infty)=\log(p_0)$ and $\mathfrak{g}'(t)>0,\ t>-\infty$ while $\mathfrak{g}'(-\infty)=\lim_{t\to\infty}\mathfrak{g}'(t)=0$.

(2) By the local LDP (see, Theorem 4.1), for $u_0=1,u_N=0$ and other positive u_n's, it holds

$$\lim_{\delta\to 0}\lim_{K\to\infty}\frac{1}{K}\log\mathsf{P}\Big(\sum_{n\le N}|x^K_n-u_n|\le\delta\Big)=-\sum_{n\le N}I(u_n,u_{n-1}).$$

(3) In order to find $(u^*_n)_{n\le N}$ such that for $u_0=1,u_n>0,u_N=0$

$$\sum_{i\le n}I(u_i,u_{i-1})\ge\sum_{i\le n}I(u^*_i,u^*_{i-1}),\tag{2.1}$$

we apply the Dynamic Programming.

Since $u_N=0$,

$$I(u_N,u_{N-1})=\sup_{t\in(-\infty,t_0)}(-u_{N-1}\mathfrak{g}(t))=-u_{N-1}\mathfrak{g}(-\infty)=:B_n(u_{N-1})\tag{2.2}$$

is the boundary condition for the Bellman equation

$$B_n(u_{n-1})=\inf_{u>0}\Big[B_{n+1}(u)+I(u,u_{n-1})\Big],\ 1\le n\le N-1.\tag{2.3}$$

For $n=N-1$, we have

$$B_{N-1}(u_{N-2})=\inf_{u>0}\Big[-u\mathfrak{g}(-\infty)+\sup_{t\in(-\infty,t_0)}\{tu-u_{N-2}\mathfrak{g}(t)\}\Big].\tag{2.4}$$

(2.4) provides the inequality,

$$B_{N-1}(u_{N-2})\ge\inf_{u>0}\big[-u\mathfrak{g}(-\infty)+tu-u_{N-2}\mathfrak{g}(t)\big],\ \forall\ t\in(-\infty,t_0)$$

which, with $t=\mathfrak{g}(-\infty)$, is transformed into

$$B_{N-1}(u_{N-2})\ge -u_{N-2}\mathfrak{g}_2(-\infty).\tag{2.5}$$

We show that the above inequality is equality. For $u,u_{N-2}>0$, "$\sup_t$" in (2.4) is attained at the point $t^*=t^*(u,u_{N-2})$, so that, for any $u>0$,

$$B_{N-1}(u_{N-2})\le u\big[t^*(u,u_{N-2})-\mathfrak{g}(-\infty)\big]-u_{N-2}\mathfrak{g}(t^*(u,u_{N-2})).$$

We choose $u=u^*_{N-1}$ such that $t^*(u^*_{N-1},u_{N-2})=\mathfrak{g}(-\infty)$. This is possible since

$$\mathfrak{g}(-\infty)=\log p_0,\quad t^*(0,u_{N-2})=-\infty,$$
$$\mathfrak{g}'(-\infty)=0,\quad t^*(\mathfrak{m},u_{n-2})=0,\quad \mathfrak{g}'(0)=\mathfrak{m},$$

so that, the existence of u^*_{N-1} follows from continuity, in u, of $t^*(u,u_{N-2})$.

The choice of u^*_{N-1} gives the inequality

$$B_{N-1}(u_{N-2}) \le -u_{N-2}\mathfrak{g}(t^*(u^*_{N-1}, u_{N-2})) = \mathfrak{g}\big(\mathfrak{g}(-\infty)\big) = \mathfrak{g}_2(-\infty).$$

Consequently, the opposite inequality for (2.5) holds true and, therefore,

$$B_{N-1}(u_{N-2}) = -u_{N-2}\mathfrak{g}_2(-\infty).$$

It is obvious too that for any $u_{N-2} > 0$,

$$u^*_{N-1} = u_{N-2}\mathfrak{g}'(t^*(u^*_{N-1}, u_{N-2})) = u_{N-2}\mathfrak{g}'(\mathfrak{g}(-\infty)).$$

Further, by induction, we find the following pairs:

$$\begin{aligned}
u^*_{N-1} &= \mathfrak{g}'\big(\mathfrak{g}(-\infty)\big)u^*_{N-2} \\
B_{N-1}(u^*_{N-2}) &= -\mathfrak{g}_2(-\infty)u^*_{N-2} \\
&\dots \\
u^*_{N-2} &= \mathfrak{g}'\big(\mathfrak{g}_2(-\infty)\big)u^*_{N-3} \\
B_{N-2}(u^*_{N-3}) &= -\mathfrak{g}_3(-\infty)u^*_{N-3} \\
&\vdots \\
u^*_1 &= \mathfrak{g}'\big(\mathfrak{g}_{n-1}(-\infty)\big)u_0 \\
B_1(u_0) &= -\mathfrak{g}_n(-\infty)u_0 \ (u_0 = 1).
\end{aligned}$$

With chosen $(u^*_n)_{1\le n\le N-1}$, the Bellman equation (2.3) is transformed into the backward recurrent equation

$$B_n(u^*_{n-1}) = B_{n+1}(u^*_n) + I(u^*_n, u^*_{n-1}), \ 1 \le n \le N-1$$

with boundary condition $-u^*_{N-1}\mathfrak{g}(-\infty)$ (see, (2.2)).

Thus, $B_1(1) = \sum_{1\le n\le N} I(u^*_n, u^*_{n-1})$.

On the other hand, the Bellman equation also yields

$$B_1(1) \ge \sum_{1\le n\le N-1} I(u_n, u_{n-1}) + B_N(u_{N-1}) = \sum_{1\le n\le N} I(u_n, u_{n-1})$$

what proves (2.1).

(4) We recall that $\sum_{n\le N} I(u^*_n, u^*_{n-1}) = -\mathfrak{g}_n(-\infty)$, that is, by Lemma 2.1 and (1.1),

$$\sum_{1\le n\le N} I(u^*_n, u^*_{n-1}) = -\log \mathfrak{f}_N(0) = -\frac{1}{K}\log \mathsf{P}(\tau \le N), \ \forall\ K > 0.$$

(5) Thus, **(1)**-**(3)** imply the statement (i); formula (1.3) follows from recurrence $u^*_n = \mathfrak{g}'\big(\mathfrak{g}_2(-\infty)\big)u^*_{n-1}$, $u^*_0 = 1$.

Finally (ii) follows from **(4)**. □

3. **Proof of Theorem 1.2**

We apply the LDP Theorem 5.1. By the local LDP, with $u_0 = 1$, $u_t > 0$ and $u_T = 0$, we have

$$\lim_{\delta\to 0}\lim_{K\to\infty}\frac{1}{K}\log \mathsf{P}\Big(\sup_{t\le T}\big|x^K_t - u_t\big| \le \delta\Big) = -J_T(u),$$

where $J_T(u) = \begin{cases} \frac{1}{2\sigma^2}\int_0^T \frac{(\dot u_t - u_t)^2}{u_t} I_{\{u_t>0\}}dt, & u_0 = 1, du_t = \dot u_t dt \\ \infty, & \text{otherwise.}\end{cases}$

Therefore (i) is reduced to minimization of $J_T(u)$ in a class of absolutely continuous test functions u_t with $u_0 = 1$, $u_t > 0$ and $u_T = 0$.

Set $w_t = \dfrac{\dot{u}_t - u_t}{\sqrt{u_t}}$, $t \in [0,T)$ and notice that the minimization of $J_T(u.)$ is equivalent to the following control problem with the controlled process u_t, solving a differential equation

$$\dot{u}_t = \alpha u_t + \sqrt{u_t} w_t, \quad t \in [0,T)$$

subject to $u_0 = 1$, where w_t the control action belongs to a class of measurable functions with $\int_0^T w_t^2 dt < \infty$ bringing u_t to zero at the time T. The control action w_t^* from this class is optimal if for any w_t,

$$\int_0^T (w_t^*)^2 dt \le \int_0^T w_t^2 dt.$$

If w_t^* exists, then the controlled process u_t^* related to w_t^* minimizes $J_T(u.)$ in the required class of continuous functions $u. = (u_t)_{t\le T}$.

In order to find w_t^*, it is convenient to deal with (recall $u_t \ge 0$) $v_t = \sqrt{u_t}$ since v_t solves the linear differential equation $\dot{v}_t = \frac{\alpha}{2} v_t + w_t, \quad v_0 = 1$. If w_t^* exists, then w_t^* brings v_t to zero at the time T, that is, $0 = v_T = e^{\frac{\alpha}{2}T} + \int_0^T e^{\frac{\alpha}{2}(T-t)} w_t^* dt$ or, equivalently,

$$-1 = \int_0^T e^{-\frac{\alpha}{2}t} w_t^* dt. \tag{3.1}$$

Hence, by the Cauchy-Schwarz inequality $1 \le \int_0^T e^{-t\alpha} dt \int_0^T (w_t^*)^2 dt$, that is, the following lower bound holds: $\int_0^T (w_t^*)^2 dt \ge \dfrac{\alpha}{1-e^{-\alpha T}}$. This lower bound is valid for any w_t providing (3.1) , so that, the condition

$$\int_0^T (w_t^*)^2 dt = \frac{\alpha}{1-e^{-\alpha T}}$$

is valid for $w_t^* = c e^{-t\frac{\alpha}{2}}$ for any constant c, bring $w_t^* = c^* e^{-t\frac{\alpha}{2}}$ with c^* solving

$$-1 = \int_0^T e^{-t\frac{\alpha}{2}} w_t^* dt = c^* \int_0^T e^{-t\alpha} dt.$$

Hence,

$$c^* = \begin{cases} -\frac{\alpha}{1-e^{-T\alpha}}, & \alpha \ne 0 \\ -\frac{1}{T}, & \alpha = 0 \end{cases} \quad \text{and} \quad w_t^* = \begin{cases} -\frac{\alpha e^{-t\frac{\alpha}{2}}}{1-e^{-T\alpha}}, & \alpha \ne 0 \\ -\frac{1}{T}, & \alpha = 0 \end{cases}$$

$$\int_0^T (w_t^*)^2 dt = \begin{cases} \frac{\alpha}{1-e^{-\alpha T}}, & \alpha \ne 0 \\ \frac{1}{T}, & \alpha = 0. \end{cases}$$

Finally, we find that

$$\begin{aligned} v_t^* &= e^{t\frac{\alpha}{2}} - \frac{\alpha}{1-e^{-T\alpha}} \int_0^t e^{(t-s)\frac{\alpha}{2}} e^{-s\frac{\alpha}{2}} ds \\ &= e^{t\frac{\alpha}{2}} \Big[1 - \frac{1-e^{-t\alpha}}{1-e^{-T\alpha}}\Big] = e^{t\frac{\alpha}{2}} \Big(\frac{e^{-t\alpha}-e^{-T\alpha}}{1-e^{-T\alpha}}\Big) \end{aligned}$$

and, since $u_t^* = (v_t^*)^2$, we obtain (1.6) and the proof of (i) is complete.

(ii) By (i),

$$J_T(u^*) = \frac{1}{2\sigma^2} \frac{\alpha}{1-e^{-\alpha T}} \tag{3.2}$$

We show that

$$\lim_{K\to\infty} \frac{1}{K} \log \mathsf{P}\big(\tau \le T\big) = J_T(u^*).$$

To this end, use the fact that $\{\tau \le T\} = \{(\omega, t) : \exists t \le T,\ x^K_t(\omega) = 0\}$. For notational convenience denote $\mathfrak{A} := \{\tau \le T\}$. Set $\mathfrak{A}^{\mathsf{cl}}$ and $\mathfrak{A}^{\mathsf{int}}$ the closure and interior of $\mathfrak{A}$. Then, by the LDP, we have

$$\varlimsup_{K\to\infty} \frac{1}{K} \log \mathsf{P}\big(\mathfrak{A}^{\mathsf{cl}}\big) \le - \inf_{u:\left\{\substack{u_s>0,s<t;\\ u_t=0\\ t\le T}\right.} J_t(u) = - \inf_{t\le T} J_t(u^*)$$

$$\varliminf_{K\to\infty} \frac{1}{K} \log \mathsf{P}\big(\mathfrak{A}^{\mathsf{int}}\big) \ge - \inf_{u:\left\{\substack{u_s>0,s<t;\\ u_t=0\\ t\le T}\right.} J_t(u) = - \inf_{t\le T} J_t(u^*).$$

Since $\varliminf_{K\to\infty} = \varlimsup_{K\to\infty}$ implies the existence of $\lim_{K\to\infty}$, it remains to show that $\inf_{t\le T} J_t(u^*) = J_T(u^*)$.

Notice that (3.2) is valid with T replaced by any $t < T$ with $u^*_\cdot$ replaced by the corresponding $u^{*,t}_\cdot = \{u^{*,t}_0 = 1; u^{*,t}_s > 0, s < t; u^{s,t}_t = 0\}$. In other words, for any t,

$$J_t(u^{*,t}) = \frac{1}{2\sigma^2}\frac{\alpha}{1 - e^{-\alpha t}},$$

and $J_t(u^{*,t})$ increases to $J_T(u^*_\cdot)$ with $t \nearrow T$. □

4. LDP in Discrete Time

Let $m = \inf\{n \le N : u_n = 0\}$ and $m = \infty$ if all $(u_n)_{n\le N}$ are positive.

$$I(y, x) = \sup_{t\in(-\infty,t_0)} [ty - x\mathfrak{g}(t)] \tag{4.1}$$

Theorem 4.1. *Assume* (1.2). *For any $N \ge 1$, the family $\big\{(x^K_n)_{n\le N}\big\}_{K\to\infty}$ obeys the LDP in $\mathbb{R}^N_+$, supplied by the Euclidian metric ϱ_N, with the speed $\frac{1}{K}$ and the rate function*

$$J_N(u_\cdot) = \begin{cases} \sum\limits_{n=1}^{m-1} I(u_n, u_{n-1}) - u_{m-1}\log(p_0), & \substack{u_0=1\\ u_n=0, n>m} \\ \sum\limits_{n=1}^{N} I(u_n, u_{n-1}), & \substack{u_0=1\\ u_n>0, n\le N}. \\ \infty, & \substack{\exists\ n: u_n=0, u_{n+1}>0\\ or\ u_0\ne 1} \end{cases}$$

Remark 1. LDP for branching processes have been considered in the literature, see, for example, [2], [4], [18]. However, they were concerned with the sequence X_n/X_{n-1}, as $n \to \infty$, whereas here we consider the LDP for X_n/X_0 processes indexed by the large initial value.

Remark 2. The nonnegativity of x^K_n provides some difficulty of the LDP verification at the "point of extinction", that is, at the point where the test function becomes zero. For sets of trajectories keeping away from zero, of which $\{\tau \le N\}$ is not, the statement of the theorem is implied by a result in Klebaner and Zeitouni, [9] and other known results that can be adapted to our setting (see, e.g. Kifer, [8], Puhalskii, [19], Klebaner and Liptser, [12], etc.). The proof of theorem is new in the part concerning the lower bound in the local LDP. However, for the sake of completeness and accuracy we give the proof below.

4.1. **Proof of Theorem 4.1.** We follow standard (necessary and sufficient) conditions for proving the LDP by showing the exponential tightness:

$$\lim_{C\to\infty} \varlimsup_{K\to\infty} \frac{1}{K} \log \mathsf{P}\big(\Omega \setminus \mathcal{K}_C\big) = -\infty$$

with compacts $\mathcal{K}_C = \{\max_{1\le n\le N} x_N \le C\}$, $C \nearrow \infty$, and the local LDP:

$$\lim_{\delta\to 0} \lim_{K\to\infty} \frac{1}{K} \log \mathsf{P}\Big(\varrho_N(x^K_\cdot, u_\cdot) \le \delta\Big) = -J_N(u_\cdot).$$

Notice that (1.2) implies the existence of a stochastic exponential, with $t_n \le Kt_0$,

$$\mathscr{E}^K_{(t_1,\dots,t_N)}(x^K_1,\dots,x^K_{N-1}) = \prod_{n=1}^{N} \mathsf{E}\Big(e^{t_n x^K_n}\big|\mathscr{F}_{n-1}\Big),$$

where $(\mathscr{F}_n)_{n\ge 0}$ is the filtration, with $\mathscr{F}_0 = \{\varnothing, \Omega\}$, generated by $(x^K_n)_{n\ge 1}$.

Set

$$\mathfrak{z}_n = e^{\sum_{i\le n} t_\ell x^K_i - \log \mathscr{E}^K_{(t_1,\dots,t_n)}(x^K_1,\dots,x^K_{n-1})}. \tag{4.2}$$

The random process $(\mathfrak{z}_n, \mathscr{F}_n)_{n\le N}$ is the (positive) martingale,

$$\mathsf{E}\mathfrak{z}_N = 1. \tag{4.3}$$

4.1.1. **Exponential tightness.** Since $\max_{1\le n\le N} x^K_i \le \sum_{1\le n\le N} x^K_n$, it is enough to show

$$\lim_{C\to\infty} \varlimsup_{K\to\infty} \frac{1}{K} \log \mathsf{P}\Big(\sum_{1\le i\le N} x^K_i \ge C\Big) = -\infty.$$

Set $t^* = \operatorname{argmax}_{t\in(-\infty,t_0)}[t - \mathfrak{g}(t)]$. Since $\mathfrak{g}(0) = 0$, we have that $t^* \in (0, t_0)$ and $\mathfrak{g}(t^*) < t^*$. We choose $t_n \equiv t^*K (< Kt_0)$, and introduce $\mathfrak{A} = \big\{\sum_{1\le i\le n} x^K_i \ge C\big\}$. With chosen t_n, we have $\mathsf{E}\mathfrak{z}_N = 1$ and, therefore, $\mathsf{E}I_{\mathfrak{A}}\mathfrak{z}_N \le 1$. Taking into account this inequality and (4.2), write

$$\begin{aligned}
1 &\ge \mathsf{E}I_{\mathfrak{A}} e^{\sum_{\{1\le n\le N\}} t^* x^K_n - \log \mathscr{E}^K_{(t^*,\dots,t^*)}(x^K_1,\dots,x^K_{N-1})} \\
&= \mathsf{E}I_{\mathfrak{A}} e^{Kt^* \sum_{\{1\le n\le N\}} x^K_n - K\mathfrak{g}(t^*)\sum_{\{1\le n\le N\}} x^K_{n-1}} \\
&\ge \mathsf{E}I_{\mathfrak{A}} e^{K\sum_{\{1\le n\le N\}}[t^* - \mathfrak{g}(t^*)]x^K_n - K|\mathfrak{g}(t^*)|} \\
&\ge \mathsf{E}I_{\mathfrak{A}} e^{KC[t^* - \mathfrak{g}(t^*)]} = e^{KC[t^* - \mathfrak{g}(t^*)] - K|\mathfrak{g}(t^*)|}\mathsf{P}\big(\mathfrak{A}\big).
\end{aligned}$$

Therefore, $\frac{1}{K}\log \mathsf{P}\big(\mathfrak{A}\big) \le -\underbrace{[t^* - \mathfrak{g}(t^*)]}_{0} C + |\mathfrak{g}(t^*)| \xrightarrow[C\to\infty]{} -\infty.$ □

4.1.2. **Local LDP. Upper bound.** We may restrict ourselves by the test function $u_\cdot = \{\underbrace{u_1,\dots,u_{N-1}}_{>0}, \underbrace{u_N}_{=0}\}$ and show that

$$\varlimsup_{\delta\to 0} \varlimsup_{K\to\infty} \frac{1}{K} \log \mathsf{P}\Big(\varrho_N(x^K_\cdot, u_\cdot) \le \delta\Big) \le -J_N(u_\cdot). \tag{4.4}$$

For the test function with all positive u_n's and $u_0 = 1$ the proof of (4.4) is similar. For test function with $u_n = 0, u_{n+1} > 0$ or $u_0 \ne 1$, (4.4) is obvious. For others test functions the verification of (4.4) is reduced to the above-mentioned ones.

Let now $\mathfrak{A} = \big\{\rho_N(x^K_\cdot, u_\cdot) \le \delta\big\}$. By (4.3), we have

$$1 \ge \mathsf{E}I_{\mathfrak{A}}\mathfrak{z}_N = \mathsf{E}I_{\mathfrak{A}} e^{\sum_{\{1\le n\le N\}}[t_n x^K_i - Kx^K_{n-1}\mathfrak{g}(t_n/K)]}. \tag{4.5}$$

Set $t^*_n = \operatorname{argmax}_{t\in(-\infty,t_0)}[tu_n - u_{n-1}\mathfrak{g}(t)]$, $n \le N-1$, and $t^*_N = -l$ $(l > 0)$, and take $t_n = Kt^*_n$, then we derive from (4.5)

$$\begin{aligned}
1 &\ge \mathsf{E}I_{\mathfrak{A}} e^{K\sum_{\{1\le n\le N\}}[t^*_n u_n - u_{n-1}\mathfrak{g}(t^*_n)] - K\sum_{1\le n\le N-1}(t^*_n + |\mathfrak{g}(t^*_n)|)\delta} \\
&= \mathsf{E}I_{\mathfrak{A}} e^{K[\sum_{\{1\le n\le N-1\}} I(u_n,u_{n-1}) - u_{N-1}\mathfrak{g}(-l)] - K\sum_{1\le n\le N-1}(|t^*_n| + |\mathfrak{g}(t^*_n)|)\delta} \\
&= \mathsf{E}I_{\mathfrak{A}} e^{K[J_{N-1}(u_\cdot) - u_{N-1}\mathfrak{g}(-l)] - K\sum_{1\le n\le N-1}(|t^*_n| + |\mathfrak{g}(t^*_n)|)\delta}.
\end{aligned}$$

Hence, taking into account that $\lim_{l\to\infty}\mathfrak{g}(-l) = \log(p_0)$, we obtain

$$\begin{aligned}
\frac{1}{K}\log \mathsf{P}\big(\mathfrak{A}\big) &\le -[J_{N-1}(u_\cdot) + u_{N-1}\mathfrak{g}(-l)] + \sum_{1\le i\le N-1}(|t^*_i| + |\mathfrak{g}(t^*_i)|)\delta \\
&\xrightarrow[\delta\to 0]{} -[J_{N-1}(u_\cdot) + u_{N-1}\mathfrak{g}(-l)] \xrightarrow[l\to\infty]{} -J_N(u_\cdot).
\end{aligned}$$

4.2. **Local LDP. Lower bound.** Obviously for $u.$ with $J_N(u.)=\infty$, it is nothing to verify. Further as in the upper bound verification, we may restrict ourselves by the test function $u.=\{\underbrace{u_1,\dots,u_{N-1}}_{>0},\underbrace{u_N}_{=0}\}$ with $\mathsf{P}(\xi_1^1=0)=p_0>0$ and show that

$$\varliminf_{\delta\to 0}\varliminf_{K\to\infty}\frac{1}{K}\log\mathsf{P}\Big(\varrho_N(x^K_\cdot,u.)\le\delta\Big)\ge -J_N(u.).$$

Write

$$\begin{aligned}
\{\varrho_N(x^k_\cdot,u.)\le\delta\} &= \big\{\varrho_{N-1}(x^k_\cdot,u.)+x^K_N\le\delta\big\}\\
&\supseteq \big\{\varrho_{N-1}(x^k_\cdot,u.)\le 0.5\delta,\quad x^K_N\le 0.5\delta\big\}\\
&\supseteq \big\{\varrho_{N-1}(x^k_\cdot,u.)\le 0.5\delta,\quad x^K_N=0\big\}\\
&\supseteq \Big\{\varrho_{N-1}(x^k_\cdot,u.)\le 0.5\delta,\quad \frac{1}{K}\sum_{j=1}^{Kx^K_{N-1}}\xi^j_N=0\Big\}\\
&\supseteq \Big\{\varrho_{N-1}(x^k_\cdot,u.)\le 0.5\delta,\quad \frac{1}{K}\sum_{j=1}^{K(u_{N-1}+\delta)}\xi^j_N=0\Big\}\\
&= \Big\{\varrho_{N-1}(x^k_\cdot,u.)\le 0.5\delta,\quad \sum_{j=1}^{K(u_{N-1}+\delta)}\xi^j_N=0\Big\}.
\end{aligned}$$

The sets $\mathfrak{A}_1=\big\{\varrho_{N-1}(x^k_\cdot,u.)\le 0.5\delta\big\}$ and $\mathfrak{A}_2=\big\{\sum_{j=1}^{K(u_{N-1}+\delta)}\xi^j_N=0\big\}$ are independent, so that,

$$\mathsf{P}\big(\varrho_N(x^k_\cdot,u.)\le\delta\big)\ge\mathsf{P}\big(\varrho_{N-1}(x^k_\cdot,u.)\le 0.5\delta\big)\mathsf{P}^{K(u_{N-1}+\delta)}\big(\xi^1_1=0\big).$$

Consequently,

$$\begin{aligned}
&\varliminf_{\delta\to 0}\varliminf_{K\to\infty}\frac{1}{K}\log\mathsf{P}\big(\varrho_N(x^k_\cdot,u.)\le\delta\big)\\
&\ge\varliminf_{\delta\to 0}\varliminf_{K\to\infty}\frac{1}{K}\log\mathsf{P}\big(\varrho_{N-1}(x^k_\cdot,u.)\le 0.5\delta\big)+u_{N-1}\log\mathsf{P}\big(\xi^1_1=0).
\end{aligned}$$

If

$$\varliminf_{\delta\to 0}\varliminf_{K\to\infty}\frac{1}{K}\log\mathsf{P}\Big(\varrho_{N-1}(x^K_\cdot,u.)\le\delta\Big)\ge -J_{N-1}(u.),\tag{4.6}$$

provided that $u_n>0$, $n\le N-1$, the the required lower bound holds true.

Thus, it is left to verify the validity of (4.6).

Set $\Lambda_{N-1}(x^K_\cdot)=\mathfrak{z}_{N-1}$, that is,

$$\Lambda_{N-1}(x^K_\cdot)=e^{\sum_{n=1}^{N-1}K\left[t^*_nx^K_n-x^K_{n-1}\mathfrak{g}(t^*_n)\right]},\quad \mathsf{E}\Lambda_{N-1}(x^K_\cdot)=1.$$

We introduce the probability measure Q^K_{N-1} with $d\mathsf{Q}^K_{N-1}=\Lambda_{N-1}(x^K_\cdot)d\mathsf{P}$. Since $\Lambda_{n-1}(x^K_\cdot)>0$, P-a.s., we also have $d\mathsf{P}=\Lambda^{-1}_{n-1}(x^K_\cdot)d\mathsf{Q}^K_{n-1}$.

In particular, for $\mathfrak{A}=\big\{\varrho_{N-1}(x^K_\cdot,u.)\le\delta\big\}$,

$$\mathsf{P}(\mathfrak{A})=\int_{\mathfrak{A}}\Lambda^{-1}_{N-1}(x^K_\cdot)d\mathsf{Q}^K_{N-1}.$$

So, the following lower bound, on the set $\mathfrak{A}$, is valid:

$$\begin{aligned}
\Lambda^{-1}_{N-1}(x^K_\cdot)&\ge e^{-KJ_{N-1}(u.)-K\delta\max_{n\le N-1}(|t^*_n|+|\mathfrak{g}(t^*_n)|)}\\
&\ge e^{-KJ_{N-1}(u.)-K\delta\max_{n\le N-1}(|t^*_n|+|\mathfrak{g}(t^*_n)|)}
\end{aligned}$$

or, equivalently,

$$\frac{1}{K}\log\mathsf{P}(\mathfrak{A})\ge -J_{N-1}(u.)-\delta\max_{n\le N-1}\big(|t^*_n|+|\mathfrak{g}(t^*_n)|\big)+\frac{1}{K}\log\mathsf{Q}^K_{N-1}(\mathfrak{A}).$$

The latter inequality implies (4.6) if

$$\lim_{K\to\infty}\frac{1}{K}\log \mathsf{Q}^K_{N-1}(\mathfrak{A})=0. \tag{4.7}$$

A simply condition, providing (4.7), is $\lim_{K\to\infty}\mathsf{Q}^K_{N-1}(\mathfrak{A})=1$ or, equivalently,

$$\lim_{K\to\infty}\mathsf{Q}^K_{N-1}\big(\varrho_{N-1}(x^K_\cdot,u_\cdot)>\delta\big)=0. \tag{4.8}$$

We verify (4.8) by showing[1]

$$\mathsf{E}^K_{N-1}\varrho^2_{N-1}(x^K\cdot,u_\cdot)=\frac{u_{N-1}}{K}\sum_{n=1}^{N-1}\frac{u_{n-1}}{u^2_n}\mathfrak{g}''(t^*_n). \tag{4.9}$$

Notice that the positiveness of $(u_n)_{n\le N-1}$ provides a boundedness for the right hand side of (4.9) and, in turn by Chebyshev's inequality, the validity of (4.8).

In order to establish (4.9), we apply the identity relative to t^*_n:

$$1=\mathsf{E}\Big(\frac{\Lambda_n(x^K_\cdot)}{\Lambda_{n-1}(x^K_\cdot)}\Big|\mathscr{F}_{n-1}\Big)=\mathsf{E}e^{K\left[t^*_nx^K_n-x^K_{n-1}\mathfrak{g}(t^*_n)\right]}. \tag{4.10}$$

Differentiating twice (4.10) in t^*_n, we find that

$$\begin{aligned}
0&=\mathsf{E}\Big([x^K_n-x^K_{n-1}\mathfrak{g}'(t^*_n)]\frac{\Lambda_i(x^K_\cdot)}{\Lambda_{n-1}(x^K_\cdot)}\Big|\mathscr{F}_{n-1}\Big)\\
0&=\mathsf{E}\Big(\big\{K[x^K_n-x^K_{n-1}\mathfrak{g}'(t^*_n)]^2-x^K_{n-1}\mathfrak{g}''(t^*_n)\big\}\frac{\Lambda_n(x^K_\cdot)}{\Lambda_{n-1}(x^K_\cdot)}\Big|\mathscr{F}_{n-1}\Big).
\end{aligned} \tag{4.11}$$

By the Bayes formula, e.g. [17], [13]: for any integrable random variable α,

$$\mathsf{E}^K_{N-1}(\alpha|\mathscr{F}_{n-1})=\mathsf{E}\Big(\alpha\frac{\Lambda_n(x^K_\cdot)}{\Lambda_{n-1}(x^K_\cdot)}\Big|\mathscr{F}_{n-1}\Big).$$

By taking $\alpha=x^K_n$ and $\alpha=[x^K_n-x^K_{n-1}\mathfrak{g}'(t^*_i)]^2$, we derive the help of (4.11) that

$$\mathsf{E}^K_{N-1}(x^K_n|\mathscr{F}_{n-1})=x^K_{n-1}\mathfrak{g}'(t^*_n) \tag{4.12}$$

$$\mathsf{E}^K_{N-1}\big([x^K_n-x^K_{n-1}\mathfrak{g}'(t^*_n)]^2|\mathscr{F}_{n-1}\big)=x^K_{n-1}\frac{\mathfrak{g}''(t^*_n)}{K} \tag{4.13}$$

Since u_n,u_{n-1} are positive, we have $\mathfrak{g}'(t^*_n)=\dfrac{u_n}{u_{n-1}}$. Hence and by (4.12), we obtain that $\mathsf{E}^K_{N-1}x^K_n=\frac{u_n}{u_{i-1}}\mathsf{E}^K_{N-1}x^K_{n-1}$. Consequently, iterating the above recursion and taking into account $u_0=1$, we find that

$$\mathsf{E}^K_{N-1}x^K_n=u_n.$$

Further, with the help of (4.13) we find a recursion

$$\mathsf{E}^K_{N-1}(x^K_n)^2=\Big(\frac{u_n}{u_{n-1}}\Big)^2\mathsf{E}^K_{N-1}(x^K_{n-1})^2+u_{n-1}\frac{\mathfrak{g}''(t^*_n)}{K}.$$

By using $\mathsf{E}^K_{N-1}(x^K_n-u_n)^2=\mathsf{E}^K_{N-1}(x^K_n)^2-u^2_n$ and and $u^2_n=\big(\frac{u_n}{u_{n-1}}\big)^2u^2_{n-1}$, we establish a recursion for $\triangle_n=\mathsf{E}^K_{N-1}(x^K_n-u_n)^2$:

$$\triangle_n=\Big(\frac{u_n}{u_{n-1}}\Big)^2\triangle_{n-1}+u_{n-1}\frac{\mathfrak{g}''(t^*_n)}{K}$$

supplied by $\triangle_0=0$. Then, $\frac{\triangle_0}{u^2_0}=0$ and

$$\frac{\triangle_n}{u^2_n}=\frac{\triangle_{n-1}}{u^2_{n-1}}+\frac{u_{n-1}}{u^2_n}\frac{\mathfrak{g}''(t^*_n)}{K},\quad \triangle_{N-1}=\frac{u_{N-1}}{K}\sum_{n=1}^{N-1}\frac{u_{n-1}}{u^2_n}\mathfrak{g}''(t^*_n).$$

[1] E^K_{N-1} denotes the expectation with respect to Q^K_{N-1}

It is left to recall that $\triangle_{N-1}=\mathsf{E}^K_{N-1}\varrho^2_{N-1}(x^K\cdot,u.)$. □

5. LDP in Continuous Time

We introduce the filtration $(\mathscr{F}^B_t)_{t\ge 0}$ generated by Brownian motion B_t, with the general conditions. All random processes considered in this section are adapted to this filtration.

Theorem 5.1. *For any $T>0$, the family $\big\{(x^K_t)_{t\le T}\big\}_{K\to\infty}$ obeys the LDP in $\mathbb{C}_{[0,T]}(\mathbb{R}_+)$, supplied by the uniform metric ϱ_T, with the speed $\frac{1}{K}$ and the rate function*

$$J_T(u.)=\begin{cases}\frac{1}{2\sigma^2}\int_0^T\frac{(\dot u_t-\alpha u_t)^2}{u_t}I_{\{u_t>0\}}dt, & u_0=1, du_t=\dot u_t dt\\ \infty, & u_0\ne 1 \ or \ du_t\ne \dot u_t dt\\ \infty, & \substack{u_0=1,\quad du_t\ne\dot u_t dt\\ u_{t''}>u_{t'}=0\ \exists\ t'<t''<T.}\end{cases}$$

Remark 3. Since $u_t\ge 0$, Freidlin-Wentzell's rate function, [6], $\frac{1}{2\sigma^2}\int_0^T\frac{(\dot u_t-\alpha u_t)^2}{u_t}dt$ is not compatible with $u_t=0$. Our branching diffusion model is a "very particular" case of a model studied by Puhalskii's in [20]. To apply the LDP analysis from [20] to the family $\big\{(x^K_t)_{t\le T}\big\}_{K\to\infty}$, one has to "disentangle" many details of the proof to make it compatible with our case. Finally, in Donati-Martin et all, [5], the LDP analysis deals with a rate function of the following type $\int_0^T\frac{(\dot u_t-\rho)^2}{u_t}dt$ for $u_t\ge 0$ related to a family of diffusion type processes without *extinction*. A reader interested in details of the direct proof can find them below.

Proof. It suffice to verify:

(i) C-exponential tightness (see [15]),

$$\lim_{C\to\infty}\varlimsup_{K\to\infty}\frac{1}{K}\log\mathsf{P}\Big(\sup_{t\le T}x^K_t\ge C\Big)=-\infty, \tag{5.1}$$

$$\lim_{\Delta\to 0}\varlimsup_{K\to\infty}\sup_{\gamma\le T}\frac{1}{K}\log\mathsf{P}\Big(\sup_{t\le\Delta}|x^K_{\gamma+t}-x^K_\gamma|\ge\eta\Big)=-\infty,\ \forall\ \eta>0, \tag{5.2}$$

where γ is stopping time relative to $(\mathscr{F}^B_t)_{t\ge 0}$,

(ii) the Local LDP,

$$\lim_{\delta\to 0}\lim_{K\to\infty}\frac{1}{K}\log\mathsf{P}\Big(\sup_{t\le T}\big|x^K_t-u_t\big|\le\delta\Big)=-J_T(u.).$$

(i)-Verification. The proof of (5.1) follows [15].

The use of Stroock's Lemma 4.12 (Ch. 4), [22], enables to reduce the proof of (5.1) to

$$\varlimsup_{K\to\infty}\frac{1}{K}\log\mathsf{E}\Big(\sup_{t\le T}x^K_t\Big)^{2K}<\infty \tag{5.3}$$

since, by Chebyshev's inequality,

$$\begin{aligned}\frac{1}{K}\log\mathsf{P}\Big(\sup_{t\le T}x^K_t\ge C\Big)&\le\frac{1}{K}\log\frac{1}{C^{2K}}\mathsf{E}\Big(\sup_{t\le T}x^K_t\Big)^{2K}\\&=-2C+\frac{1}{K}\log\mathsf{E}\Big(\sup_{t\le T}x^K_t\Big)^{2K}\xrightarrow[\substack{K\to\infty\\C\to\infty}]{}-\infty.\end{aligned}$$

We introduce a continuous martingale and its variation process

$$M^K_t=\int_0^t\frac{\sigma}{\sqrt{K}}\sqrt{x^K_s}dB_s\quad\text{and}\quad\langle M^K\rangle_t=\frac{\sigma^2}{K}\int_0^t x^K_s ds \tag{5.4}$$

respectively. By (1.5), $x^K_t\le 1+\int_0^t|\alpha|x^K_s\,ds+\sup_{t'\le t}|M^K_{t'}|$, so that, due to Bellman-Gronwall's inequality,

$$\sup_{t'\le t}x^K_{t'}\le e^{|\alpha|T}\Big(1+\sup_{t'\le t}|M^K_{t'}|\Big),\ t\le T, \tag{5.5}$$

that is, (5.3) can be reduced to

$$\varlimsup_{K\to\infty}\frac{1}{K}\log\mathsf{E}\Big(\sup_{t\le T}|M^K_t|\Big)^{2K}<\infty. \tag{5.6}$$

We continue the proof by verifying of (5.6).

Denote by V^K_t the compensator of $|M^K_t|^{2K}$ in a sense that V^K_t is a continuous increasing process compensating $|M^K_t|^{2K}$ up to a local martingale. Then, by a version of Doob's inequality (see, Theorem 19.2 in [16])

$$\mathsf{E}\sup_{t'\le t}|M^K_{t'}|^{2K}\le\Big(\frac{2K}{2K-1}\Big)^{2K}\mathsf{E}V^K_t. \tag{5.7}$$

We determine V^K_t with the help of Itô's formula applied to $(M^K_t)^{2K}$:

$$(M^K_t)^{2K}=2K\int_0^t(M^K_s)^{2K-1}dM^K_s+\underbrace{K(2K-1)\int_0^t(M^K_s)^{2(K-1)}d\langle M^K\rangle_s}_{=V^K_t}$$

and, due to (5.4), $V^K_t=\sigma^2(2K-1)\int_0^t(M^K_s)^{2(K-1)}x^K_s\,ds$. Consequently, by (5.5),

$$\begin{aligned}V^K_t&\le\sigma^2(2K-1)e^{|\alpha|T}\int_0^t\Big(1+\sup_{s'\le s}|M^K_{s'}|^{2K-1}\Big)ds\\&\le 2\sigma^2Ke^{|\alpha|T}\int_0^t\Big(2+\sup_{s'\le s}|M^K_{s'}|^{2K}\Big)ds=:\widehat{V}^K_t.\end{aligned}$$

The use of $\widehat{V}^K_t$ instead of V^K_t in (5.7) brings

$$\mathsf{E}\sup_{t'\le t}|M^K_{t'}|^{2K}\le\Big(\frac{2K}{2K-1}\Big)^{2K}2\sigma^2Ke^{|\alpha|T}\int_0^t\Big(2+\mathsf{E}\sup_{s'\le s}|M^K_{s'}|^{2K}\Big)ds.$$

Without loss of generality one may assume that $\mathsf{E}\sup_{t'\le T}|M^K_{t'}|^{2K}<\infty$ (otherwise a localization arguments can be applied). Then, by Bellman-Gronwall's inequality, we find that

$$\mathsf{E}\sup_{t'\le T}|M^K_{t'}|^{2K}\le 4\sigma^2KTe^{|\alpha|T}\Big(\frac{2K}{2K-1}\Big)^{2K}\exp\Big\{2\sigma^2KT\Big(\frac{2K}{2K-1}\Big)^{2K}\Big\}$$

and, in turn,

$$\begin{aligned}\frac{1}{K}\log\mathsf{E}\sup_{t'\le T}|M^K_{t'}|^{2K}&\le\frac{1}{K}\log\big(4\sigma^2Te^{|\alpha|T}\big)+\frac{1}{K}\log K+2\log\Big(\frac{2K}{2K-1}\Big)\\&+2\sigma^2T\Big(\frac{2K}{2K-1}\Big)^{2K}\xrightarrow[K\to\infty]{}2\sigma^2T\mathsf{e},\ \text{(i.e. (5.6) holds true).}\end{aligned}$$

By (5.1), the proof of (5.2) is reduced to the verification of two conditions: for any $\eta,C>0$,

$$\lim_{\Delta\to0}\varlimsup_{K\to\infty}\sup_{\gamma\le T}\frac{1}{K}\log\mathsf{P}\Big(\sup_{t\le\Delta}\int_\gamma^{\gamma+t}x^K_s\,ds\ge\eta,\ \sup_{s\le T}x^K_s\le C\Big)=-\infty$$

$$\lim_{\Delta\to0}\varlimsup_{K\to\infty}\sup_{\gamma\le T}\frac{1}{K}\log\mathsf{P}\Big(\frac{\sigma}{\sqrt{K}}\sup_{t\le\Delta}\Big|\int_\gamma^{\gamma+t}\sqrt{x^K_s}\,dB_s\Big|\ge\eta,\ \sup_{s\le T}x^K_s\le C\Big)=-\infty.$$

The first is obvious while the second is equivalent to

$$\lim_{\delta\to 0}\varlimsup_{K\to\infty}\sup_{\gamma\le T}\frac{1}{K}\log \mathsf{P}\Big(\sup_{t\le\delta} I_{T,C}\big|M^K_{\gamma+t}-M^K_{\gamma}\big|\ge\eta\Big)=-\infty, \tag{5.8}$$

where $I_{t,C}=I_{\{\sup_{s\le t}x^K_s\le C\}}$, $t\le T$.

Set $N^K_t=M^K_{\gamma+t}-M^K_\gamma$ and notice that $(N^K_t,\mathscr{F}^B_{\gamma+t})_{t\ge 0}$ is the local martingale with the variation process $\langle N^K\rangle_t=\frac{\sigma^2}{K}\int_\gamma^{\gamma+t}x^K_s ds$.

Further, the use of $I_{T,C}N^K_t=I_{T,C}\int_0^t I_{s,C}dN^K_s$ simplifies (5.8) up to

$$\lim_{\Delta\to 0}\varlimsup_{K\to\infty}\sup_{\gamma\le T}\frac{1}{K}\log \mathsf{P}\Big(\sup_{t\le\Delta}\Big|\int_0^t I_{s,C}dN^K_s\Big|\ge\eta\Big)=-\infty. \tag{5.9}$$

The local martingale $N^{K,C}_t:=\int_0^t I_{s,C}dN^K_s$ possesses the variation process

$$\langle N^{K,C}\rangle_t=\int_0^t I_{s,C}d\langle N^K\rangle_s=\frac{\sigma^2}{K}\int_0^t I_{s,C}x^K_s ds,$$

that is, $\mathrm{d}\langle N^{K,C}\rangle_t\le\frac{\sigma^2 C}{K}dt$.

Now, we are able to verify (5.9) with the help of stochastic exponential technique. Let

$$\mathfrak{z}_t(\lambda)=e^{\lambda N^{K,C}_t-\frac{\lambda^2}{2}\langle N^{K,C}\rangle_t},\quad \lambda\in\mathbb{R}.$$

Since $\mathfrak{z}_t(\lambda)$ is a continuous local martingale and supermartingale too, for any stopping time ϑ, $\mathsf{E}\mathfrak{z}_\vartheta(\lambda)\le 1$. Let $\theta=\inf\{t\le\Delta: N^{K,C}_t\ge\eta\}$. Taking into account that $\{\theta\le\Delta\}=\{N^{K,C}_\theta\ge\eta\}$, write $1\ge\mathsf{E}I_{\{\theta\le\Delta\}}\mathfrak{z}_\theta(\lambda)$. The value $\mathfrak{z}_\theta(\lambda)$ is evaluated below on the set $\{\theta\le\Delta\}$ as follows: with $\lambda>0$ and $\langle N^{K,C}\rangle_\theta\le\frac{\sigma^2C}{K}\theta\le\frac{\sigma^2C}{K}\Delta$,

$$\mathfrak{z}_\theta(\lambda)\ge e^{\lambda\eta-\frac{\lambda^2\sigma^2C}{2K}\Delta}.$$

Therefore, $\log\mathsf{P}(\theta\le\Delta)\le-\big[\lambda\eta-\frac{\lambda^2\sigma^2C}{2K}\Delta\big]$ and the choice of $\lambda=\frac{K\eta}{\sigma^2C\Delta}$ provides

$$\frac{1}{K}\log\mathsf{P}(\theta\le\Delta)\le-\frac{\eta^2}{2\sigma^2C\Delta}\xrightarrow[\Delta\to 0]{}-\infty.$$

It is clear that the same result remains valid for $\theta=\inf\{t:-N^{K,C}_t\ge\eta\}$. Combining both, we obtain (5.9).

(ii)-Verification.

1. $\underline{u_0=1,\ du_t=\dot u_t dt,\ (u_t>0)_{t<T},\ u_T\ge 0.}$

Set $0<\triangle<T$ and

$$\mathfrak{A}_{\triangle,\delta}=\Big\{\sup_{t\le T-\triangle}|x^K_t-u_t|\le\delta\Big\}\quad\text{and}\quad\mathfrak{A}_\delta=\Big\{\sup_{t\le T}|x^K_t-u_t|\le\delta\Big\}.$$

Since $u_t>0$, $t\le T-\triangle$, in the accordance to Freidlin and Wentzell, [6], it holds

$$\varlimsup_{K\to\infty}\frac{1}{K}\log\mathsf{P}\big(\mathfrak{A}_{\triangle,\delta}\big)\le-\frac{1}{2\sigma^2}\int_0^{T-\triangle}\frac{(\dot u_t-\alpha u_t)^2}{u_t}dt+O(\delta) \tag{5.10}$$

$$\varliminf_{K\to\infty}\frac{1}{K}\log\mathsf{P}\big(\mathfrak{A}_{\triangle,\delta}\big)\ge-\frac{1}{2\sigma^2}\int_0^{T-\triangle}\frac{(\dot u_t-\alpha u_t)^2}{u_t}dt+O(\delta). \tag{5.11}$$

We show that "(5.10), (5.11)" imply the desired result

$$\varlimsup_{\delta\to 0}\varlimsup_{K\to\infty}\frac{1}{K}\log\mathsf{P}\big(\mathfrak{A}_\delta\big)\le-\frac{1}{2\sigma^2}\int_0^T\frac{(\dot u_t-\alpha u_t)^2}{u_t}I_{\{u_t>0\}}dt \tag{5.12}$$

$$\varliminf_{\delta\to 0}\varliminf_{K\to\infty}\frac{1}{K}\log\mathsf{P}\big(\mathfrak{A}_\delta\big)\ge-\frac{1}{2\sigma^2}\int_0^T\frac{(\dot u_t-\alpha u_t)^2}{u_t}I_{\{u_t>0\}}dt. \tag{5.13}$$

First of all notice that $\int_0^{T-\triangle}\frac{(\dot u_t-\alpha u_t)^2}{u_t}dt$ increases to $\int_{[0,T)}\frac{(\dot u_t-\alpha u_t)^2}{u_t}dt$ with $\triangle$ decreasing to zero and $\int_{[0,T)}\frac{(\dot u_t-\alpha u_t)^2}{u_t}dt=\int_{[0,T]}\frac{(\dot u_t-\alpha u_t)^2}{u_t}I_{\{u_t>0\}}dt$.

(5.10)$\Rightarrow$ (5.12) since $\mathfrak{A}_\delta\subseteq\mathfrak{A}_{\triangle,\delta}$.

The implication (5.11)$\Rightarrow$(5.13) is verified as follows. Set

$$x^{K,\triangle}_t=\begin{cases}x^K_{t-\triangle}, & t\ge\triangle\\ 0, & \text{otherwise}\end{cases}\quad\text{and}\quad u^\triangle_t=\begin{cases}u_{t-\triangle}, & t\ge\triangle\\ 0, & \text{otherwise.}\end{cases}$$

The obvious inequality

$$\sup_{t\le T}|x^K_t-u_t|\le\sup_{t\le T}|x^{K,\triangle}_t-u^\triangle_t|+\sup_{t\le T}\big|x^K_t-x^{K,\triangle}_t\big|+\sup_{t\le T}\big|u_t-u^\triangle_t\big|.$$

provides

$$\begin{aligned}\mathsf{P}\big(\mathfrak{A}_\delta\big)&\ge\mathsf{P}\Big(\sup_{t\le T}|x^{K,\triangle}_t-u^\triangle_t|+\sup_{t\le T}\big|x^K_t-x^{K,\triangle}_t\big|+\sup_{t\le T}\big|u_t-u^\triangle_t\big|\le\delta\Big)\\&\ge\mathsf{P}\Big(\sup_{t\le T}|x^{K,\triangle}_t-u^\triangle_t|\le0.5\delta,\quad\sup_{t\le T}\big|x^K_t-x^{K,\triangle}_t\big|+\sup_{t\le T}\big|u_t-u^\triangle_t\big|\le\delta\Big)\\&\ge\mathsf{P}\big(\mathfrak{A}_{\triangle,05\delta}\big)-\mathsf{P}\Big(\sup_{t\le T}\big|x^K_t-x^{K,\triangle}_t\big|+\sup_{t\le T}\big|u_t-u^\triangle_t\big|>\delta\Big).\end{aligned}$$

Consequently,

$$\mathsf{P}\big(\mathfrak{A}_\delta\big)\vee\mathsf{P}\Big(\sup_{t\le T}\big|x^K_t-x^{K,\triangle}_t\big|+\sup_{t\le T}\big|u_t-u^\triangle_t\big|>\delta\Big)\ge\mathsf{P}\big(\mathfrak{A}_{\triangle,0.5\delta}\big).$$

Now, by (5.11), we have

$$\begin{aligned}&\Big\{\varliminf_{K\to\infty}\frac{1}{K}\log\mathsf{P}\big(\mathfrak{A}_\delta\big)\Big\}\bigvee\Big\{\varliminf_{K\to\infty}\frac{1}{K}\log\mathsf{P}\Big(\sup_{t\le T}\big|x^K_t-x^{K,\triangle}_t\big|+\sup_{t\le T}\big|u_t-u^\triangle_t\big|>\delta\Big)\Big\}\\&\qquad\ge-\frac{1}{2\sigma^2}\int_0^{T-\triangle}\frac{(\dot u_t-\alpha u_t)^2}{u_t}dt+O(\delta).\end{aligned}\tag{5.14}$$

Finally by (5.2), as long as

$$\lim_{\triangle\to0}\varlimsup_{K\to\infty}\frac{1}{K}\log\mathsf{P}\Big(\sup_{t\le T}\big|x^K_t-x^{K,\triangle}_t\big|+\sup_{t\le T}\big|u_t-u^\triangle_t\big|>\delta\Big)=-\infty,$$

the desired result, (5.13), follows by taking "$\lim_{\delta\to0}\lim_{\triangle\to0}\varliminf_{K\to\infty}$" from both sides of (5.14) and taking into account that $\varliminf_{K\to\infty}\frac{1}{K}\log\mathsf{P}\big(\mathfrak{A}_\delta\big)$ is independent of $\triangle$.

2. $\underline{u_0\ne1\text{ or }du_t\ne\dot u_tdt.}$ This part of proof is standard and is omitted.

3. $\underline{u_{t''}>u_{t'}=0\ \exists\ t'<t''\le T\text{ and }u_0=1,du_t=\dot u_tdt}$. Without loss of generality we may restrict ourselves by the following setting:

$$u_t\begin{cases}=0, & t=t'\\ >0, & t\in(0,t''],\end{cases}\qquad\dot u_t>0,\ t\in[t',t''].$$

The use an obvious inequality

$$\begin{aligned}\mathsf{P}\Big(\sup_{t\le T}|x^K_t-u_t|\le\delta\Big)&\le\mathsf{P}\Big(x^K_{t'}\vee|x^K_{t_\varepsilon}-u_{t''}|\le\delta\Big)\\&\le\mathsf{P}\Big(x^K_{t'}+|x^K_{t''}-u_{t''}|\le2\delta\Big)\end{aligned}$$

enables to derive the desired property from

$$\varlimsup_{K\to\infty}\varlimsup_{\delta\to0}\frac{1}{K}\log\mathsf{P}\Big(x^K_{t'}+|x^K_{t''}-u_{t''}|\le2\delta\Big)=-\infty.\tag{5.15}$$

For (5.15) verification, one applies a stochastic exponential technique: with continuously differentiable function $\lambda(t)$ set

$$\mathfrak{z}_t(\lambda)=\exp\Big(\int_{t'}^{t}\lambda(s)\big[dx_s^K-\alpha x_s^K ds\big]-\frac{1}{2}\int_{t'}^{t}\frac{\sigma^2}{K}x_s^K ds\Big).$$

The random process $\mathfrak{z}_t(\lambda)$ is a local martingale and supermartingale too, so that, in particular, $\mathsf{E}\mathfrak{z}_{t''}(\lambda)\le 1$. This inequality implies new one: for any measurable set $\mathfrak{A}$,

$$1\ge \mathsf{E}I_{\mathfrak{A}}\mathfrak{z}_{t''}(\lambda). \tag{5.16}$$

Now, we find the lower bound for $\mathfrak{z}_{t''}(\lambda)$ on $\mathfrak{A}=\{x_{t'}^K+|x_{t''}^K-u_{t''}|\le 2\delta\}$. Write, by taking $\theta(s)=\lambda(s)/K$,

$$\begin{aligned}\log\mathfrak{z}_t(\lambda)&=K\theta(t'')[x_{t''}^K-u_{t''}]-K\int_{t'}^{t''}\Big(\dot\theta(s)+\alpha\theta(s)+\frac{\theta^2(s)\sigma^2}{2}\Big)[x_s^K-u_s]ds\\&+K\int_{t'}^{t''}\Big(\theta(s)[\dot u_s-\alpha u_s]-\frac{\theta^2(s)\sigma^2}{2}u_s\Big)ds\\&\ge -2K\delta\Big[|\theta(t'')|+K\int_{t'}^{t''}\Big|\dot\theta(s)+\alpha\theta(s)+\frac{\theta^2(s)\sigma^2}{2}\Big|ds\Big]\\&+K\int_{(t',t'']}\Big(\theta(s)[\dot u_s-\alpha u_s]-\frac{\theta^2(s)\sigma^2}{2}u_s\Big)ds.\end{aligned}$$

This lower bound, jointly with (5.16), imply

$$\begin{aligned}&\varlimsup_{\delta\to 0}\varlimsup_{K\to\infty}\frac{1}{K}\log\mathsf{P}\Big(x_{t'}^K+|x_{t''}^K-u_{t''}|\le 2\delta\Big)\\&\qquad\le-\int_{(t',t'']}\Big(\theta(s)[\dot u_s-\alpha u_s]-\frac{\theta^2(s)\sigma^2}{2}u_s\Big)ds.\end{aligned}$$

Finally, by taking $\theta(s)=\dfrac{\dot u_s-u_s}{u_s\sigma^2}$, we find the following upper bound

$$\begin{aligned}\varlimsup_{\delta\to 0}\varlimsup_{K\to\infty}\frac{1}{K}\log\mathsf{P}\Big(x_{t'}^K+|x_{t''}^K-u_{t''}|\le 2\delta\Big)&\le-\frac{1}{2\sigma^2}\int_{(t',t'']}\frac{(\dot u_s-\alpha u_s)^2}{u_s}ds\\&\le-\frac{1}{2\sigma^2}\int_{(t',t'']}\frac{\dot u_s^2}{u_s}ds+\frac{\alpha}{\sigma^2}[u_{t''}-u_{t'}]+\frac{|\alpha|}{2\sigma^2}\int_{(t',t'']}u_s ds.\end{aligned}$$

The second and third terms in the right hand side of the above inequality are bounded. So, it is left to show that

$$\int_{(t',t'']}\frac{\dot u_s^2}{u_s}ds=\infty.$$

Let $t_*=\inf\{t>t':\dot u_t\le 0.5\dot u_0\}\wedge t''$. Then

$$\int_{(t',t'']}\frac{\dot u_s^2}{u_s}ds\ge 0.5\dot u_0\int_{(t',t_*]}\frac{\dot u_s}{u_s}ds=0.5\dot u_0[\log(u_{t_*})-\lim_{s\downarrow t'}\log(u_s)]=\infty.$$

$\square$

References

[1] Athreya, K. and Ney, P. *Branching Processes.* Springer, 2004.

[2] Athreya, K. B. (1994). Large deviation rates for branching processes I. Single type case. *Ann. Appl. Probab.* 4, 779-790.

[3] Barbour, A. D. (1980) Density dependent Markov population processes. *Lecture Notes in Biomathematics* 38, 36–49, Springer.

[4] Biggins, J.D. and Bingham, N.H. (1994) Large Deviation in the supercritical branching process. *Adv. in Appl. Probab.* 25, 757-772.

[5] Donati-Martin, C., Rouault, A., Yor, M. and Zani, M. (2004) Large deviations for squares of Bessel and Ornstein-Uhelnbeck processes. *Probab. Theory and Related Fields.* **129**, no 2, 261-289.

[6] Freidlin, M.I., Wentzell A.D. (1984)*Random Perturbations of Dynamical Systems.* N.Y. Springer.

[7] Harris, T. *The theory of branching rrocesses.* Springer, 1963.

[8] Kifer, Yu. (1990) A discrete-time version of the Wentzell-Freidlin theory. *The Annals of Probability.* **18**, no. 4, 1676-1692.

[9] Klebaner F. and Zeitouni O. (1994) The exit problem for a class of density dependent branching systems. *Annals of Applied Probability.* 4, 1188-1205.

[10] Klebaner F.C. (1993) Population-dependent branching processes with a threshold. *Stochastic processes and their applications.* 46, 115-127.

[11] Klebaner F.C. and Nerman O. (1994) Autoregresive approximation in branching processes with a treshhold. *Stochastic processes and their applications.* 51, 1-7.

[12] Klebaner F.C. and Liptser R. (1996) Large deviations for recurrent sequences with aftereffect, *Problems of Information Transmission.* **32**, 4, 23-34.

[13] Klebaner, F.C. (2005)*Introduction to Stochastic Calculus with Applications. Second Ed.* Imperial College Press, London.

[14] Klebaner, F. and Liptser, R. (2001) *Asymptotic Analysis and Extinction in a Stochastic Lotka-Volterra model*. Annals of Applied Probability. **11** 4 pp. 1263-1291.

[15] Liptser, R.S., Pukhalskii, A.A. (1992) Limit theorems on large deviations for semimartingales. Stochastic and Stochastics Reports **38**, 201–249.

[16] Liptser, R.Sh. and Shiryayev, A.N. (1989) *Theory of Martingales.* Kluwer Acad. Publ.

[17] Liptser, R. and Shiryaev, A. (1978) *Statistics of Random Processes II. Applications* , Applications of Mathematics, vol. 6, Springer-Verlag,New York Heidelberg Berlin.

[18] Ney P.E. and Vidyashankar A.N. (2004) Local Limit Theory and Large Deviations for Supercritical Branching Processes. *Ann. Appl. Probab.* Vol. 14, No. 3, 11351166.

[19] Puhalskii, A. (2001) Large Deviations and Idempotent Probability, Chapman & Hall/CRC Press.

[20] Puhalskii, A. (2004) On some degenerate large deviation problems. *Electronic J. Probab.* 9:862-886.

[21] Rogers, L.C.G. and Williams, D. (1990) Diffusion, Markov Processes, and Martingales. Volume 2. Itô calculus. Willey.

[22] Stroock, D.W (1984)*An Introduction to the Theory of Large Deviations.* Springer-Verlag, Berlin.

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