

Continuity theorems for the $M/M/1/n$ queueing system

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Abstract In this paper continuity theorems are established for the number of losses during a busy period of the $M/M/1/n$ queue. We consider an $M/GI/1/n$ queueing system where the service time probability distribution, slightly different in a certain sense from the exponential distribution, is approximated by that exponential distribution. Continuity theorems are obtained in the form of one or two-sided stochastic inequalities. The paper shows how the bounds of these inequalities are changed if further assumptions, associated with specific properties of the service time distribution (precisely described in the paper), are made. Specifically, some parametric families of service time distributions are discussed, and the paper establishes uniform estimates (given for all possible values of the parameter) and local estimates (where the parameter is fixed and takes only the given value). The analysis of the paper is based on the level crossing approach and some characterization properties of the exponential distribution.

Keywords Continuity theorems · Loss systems · $M/GI/1/n$ and $M/M/1/n$ queues · Busy period · Branching process · Number of level crossings · Kolmogorov (uniform) metric · Stochastic ordering · Stochastic inequalities

Mathematics Subject Classification (2000) 60K25 · 60B05 · 60E15 · 62E17

Dedicated to Vladimir Mikhailovich Zolotarev, Victor Makarovich Kruglov, and to the memory of Vladimir Vyacheslavovich Kalashnikov.

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1 Introduction

1.1 Motivation, the problem formulation and review of the related literature

In a large number of engineering applications of queueing theory it is pertinent to know what to expect when we replace one probability distribution function of a given stochastic model, by another probability distribution with simpler/concise properties. Many of these applications are associated with the case where a probability distribution is replaced by an exponential distribution if it is known that the given distribution is close (in a certain sense) to the exponential distribution. These problems are closely related to characterization problems associated with the exponential distribution and its stability (e.g. Azlarov and Volodin [13]).

Continuity analysis of queueing systems is a difficult problem when compared to general characterization and continuity problems for random variables, and it is relevant to know the expected behavior of the system under a range of *different* conditions. The paper discusses three different conditions under which the probability distribution function of a service time, which differs slightly from the exponential distribution in the uniform metric, can be approximated by that exponential distribution.

This paper considers an $M/GI/1/n$ queueing system, where n is the buffer size excluding a customer in service, λ is the rate of Poisson input, $B(x) = \Pr\{\chi \leq x\}$ is the probability distribution of a service time χ , and the parameter μ is the reciprocal of the expected service time. It is assumed then that the probability distribution function $B(x)$ is close (under specific cases described below) to the exponential distribution with the same parameter μ .

Let L_n denote the number of losses during a busy period of this queueing system. The classic explicit results for the $M/GI/1/n$ queueing system, for example the recurrence relations for the expectations EL_n , as well as a number of results on the asymptotic behavior of EL_n as $n \rightarrow \infty$, can be found in Abramov [1, 4], Cooper and Tilt [16] and Tomko [35]. An application of these explicit results to asymptotic analysis of lost messages in communication networks is given in Abramov [9], and their application to problems of optimal control of large dams is given in [11, 12]. Some simple stochastic inequalities for different finite buffer queueing systems are obtained in Abramov [5, 10], Richter [32], Peköz et al. [29], Wolff [37] and other papers. Other relevant studies of the loss probability in finite buffer systems can be found in Abramov [2, 8] and Choi et al. [14].

The present paper establishes continuity theorems in the form of *stochastic inequalities* for L_n similar to the stochastic inequalities established in Abramov [5, 10]. The main difference between the results of this paper and those of Abramov [5, 10] is that the stochastic inequalities established in the present paper depend on specific properties of the service time distribution, whereas the stochastic bounds of the earlier papers [5, 10] are obtained without any special assumption on that service time distribution. Then the specific feature of the present paper is that the stochastic inequalities obtained can be considered as *approximation bounds* or *continuity bounds* for the queueing system in the uniform metric, and these approximation bounds can have real application for stochastic systems arising in practice. The phrase *continuity theorems* used in this paper is associated with the traditional “ ϵ - δ language” of

mathematical analysis”. The input characteristic of the system is assumed to have a small variation ϵ , and then the output characteristic has a small variation δ continuously depending on ϵ . Both these ϵ and δ variations are assumed to be described by some special metrics. A large variety of metrics for the continuity analysis of complex systems can be found in Dudley [19], Kalashnikov and Rachev [26], Rachev [30], Zolotarev [40] and many others. In the cases where an application of the well-known traditional metrics such as the uniform metric or the Levy metric becomes unavailable, very difficult or not profitable, the continuity analysis of a complex system requires to use the special type of metrics, appropriate for analysis of a given system or associated with given special conditions.

The continuity analysis of queueing systems goes back to the papers of Kennedy [27] and Whitt [36] and further developed by Kalashnikov [25], Zolotarev [38, 39] and Gordienko and Ruiz de Chávez [22, 23] to mention a few.

The continuity analysis of such standard characteristics as *the number of losses during a busy period* is a very difficult problem. One of the advantages of the method in this paper, is that the continuity of this characteristic is studied in terms of the *uniform probability metric*. As a result, the final results are simple and clear, and the conditions of the theorems are verifiable. (The uniform metric of one-dimensional distributions is usually called the Kolmogorov metric.) Furthermore, under different assumptions that the service time distribution belongs to its parametric family of distributions depending on parameter p we establish *uniform* estimates for the number of losses during a busy period given for all possible values p . That is, stochastic inequalities depending only on ϵ but not on p are obtained. In the case where parameter p of this family is fixed (specifically the case $p = \frac{1}{2}$ is considered) we establish essentially stronger estimates than these corresponding uniform estimates.

For the purpose of the continuity analysis, this paper essentially develops the level crossing approach of the earlier papers of Abramov [1, 3, 5, 6, 10] and some characterization theorems associated with the exponential distribution, that are then used together with the aforementioned metrical approach. We also develop the continuity theorem of Azlarov and Volodin [13] under some additional conditions and use the properties of the known classes of special probability distributions such as NBU (New Better than Used), NWU (New Worse than Used) (e.g. Stoyan [34] for the properties of these classes of distribution and related results). The special parametric order relation \mathcal{C}_λ , which was introduced and originally studied in Abramov [1], is also used. For the reader’s convenience all necessary facts and concepts are recalled in this paper.

1.2 The main conditions for the probability distribution of a service time

The main conditions for the probability distribution function $B(x)$ ($B(0) = 0$) of a service time under which we obtain the stochastic bounds are as follows.

Condition (A) The probability distribution of a service time has the representation

$$B(x) = pF(x) + (1 - p)(1 - e^{-\mu x}), \quad 0 < p \leq 1, \quad (1.1)$$

where $F(x) = \Pr\{\zeta \leq x\}$ is a probability distribution function of a nonnegative random variable having the expectation $\frac{1}{\mu}$, and

$$\sup_{x, y \geq 0} |F_y(x) - F(x)| \leq \epsilon, \quad \epsilon \geq 0, \quad (1.2)$$

where $F_y(x) = \Pr\{\zeta \leq x + y \mid \zeta > y\}$. Relation (1.2) says that the distance in the uniform metric between $F(x)$ and $E_\mu(x) = 1 - e^{-\mu x}$ is not greater than ϵ , the case $\epsilon = 0$ corresponds to the equality $F(x) = E_\mu(x)$ for all x . In the following we assume that $\epsilon > 0$ and write the strong inequalities (i.e. the right-hand side of (1.2) is less than ϵ).

Condition (B) Along with (1.1) and (1.2) it is assumed that $F(x)$ belongs either to the class NBU or to the class NWU.

Recall that a probability distribution function $\Xi(x)$ of a nonnegative random variable is said to belong to the class NBU if for all $x \geq 0, y \geq 0$ we have $\overline{\Xi}(x + y) \leq \overline{\Xi}(x)\overline{\Xi}(y)$. (Throughout the paper, for any probability distribution function $\Xi(x)$ we use the notation $\overline{\Xi}(x) = 1 - \Xi(x)$.) If the opposite inequality holds i.e. $\overline{\Xi}(x + y) \geq \overline{\Xi}(x)\overline{\Xi}(y)$ then $\Xi(x)$ is said to belong to the class NWU.

Condition (C) The probability distribution function $B(x)$ of a service time belongs to the class NBU, and

$$\sup_{x, y \geq 0} |B_y(x) - B(x)| < \epsilon, \quad (1.3)$$

where $B_y(x) = \Pr\{\chi \leq x + y \mid \chi > y\}$.

Condition (D) Let $F_y(x) = \Pr\{\zeta \leq x + y \mid \zeta > y\}$ be a family of given probability distributions. It is assumed that there exists the probability distribution $F_{y,0}(x)$ of this family satisfying the relation $F_{y,0} \leq_{\mathcal{C}_\lambda} F_y$ for all $y \geq 0$.

The definition and main property of the parametric order relation \mathcal{C}_λ is recalled in Appendix A. For a more detailed consideration see Abramov [1].

Condition (E) Let $B_y(x) = \Pr\{\chi \leq x + y \mid \chi > y\}$ be a family of given probability distributions. It is assumed that there exists the probability distribution $B_{y,0}(x)$ of this family satisfying the relation $B_{y,0} \leq_{\mathcal{C}_\lambda} B_y$ for all $y \geq 0$.

If the family of probability distributions $F_y(x)$ (or $B_y(x)$) is partially ordered with respect to the order relation \mathcal{C}_λ , then according to Zorn's lemma (e.g. Ciesielski [15]) $F_{y,0}$ (or correspondingly $B_{y,0}$) is a minimal element of this family (regarding this order relation \mathcal{C}_λ).

The only difference between Conditions (D) and (E) is that Condition (E) is related to the probability distribution function $B(x)$ of a service time, whereas Condition (D) is related to the associated probability distribution function $F(x)$.

Conditions (A), (B) and (C) are the main conditions for our consideration, while Conditions (D) and (E) are additional (associated) conditions. In other words one of Conditions (A), (B) and (C) are always present in our consideration. Condition (D) can be present only with Conditions (A) and (B), and Condition (E) can be present only with Condition (C).

It is worth noting that in certain known cases the minimal element $F_{y,0}$ can be determined easily. For example, in the case where $F(x)$ belongs to the class NWU, the minimal element of the family of probability distributions $F_y(x)$ is $F_{y,0} = F_0 = F$

in the sense of the order relation \mathcal{C}_λ . (By replacing NWU with NBU and respectively ‘minimal’ with ‘maximal’, this fact is explicitly used in Theorem 4.2, Sect. 4.)

1.3 Further discussion of the main conditions

Conditions (A), (B) and (C) are three different conditions where if ϵ is small, then the probability distribution function $B(x)$ is close to the exponential distribution in the sense of the uniform metric. Therefore it is very significant to know what one can expect if Conditions (A), (B) or (C) are satisfied.

In most queueing problems representation (1.1) is not standard. The standard assumption appearing in characterization problems associated with the exponential distribution is (1.3) (e.g. Azlarov and Volodin [13]; cf. Daley [18] for some earlier results). Representation of the probability distribution $B(x)$ in its special form (1.1) matches the usual case $B(x) = F(x)$ (that is the case $p = 1$). It is shown below (see Lemma 2.1) that assumptions (1.1) and (1.2) of Condition (A) allow us only to prove that for all $0 < p \leq 1$

$$\sup_{x,y \geq 0} |B_y(x) - B(x)| < 5\epsilon, \tag{1.4}$$

and only in the special case where $p = \frac{1}{2}$ we have

$$\sup_{x,y \geq 0} |B_y(x) - B(x)| < 2\epsilon \tag{1.5}$$

(see Remark 2.3). As a result, the approximation bounds under Conditions (A), (B) and (C) are all different, and more specifically representation (1.1) together with assumption (1.2) lead to relatively worse estimates than those in the special case $B(x) = F(x)$ under assumption (1.3). However (1.4) is a uniform estimate for all $0 < p \leq 1$, whereas (1.5) is a usual local estimate better than (1.4). In the special case $p = 1$ it is a local estimate too, which in fact is (1.3).

The advantage of (1.1) is also as follows. When the parameter p is small, the service time distribution is close to the exponential one in the uniform metric. Thus, our results are associated with a wider class of approximations and based on a two-parameter family of distributions. In practice, this can help us find a more appropriate representation and approximation for the initial probability distribution function $B(x)$ given empirically. The main results of this paper establish explicit dependence of the parameter ϵ only and are uniform in p . However there are examples related to the special case of $p = \frac{1}{2}$ where the bounds obtained are essentially better than in the general case of arbitrary p . In a similar manner the case of any given p can be considered, where the specific estimates are expected to be better than the corresponding uniform estimates.

Notice also that the class of stochastic inequalities is wider. On the one hand, only the parameter ϵ can be assumed to be small, and on the other we can assume that both ϵ and p are small values. The variety of these assumptions enables us to choose the possible appropriate parameters of the model for the further approximations in order to obtain a relevant conclusion.

1.4 Organization and methodology of this paper

This paper is structured into 4 sections. Following this introduction, Sects. 2, 3 and 4 study the number of losses during a busy period under Conditions (A), (B) and (C), respectively.

In Sect. 2 we give a deeper analysis of the intervals obtained under the special procedure of deleting subintervals and connecting the ends as is explained in a number of the earlier papers of the author (see Abramov [1, 3, 5, 6, 10]). Specifically we prove the following results. The first result of this section, Lemma 2.1, provides the uniform estimate (1.4) for the parametric family of distributions. Remark 2.3 provides local estimate (1.5) in the special case $p = \frac{1}{2}$. The proof of both these estimates is based on the characterization theorem of Azlarov and Volodin [13], which is formulated in Lemma 2.2. The proof of Theorem 2.4 is based on the level crossing approach which in its present form is originated and developed by the author (see Abramov [1, 3, 5, 6, 10]).

In Sect. 3, under Conditions (B) and (D) stronger results than those under Conditions (A) and (D) are established. The results are based on similar proofs to those in Sect. 2.

In Sect. 4, under Condition (C) the two-sides stochastic inequalities are given in Theorem 4.2. The proofs of the statements under Condition (C) are completely analogous to the earlier proofs under Condition (A).

The partial order relation \mathcal{C}_λ and its main property are discussed in Appendix A. The proof of both Lemmas 2.2 and 3.1 is given in Appendix B. In Appendix C, the proof of Lemma 2.6 is given.

2 The number of losses under Condition (A)

Throughout the paper we use Kolmogorov's (uniform) metric between two one-dimensional probability distributions. Recall the definition of Kolmogorov's metric (e.g. Kalashnikov and Rachev [26], Rachev [30]). Let $G(x) = \Pr\{\xi \leq x\}$ and $H(x) = \Pr\{\eta \leq x\}$ be probability distribution functions of the random variables ξ and η . Kolmogorov's metric $\mathcal{K}(G, H)$ between two probability distribution functions $G(x)$ and $H(x)$ is defined by

$$\mathcal{K}(G, H) = \sup_{x \in \mathbb{R}^1} |G(x) - H(x)|.$$

In the following for Kolmogorov's metric the notation $\mathcal{K}(\xi, \eta)$ is also used, where $\mathcal{K}(\xi, \eta) = \mathcal{K}(G, H)$. All exponential distributions are denoted $E_\alpha = E_\alpha(x) = 1 - e^{-\alpha x}$, where $\alpha > 0$ is the parameter of distribution.

Lemma 2.1 *Under assumptions (1.1) and (1.2) we have*

$$\sup_{x \geq 0, y \geq 0} |B_y(x) - B(x)| < 5\epsilon.$$

In order to prove this lemma we use the following result of Azlarov and Volodin [13], the proof of which is also provided in Appendix B.

Lemma 2.2 (Azlarov and Volodin [13]) *Let $F(x) = \Pr\{\zeta \leq x\}$ be a probability distribution function of a nonnegative random variable with the expectation $1/\mu$, and*

$$\sup_{x \geq 0, y \geq 0} |F_y(x) - F(x)| < \epsilon.$$

Then

$$\mathcal{K}(F, E_\mu) < 2\epsilon.$$

Proof of Lemma 2.1 We have

$$\begin{aligned} \bar{B}_y(x) &= \frac{1 - pF(x + y) - (1 - p)E_\mu(x + y)}{1 - pF(y) - (1 - p)E_\mu(y)} \\ &= \frac{p\bar{F}(x + y) + (1 - p)e^{-\mu(x+y)}}{p\bar{F}(y) + (1 - p)e^{-\mu y}}. \end{aligned} \tag{2.1}$$

Therefore,

$$\begin{aligned} |B_y(x) - B(x)| &= \left| \frac{p\bar{F}(x + y) + (1 - p)e^{-\mu(x+y)}}{p\bar{F}(y) + (1 - p)e^{-\mu y}} - p\bar{F}(x) - (1 - p)e^{-\mu x} \right| \\ &= \left| \frac{p\bar{F}(x + y) + (1 - p)e^{-\mu(x+y)}}{p\bar{F}(y) + (1 - p)e^{-\mu y}} - \frac{p^2\bar{F}(x)\bar{F}(y)}{p\bar{F}(y) + (1 - p)e^{-\mu y}} \right. \\ &\quad \left. - \frac{p(1 - p)[e^{-\mu x}\bar{F}(y) + e^{-\mu y}\bar{F}(x)]}{p\bar{F}(y) + (1 - p)e^{-\mu y}} - \frac{(1 - p)^2e^{-\mu(x+y)}}{p\bar{F}(y) + (1 - p)e^{-\mu y}} \right|. \end{aligned} \tag{2.2}$$

According to (1.2) we have

$$\left| \frac{\bar{F}(x + y)}{\bar{F}(y)} - \bar{F}(x) \right| < \epsilon, \tag{2.3}$$

where the denominator in the fraction of (2.3) is assumed to be positive, and if it is equal to zero, then we use the convention $\frac{0}{0} = 0$. It follows from Lemma 2.2 that

$$|\bar{F}(x) - e^{-\mu x}| < 2\epsilon. \tag{2.4}$$

Therefore from (2.2), (2.3), (2.4) and the triangle inequality for all $x \geq 0, y \geq 0$, we obtain:

$$\begin{aligned} &|p\bar{F}(x + y) - p^2\bar{F}(x)\bar{F}(y) - p(1 - p)e^{-\mu x}\bar{F}(y)| \\ &\leq p\bar{F}(y) \left| \frac{\bar{F}(x + y)}{\bar{F}(y)} - e^{-\mu x} \right| + p^2\bar{F}(y) \left| \bar{F}(x) - e^{-\mu x} \right| \end{aligned}$$

$$\begin{aligned}
&= p\bar{F}(y) \left| \frac{\bar{F}(x+y)}{\bar{F}(y)} - \bar{F}(x) + \bar{F}(x) - e^{-\mu x} \right| \\
&\quad + p^2\bar{F}(y) |\bar{F}(x) - e^{-\mu x}| \\
&\leq p\bar{F}(y) \left(\left| \frac{\bar{F}(x+y)}{\bar{F}(y)} - \bar{F}(x) \right| + |\bar{F}(x) - e^{-\mu x}| \right) \\
&\quad + p^2\bar{F}(y) |\bar{F}(x) - e^{-\mu x}| \\
&< 3\epsilon p\bar{F}(y) + 2\epsilon p^2\bar{F}(y) \leq 5\epsilon p\bar{F}(y), \tag{2.5} \\
&|(1-p)e^{-\mu(x+y)} - p(1-p)e^{-\mu y}\bar{F}(x) - (1-p)^2e^{-\mu(x+y)}| \\
&\leq (1-p)e^{-\mu y} \left(|e^{-\mu x} - \bar{F}(x)| + |(1-p)\bar{F}(x) - (1-p)e^{-\mu x}| \right) \\
&< 2\epsilon(1-p)e^{-\mu y} + 2\epsilon(1-p)^2e^{-\mu y} \\
&< 3\epsilon(1-p)e^{-\mu y} + 2\epsilon(1-p)^2e^{-\mu y} < 5\epsilon(1-p)e^{-\mu y}, \tag{2.6}
\end{aligned}$$

and then it follows from (2.5) and (2.6) that for all $x \geq 0, y \geq 0$,

$$|B_y(x) - B(x)| < \frac{5\epsilon p\bar{F}(y) + 5\epsilon(1-p)e^{-\mu y}}{p\bar{F}(y) + (1-p)e^{-\mu y}} = 5\epsilon.$$

Hence, Lemma 2.1 is proved. \square

Remark 2.3 The statement of Lemma 2.1 is a uniform estimate for all p . In special cases where p is given, one can obtain stronger estimates. For example, in the case $p = \frac{1}{2}$ we have the following inequalities:

$$\begin{aligned}
&\left| \frac{1}{2}\bar{F}(x+y) - \frac{1}{4}\bar{F}(x)\bar{F}(y) - \frac{1}{4}e^{-\mu x}\bar{F}(y) \right| \\
&\leq \frac{1}{4}\bar{F}(y) \left| \frac{\bar{F}(x+y)}{\bar{F}(y)} - \bar{F}(x) \right| + \frac{1}{4}\bar{F}(y) \left| \frac{\bar{F}(x+y)}{\bar{F}(y)} - e^{-\mu x} \right| \\
&\leq \frac{1}{2}\bar{F}(y) \left| \frac{\bar{F}(x+y)}{\bar{F}(y)} - \bar{F}(x) \right| + \frac{1}{4}\bar{F}(y) \left| \bar{F}(x) - e^{-\mu x} \right| \\
&< \frac{1}{2}\epsilon\bar{F}(y) + \frac{1}{2}\epsilon\bar{F}(y) = \epsilon\bar{F}(y), \tag{2.7}
\end{aligned}$$

$$\begin{aligned}
&\left| \frac{1}{2}e^{-\mu(x+y)} - \frac{1}{4}e^{-\mu y}\bar{F}(x) - \frac{1}{4}e^{-\mu(x+y)} \right| \\
&= \frac{1}{4}e^{-\mu y} |e^{-\mu x} - \bar{F}(x)| < \frac{1}{2}\epsilon e^{-\mu y} < \epsilon e^{-\mu y}. \tag{2.8}
\end{aligned}$$

Therefore, from (2.7) and (2.8) we obtain:

$$|B_y(x) - B(x)| < \frac{\epsilon \overline{F}(y) + \epsilon e^{-\mu y}}{\frac{1}{2} \overline{F}(y) + \frac{1}{2} e^{-\mu y}} = 2\epsilon. \tag{2.9}$$

Theorem 2.4 *Under Conditions (A) and (D) we have*

$$L_n \geq_{st} \sum_{i=1}^{Z_n} \varsigma_i.$$

Z_n denotes the number of offspring in the n th generation of the Galton–Watson branching process with $Z_0 = 1$ and the offspring generating function

$$g(z) < \frac{\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)} + 5\epsilon \cdot \frac{1 - z + 2z\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)}, \quad |z| \leq 1, \tag{2.10}$$

where

$$\widehat{B}(\lambda) = p\widehat{F}(\lambda) + (1 - p)\frac{\mu}{\mu + \lambda},$$

$$\widehat{F}(\lambda) = \int_0^\infty e^{-\lambda x} dF(x).$$

$\varsigma_i, i = 1, 2, \dots$, is the sequence of nonnegative integer-valued independent identically distributed random variables all having the probability law

$$\Pr\{\varsigma_i = m\} = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^m}{m!} dG(x), \tag{2.11}$$

and $G(x)$ is a probability distribution that satisfies the inequalities $G \leq_{c_\lambda} F_{y,0}$ and $G \leq_{c_\lambda} E_\mu$, where $F_{y,0} \leq_{c_\lambda} F_y$ for all $y \geq 0$.

Remark 2.5 The condition (2.11) of Theorem 2.4 involves a probability distribution function $G(x)$. If Condition (D) is not satisfied, or the probability distribution $F_{y,0}(x)$ is unknown and cannot be evaluated, then a probability distribution function $G(x)$ should be chosen such that $G \leq_{c_\lambda} F_y$ for all $y \geq 0$ and $G \leq_{c_\lambda} E_\mu$.

Proof Consider a busy period of an $M/GI/1/n$ queueing system. Let $f(j), 0 \leq j \leq n + 1$, denote the number of customers arriving during a busy period who, upon their arrival, meet j customers in the system. It is clear that $f(0) = 1$ with probability 1. Let $t_{j,1}, t_{j,2}, \dots, t_{j,f(j)}$ be the instants of arrival of these $f(j)$ customers, and let $s_{j,1}, s_{j,2}, \dots, s_{j,f(j)}$ be the instants of service completions (the case $j \leq n$) or losses (the case $j = n + 1$) at which there remain only j customers in the system. Note that $t_{n+1,k} = s_{n+1,k}, 1 \leq k \leq f(n + 1)$, and $f(n + 1) = L_n$, the number of losses during a busy period.

For $0 \leq j \leq n$ consider the intervals:

$$[t_{j,1}, s_{j,1}), [t_{j,2}, s_{j,2}), \dots, [t_{j,f(j)}, s_{j,f(j)}). \tag{2.12}$$

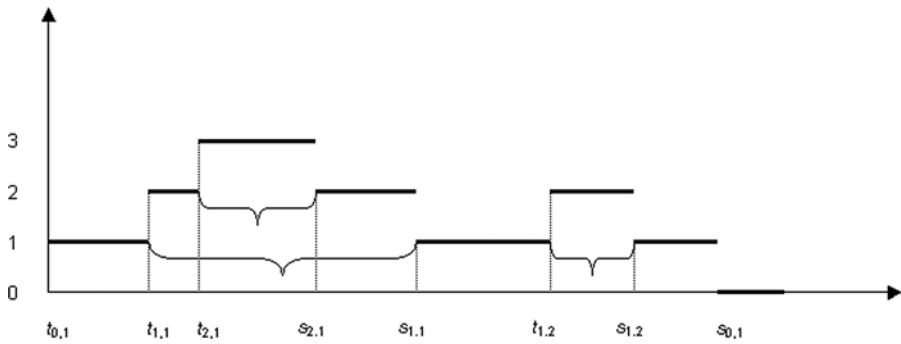


Fig. 1 Up- and down-crossings on a busy period

It is clear that the intervals

$$[t_{j+1,1}, s_{j+1,1}), [t_{j+1,2}, s_{j+1,2}), \dots, [t_{j+1,f(j+1)}, s_{j+1,f(j+1)}) \tag{2.13}$$

are contained in intervals (2.12). Let us delete the intervals in (2.13) from those in (2.12) and connect the ends, that is every point $t_{j+1,k}$ with the corresponding point $s_{j+1,k}$, $1 \leq k \leq f(j+1)$, if the set of intervals (2.13) is not empty. In other words, in the interval of the form $[t_{j,k}, s_{j,k})$, the inserted points are of the form $t_{j+1,m}$. The random variable $\xi_{j,k}$ is then the length of interval $[t_{j,k}, s_{j,k})$ minus the intervals $[t_{j+1,m}, s_{j+1,m})$ contained in this interval.

A typical example of the level crossings on a busy period is given in Fig. 1. The arc braces in the figure indicate the places of connection of the points after deleting the intervals.

In the example, given in Fig. 1, $n = 3$ and the number of inserted points of level 1 is equal to 2, that is $f(1) = 2$. Similarly, $f(2) = 1$.

Let us take one of the intervals, $[t_{j,k}, s_{j,k})$ say, and a customer in service at time $t_{j,k}$. Let $\tau_{j,k}$ denote the time elapsed from the moment of a service start for that customer until time $t_{j,k}$, and let $\vartheta_{j,k}$ denote the residual service time of the tagged customer. The analysis of a residual service time is well-known in the literature, and in the simplest cases is associated with the *inspection paradox* or *waiting time paradox* (see e.g. [17, 20, 21, 24, 28, 31, 33] and many others).

The sum $\tau_{j,k} + \vartheta_{j,k}$ has a more complicated structure than that in the standard inspection or waiting time paradox, and deriving an explicit representation for $\Pr\{\vartheta_{j,k} \leq x\}$ in terms of the probability distributions $B(x)$ or $B_y(x)$ is very difficult. By using Lemma 2.1 above, the probability distribution function $B_{j,k}(x) = \Pr\{\vartheta_{j,k} \leq x\}$ can be evaluated nevertheless.

Note first the following technical result.

Lemma 2.6 *For any integer $j \geq 1$ and $k \geq 1$, and any nonnegative real x and y ,*

$$\Pr\{\vartheta_{j,k} \leq x \mid \tau_{j,k} = y\} = B_y(x).$$

The proof of this lemma is given in Appendix C.

Therefore, from Lemma 2.6 and Lemma 2.1 for any nonnegative y we have:

$$|\Pr\{\vartheta_{j,k} \leq x \mid \tau_{j,k} = y\} - B(x)| = |B_y(x) - B(x)| < 5\epsilon. \tag{2.14}$$

We will show below, that by using the formula for the total probability one arrive at

$$|\Pr\{\vartheta_{j,k} \leq x\} - B(x)| < 5\epsilon. \tag{2.15}$$

For this purpose we should show that the random variable $\tau_{j,k}$ is proper, i.e. $\Pr\{\tau_{j,k} < \infty\} = 1$, and then use the formula for the total probability:

$$\begin{aligned} \Pr\{\vartheta_{j,k} \leq x\} &= \int_0^\infty \Pr\{\vartheta_{j,k} \leq x \mid \tau_{j,k} = y\} d\Pr\{\tau_{j,k} \leq y\} \\ &= \int_0^\infty B_y(x) d\Pr\{\tau_{j,k} \leq y\}. \end{aligned} \tag{2.16}$$

(Notice, that the random variables $\vartheta_{j,k}$ and $\tau_{j,k}$ are generally dependent, and we cannot derive a representation for $\Pr\{\tau_{j,k} \leq y\}$.)

Let us show that the random variable $\tau_{j,k}$ is proper for any j and k . Indeed, let \mathcal{A}_i denote the event that there are i arrivals during the time interval from the service beginning of the customer, who is currently being served at the moment $t_{j,k}$, until the time moment before the arrival at $t_{j,k}$ (the arrival at time $t_{j,k}$ and possible arrivals at the moment of the service beginning are excluded). Then,

$$\Pr\{\tau_{j,k} \leq x\} = \sum_{i=0}^{j-1} \Pr\{\tau_{j,k} \leq x \mid \mathcal{A}_i\} \Pr\{\mathcal{A}_i\} \tag{2.17}$$

$$\geq \sum_{i=0}^{j-1} \Pr\{\text{sum of } i + 1 \text{ interarrival times} \leq x\} \Pr\{\mathcal{A}_i\} \tag{2.18}$$

$$\geq \Pr\{\text{sum of } j \text{ interarrival times} \leq x\}. \tag{2.19}$$

(The expression for the probability distribution of (2.19) can be written explicitly.) Inequality (2.18) follows from (2.17) due to the following property:

$$\begin{aligned} &\Pr\{\tau_{j,k} \leq x \mid \mathcal{A}_i\} \\ &:= \Pr\{\text{sum of } i + 1 \text{ interarrival times} \leq x \mid \text{all of them occur during a service time}\} \\ &\geq \Pr\{\text{sum of } i + 1 \text{ interarrival times} \leq x\}. \end{aligned}$$

Inequality (2.19) follows from (2.18) due to the elementary fact that for $i \leq j$

$$\begin{aligned} &\Pr\{\text{sum of } i \text{ interarrival times} \leq x\} \\ &\geq \Pr\{\text{sum of } j \text{ interarrival times} \leq x\}. \end{aligned}$$

Thus, $\tau_{j,k}$ is a proper random variable, and (2.16) is valid. Therefore, denoting the probability of (2.16) by $B_{j,k}(x)$, we see that under Conditions (A) and (D):

$$\begin{aligned}
 |B_{j,k}(x) - B(x)| &\leq \int_0^\infty \underbrace{\sup_{x \geq 0, y \geq 0} |B_y(x) - B(x)|}_{< 5\epsilon \text{ due to Lemma 2.1}} d\Pr\{\tau_{j,k} \leq y\} \\
 &< 5\epsilon \int_0^\infty d\Pr\{\tau_{j,k} \leq y\} = 5\epsilon,
 \end{aligned}
 \tag{2.20}$$

and (2.15) follows.

Let $\kappa_{j,k}$ denote the number of inserted points within the interval $[t_{j,k}, s_{j,k})$ ($\sum_{k=1}^{f(j)} \kappa_{j,k} := f(j+1)$). Then for $1 \leq j < n$ ($n > 1$)

$$\Pr\{\kappa_{j,k} = 0\} = \int_0^\infty e^{-\lambda x} dB_{j,k}(x) = \widehat{B}_{j,k}(\lambda),
 \tag{2.21}$$

$$\Pr\{\kappa_{j,k} = m\} = [1 - \Pr\{\kappa_{j,k} = 0\}][1 - \widehat{B}(\lambda)]^{m-1} \widehat{B}(\lambda), \quad m = 1, 2, \dots,$$

where $\widehat{B}(\lambda) = \int_0^\infty e^{-\lambda x} dB(x)$. Notice, that according to (2.20) for any $1 \leq j < n$ ($n > 0$)

$$\begin{aligned}
 |\widehat{B}_{j,k}(\lambda) - \widehat{B}(\lambda)| &= \left| \int_0^\infty e^{-\lambda x} dB_{j,k}(x) - \int_0^\infty e^{-\lambda x} dB(x) \right| \\
 &\leq \int_0^\infty \lambda e^{-\lambda x} \underbrace{|B_{j,k}(x) - B(x)|}_{< 5\epsilon \text{ due to (2.20)}} dx < 5\epsilon.
 \end{aligned}
 \tag{2.22}$$

The term $1 - \Pr\{\kappa_{j,k} = 0\}$ of (2.21) is the probability that during the residual service time of the tagged customer there is at least one arrival (and therefore at least one inserted point). The next term $[1 - \widehat{B}(\lambda)]^{m-1} \widehat{B}(\lambda)$ means that along with the first inserted point associated with the tagged customer, there are $m - 1$ other inserted points.

For $j = n$ we have

$$\Pr\{\kappa_{n,k} = m\} = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^m}{m!} dB_{n,k}(x), \quad m = 0, 1, \dots
 \tag{2.23}$$

Let us now consider the time interval $[t_{0,1}, s_{0,1})$ and the number of inserted points $\kappa_{0,1}$. Taking into account that $B_{0,1}(x) = B(x)$ we have

$$\Pr\{\kappa_{0,1} = m\} = [1 - \widehat{B}(\lambda)]^m \widehat{B}(\lambda), \quad m = 0, 1, \dots,
 \tag{2.24}$$

in the case when the loss queueing system is *not* $M/GI/1/0$, and

$$\Pr\{\kappa_{0,1} = m\} = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^m}{m!} dB(x), \quad m = 0, 1, \dots$$

in the case of the $M/GI/1/0$ queueing system.

In the case when the queueing system is not $M/GI/1/0$, for the probability generating function of $\kappa_{0,1}$ from (2.24) we have the representation

$$\sum_{m=0}^{\infty} z^m \Pr\{\kappa_{0,1} = m\} = \frac{\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)}, \quad |z| \leq 1. \tag{2.25}$$

Relation (2.25) has been established in [10] as a particular case of a more general result.

Let us now prove the inequality (2.10). For all $j < n$ and $k \geq 1$ denote:

$$g_{j,k}(z) = \sum_{m=0}^{\infty} z^m \Pr\{\kappa_{j,k} = m\}, \quad |z| \leq 1.$$

Comparing this expression with (2.25) we have as follows.

Let us denote

$$g^*(z) = \frac{\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)}.$$

Then from (2.22) and (2.25) we have

$$\begin{aligned} |g^*(z) - g_{j,k}(z)| &= \left| \sum_{m=0}^{\infty} z^m [\Pr\{\kappa_{0,1} = m\} - \Pr\{\kappa_{j,k} = m\}] \right| \\ &= \left| \widehat{B}(\lambda) - \widehat{B}_{j,k}(\lambda) + \sum_{m=1}^{\infty} z^m [1 - \widehat{B}(\lambda)]^{m-1} \widehat{B}(\lambda) [\widehat{B}_{j,k}(\lambda) - \widehat{B}(\lambda)] \right| \\ &\leq \underbrace{|\widehat{B}(\lambda) - \widehat{B}_{j,k}(\lambda)|}_{<5\epsilon \text{ due to (2.22)}} + \underbrace{|\widehat{B}(\lambda) - \widehat{B}_{j,k}(\lambda)|}_{<5\epsilon \text{ due to (2.22)}} z \sum_{m=0}^{\infty} z^m [1 - \widehat{B}(\lambda)]^m \widehat{B}(\lambda) \\ &< 5\epsilon [1 + zg^*(z)]. \end{aligned}$$

Therefore,

$$\begin{aligned} g_{j,k}(z) &< 5\epsilon [1 + zg^*(z)] + g^*(z) \\ &= \frac{\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)} + 5\epsilon \cdot \frac{1 - z + 2z\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)}, \end{aligned}$$

for all $j < n$ and $k \geq 1$, and we arrive at

$$g(z) = \sup_{j < n, k \geq 1} g_{j,k}(z) < \frac{\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)} + 5\epsilon \cdot \frac{1 - z + 2z\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)}, \tag{2.26}$$

and (2.10) follows. Notice, that from the detailed calculations of (2.6), inequality (2.26) is strong.

Now, in order to finish the proof of the theorem, let us first establish some elementary properties of the class of probability distribution $\{B_y(x)\}$. Recall that the explicit representation for $B_y(x)$ follows from (2.1).

Property 2.7 *The probability distribution $B_y(x)$ can be represented as*

$$B_y(x) = r_y F_y(x) + (1 - r_y) E_\mu(x), \tag{2.27}$$

with parameter $r_y < 1$ depending on y .

The equivalent form of (2.27) is

$$\bar{B}_y(x) = r_y \bar{F}_y(x) + (1 - r_y) e^{-\mu x}.$$

Proof Indeed, we have:

$$r_y \left(\frac{\bar{F}(x+y)}{\bar{F}(y)} \right) + (1 - r_y) e^{-\mu x} = \frac{p \bar{F}(x+y) + (1 - p) e^{-\mu(x+y)}}{p \bar{F}(y) + (1 - p) e^{-\mu(y)}}.$$

After some algebra we see that

$$r_y = \frac{p \bar{F}(y)}{p \bar{F}(y) + (1 - p) e^{-\mu y}},$$

which clearly constitutes that $r_y < 1$. □

We have the following property.

Property 2.8 $G \leq_{c_\lambda} B_y$ for all $y \geq 0$.

Proof Indeed, since $G \leq_{c_\lambda} F_{y^0}$, $G \leq_{c_\lambda} E_\mu$, then, because of $F_{y^0} \leq_{c_\lambda} F_y$, we also have $G \leq_{c_\lambda} F_y$ for all $y \geq 0$. Therefore,

$$r_y G + (1 - r_y) G \leq_{c_\lambda} r_y F_y + (1 - r_y) E_\mu. \tag{2.28}$$

The left-hand side of (2.28) is $G(x)$, while the right-hand side is $B_y(x)$. Therefore, $G \leq_{c_\lambda} B_y$. □

Property 2.8 is crucial. Its consequence is the following result $G \leq_{c_\lambda} B_{n,k}$, where $B_{n,k}(x) = \Pr\{\vartheta_{n,k} \leq x\}$, which follows immediately from this Property 2.8 by application of the formula for the total probability. Indeed,

$$B_{n,k}(x) = \Pr\{\vartheta_{n,k} \leq x\} = \int_0^\infty B_y(x) d\Pr\{\tau_{n,k} \leq y\},$$

and consequently for $j = 0, 1, \dots$,

$$\begin{aligned} \int_0^\infty e^{-\lambda x} x^j B_{n,k}(x) dx &= \int_0^\infty e^{-\lambda x} x^j \left(\int_0^\infty B_y(x) d\Pr\{\tau_{n,k} \leq y\} \right) dx \\ &\leq \int_0^\infty e^{-\lambda x} x^j G(x) dx \int_0^\infty d\Pr\{\tau_{n,k} \leq y\} \\ &= \int_0^\infty e^{-\lambda x} x^j G(x) dx. \end{aligned}$$

Hence, $G \leq_{\mathcal{C}_\lambda} B_{n,k}$.

Therefore, going back to (2.23) we conclude that, according to Lemma A.1 of Appendix A, there exists the minimal random variable ζ in the sense of the stochastic order relation, i.e. $\zeta \leq_{st} \kappa_{n,k}$ for all $k = 1, 2, \dots$, corresponding to the minimal probability distribution function $G(x)$ in the sense of the order relation \mathcal{C}_λ . Specifically,

$$\Pr\{\zeta = m\} = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^m}{m!} dG(x). \tag{2.29}$$

Thus, taking the sequence of independent identically distributed random variables ζ_i , $i = 1, 2, \dots$, all having the same distribution as ζ , one can conclude

$$L_n \geq_{st} \sum_{i=1}^{f(n)} \zeta_i.$$

Therefore, the statement of Theorem 2.4 follows due to stochastic comparison. \square

Corollary 2.9 *In the case $p = \frac{1}{2}$ under Conditions (A) and (D) we have*

$$L_n \geq_{st} \sum_{i=1}^{Z_n} \zeta_i.$$

Z_n denotes the number of offspring in the n th generation of the Galton–Watson branching process with $Z_0 = 1$ and the offspring generating function

$$g(z) < \frac{\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)} + 2\epsilon \cdot \frac{1 - z + 2z\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)}, \quad |z| \leq 1$$

where

$$\widehat{B}(\lambda) = p\widehat{F}(\lambda) + (1 - p) \frac{\mu}{\mu + \lambda},$$

$$\widehat{F}(\lambda) = \int_0^\infty e^{-\lambda x} dF(x).$$

ζ_i , $i = 1, 2, \dots$, is the sequence of nonnegative integer-valued independent identically distributed random variables all having the probability law

$$\Pr\{\zeta_i = m\} = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^m}{m!} dG(x),$$

and $G(x)$ is a probability distribution function that satisfies the inequalities $G \leq_{\mathcal{C}_\lambda} F_{y,0}$ and $G \leq_{\mathcal{C}_\lambda} E_\mu$, where $F_{y,0} \leq_{\mathcal{C}_\lambda} F_y$ for all $y \geq 0$.

Proof As we can see the only difference between the statements of Theorem 2.4 and Corollary 2.9 is in the coefficient before ϵ . Specifically, similarly to (2.20),

$$|B_{j,k}(x) - B(x)| \leq \underbrace{\int_0^\infty \sup_{x \geq 0, y \geq 0} |B_y(x) - B(x)| d\Pr\{\tau_{j,k} \leq y\}}_{< 2\epsilon \text{ due to (2.9)}} < 2\epsilon,$$

and then similarly to (2.22)

$$|\widehat{B}_{i,j}(\lambda) - \widehat{B}(\lambda)| < 2\epsilon$$

as well, where

$$B_{j,k}(x) = \Pr\{\vartheta_{j,k} \leq x\}.$$

The rest part of the proof of the corollary is similar to the proof of Theorem 2.4. \square

3 The number of losses under Condition (B)

In this section it is assumed, additionally to Condition (A) that the probability distribution function $F(x)$ belongs either to the class NBU or to the class NWU. Having this additional assumption, Condition (B) enables us to obtain stronger inequalities than in the previous section under Condition (A).

The lemma below gives a stronger result than characterization Lemma 2.2. Specifically, we have the following lemma.

Lemma 3.1 *Under Condition (B) we have*

$$\mathcal{K}(F, E_\mu) < \epsilon.$$

Proof The proof of this lemma is given in Appendix B. \square

Lemma 3.2 *Under Condition (B) we have*

$$\sup_{x \geq 0, y \geq 0} |B_y(x) - B(x)| < 3\epsilon.$$

Proof We start the proof from representation (2.2). Then, as in the proof of Lemma 2.1, we have (2.3). However, taking into account that $F(x)$ belongs to one of the aforementioned classes NBU and NWU, then instead of the earlier inequality (2.4) given in the proof of Lemma 2.1 we have the stronger inequality

$$|\overline{F}(x) - e^{-\mu x}| < \epsilon, \tag{3.1}$$

which in turn is the result of the application of Lemma 3.1. From (2.2), (2.3), (3.1) and the triangle inequality, for all $x \geq 0, y \geq 0$ we obtain:

$$\begin{aligned}
 & |p\bar{F}(x+y) - p^2\bar{F}(x)\bar{F}(y) - p(1-p)e^{-\mu x}\bar{F}(y)| \\
 & \leq p\bar{F}(y) \left| \frac{\bar{F}(x+y)}{\bar{F}(y)} - e^{-\mu x} \right| + p^2\bar{F}(y) |\bar{F}(x) - e^{-\mu x}| \\
 & \leq p\bar{F}(y) \left| \frac{\bar{F}(x+y)}{\bar{F}(y)} - \bar{F}(x) + \bar{F}(x) - e^{-\mu x} \right| + p^2\bar{F}(y) |\bar{F}(x) - e^{-\mu x}| \\
 & \leq p\bar{F}(y) \left(\left| \frac{\bar{F}(x+y)}{\bar{F}(y)} - \bar{F}(x) \right| + |\bar{F}(x) - e^{-\mu x}| \right) + p^2\bar{F}(y) |\bar{F}(x) - e^{-\mu x}| \\
 & < 2\epsilon p\bar{F}(y) + \epsilon p^2\bar{F}(y) < 3\epsilon p\bar{F}(y), \tag{3.2}
 \end{aligned}$$

$$\begin{aligned}
 & |(1-p)e^{-\mu(x+y)} - p(1-p)e^{-\mu y}\bar{F}(x) - (1-p)^2e^{-\mu(x+y)}| \\
 & \leq (1-p)e^{-\mu y} |e^{-\mu x} - \bar{F}(x)| + (1-p)^2e^{-\mu y} |e^{-\mu x} - \bar{F}(x)| \\
 & < \epsilon(1-p)e^{-\mu y} + \epsilon(1-p)^2e^{-\mu y} < 3\epsilon(1-p)e^{-\mu y}. \tag{3.3}
 \end{aligned}$$

Therefore, from (3.2) and (3.3) for all $x \geq 0, y \geq 0$ we have

$$|B_y(x) - B(x)| < \frac{3\epsilon p\bar{F}(y) + 3\epsilon(1-p)e^{-\mu y}}{p\bar{F}(y) + (1-p)e^{-\mu y}} = 3\epsilon.$$

Lemma 3.2 is proved. □

Remark 3.3 In the special case where $p = \frac{1}{2}$ we have:

$$\begin{aligned}
 & \left| \frac{1}{2}\bar{F}(x+y) - \frac{1}{4}\bar{F}(x)\bar{F}(y) - \frac{1}{4}e^{-\mu x}\bar{F}(y) \right| \\
 & \leq \frac{1}{4}\bar{F}(y) \left| \frac{\bar{F}(x+y)}{\bar{F}(y)} - \bar{F}(x) \right| + \frac{1}{4}\bar{F}(y) \left| \frac{\bar{F}(x+y)}{\bar{F}(y)} - e^{-\mu x} \right| \\
 & \leq \frac{1}{2}\bar{F}(y) \left| \frac{\bar{F}(x+y)}{\bar{F}(y)} - \bar{F}(x) \right| + \frac{1}{4}\bar{F}(y) |\bar{F}(x) - e^{-\mu x}| \\
 & < \frac{3}{4}\epsilon\bar{F}(y), \tag{3.4}
 \end{aligned}$$

$$\begin{aligned}
 & \left| \frac{1}{2}e^{-\mu(x+y)} - \frac{1}{4}e^{-\mu y}\bar{F}(x) - \frac{1}{4}e^{-\mu(x+y)} \right| \\
 & = \frac{1}{4}e^{-\mu y} |e^{-\mu x} - \bar{F}(x)| < \frac{1}{4}\epsilon e^{-\mu y} < \frac{3}{4}\epsilon e^{-\mu y}. \tag{3.5}
 \end{aligned}$$

Therefore, from (3.4) and (3.5) we have:

$$|B_y(x) - B(x)| < \frac{\frac{3}{4}\epsilon\bar{F}(y) + \frac{3}{4}\epsilon e^{-\mu y}}{\frac{1}{2}\bar{F}(y) + \frac{1}{2}e^{-\mu y}} = \frac{3}{2}\epsilon.$$

The main result of this section is the following

Theorem 3.4 *Under Conditions (B) and (D) we have*

$$L_n \geq_{st} \sum_{i=1}^{Z_n} \zeta_i.$$

Z_n denotes the number of offspring in the n th generation of the Galton–Watson branching process with $Z_0 = 1$ and the offspring generating function

$$g(z) < \frac{\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)} + 3\epsilon \cdot \frac{1 - z + 2z\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)}, \quad |z| \leq 1,$$

where $\widehat{B}(\lambda)$ is as in Theorem 2.4. The sequence of independent identically distributed random variables ζ_i , $i = 1, 2, \dots$, is as in Theorem 2.4.

Remark 3.5 If Condition (D) is not satisfied, or the probability distribution $F_{y,0}(x)$ is unknown and cannot be evaluated, then a probability distribution function $G(x)$ should be chosen such that $G \leq_{c_\lambda} F_y$ for all $y \geq 0$ and $G \leq_{c_\lambda} E_\mu$.

Proof The proof of the theorem repeats the proof of corresponding Theorem 2.4. The only difference involves using the estimate given by Lemma 3.2 rather than the estimate given by Lemma 2.1. \square

Corollary 3.6 *In the case $p = \frac{1}{2}$ under Conditions (B) and (D) we have*

$$L_n \geq_{st} \sum_{i=1}^{Z_n} \zeta_i.$$

Z_n denotes the number of offspring in the n th generation of the Galton–Watson branching process with $Z_0 = 1$ and the offspring generating function

$$g(z) < \frac{\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)} + \frac{3\epsilon}{2} \cdot \frac{1 - z + 2z\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)}, \quad |z| \leq 1,$$

where $\widehat{B}(\lambda)$ is as in Theorem 2.4. The sequence of independent identically distributed random variables ζ_i , $i = 1, 2, \dots$, is as in Theorem 2.4.

The proof of Corollary 3.6 is similar to the proof of Theorems 2.4 or 3.4, or Corollary 2.9. The only difference is that the proof of this corollary uses the estimate obtained in Remark 3.3 rather than the estimate of Lemma 3.2.

4 The number of losses under Condition (C)

In this section we study the number of losses under Condition (C). Applying Lemma 3.1 we have the following.

Lemma 4.1 *Under Condition (C)*

$$\mathcal{K}(B, E_\mu) < \epsilon.$$

The main result of the section is the following theorem.

Theorem 4.2 *Under Conditions (C) and (E) we have*

$$\sum_{i=1}^{X_n} \zeta_i \leq_{st} L_n \leq_{st} \sum_{i=1}^{Y_n} v_i. \tag{4.1}$$

X_n denotes the number of offspring in the n th generation of the Galton–Watson branching process with $X_0 = 1$ and the offspring generating function

$$g_X(z) < \frac{\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)} + \epsilon \cdot \frac{1 - z + 2z\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)}, \quad |z| \leq 1, \tag{4.2}$$

and Y_n denotes the number of offspring in the n th generation of the Galton–Watson branching process with $Y_0 = 1$ and the offspring generating function

$$g_Y(z) = \frac{\widehat{B}(\lambda)}{1 - z + z\widehat{B}(\lambda)}, \quad |z| \leq 1.$$

The sequence of independent identically distributed random variables ζ_i , $i = 1, 2, \dots$, is determined as follows:

$$\Pr\{\zeta_i = m\} = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^m}{m!} dB_{y,0}(x), \quad m = 0, 1, \dots, \tag{4.3}$$

where $B_{y,0}(x)$ is the minimal probability distribution function in the sense that $B_{y,0} \leq_{c_\lambda} B_y$ for all $y \geq 0$. The sequence of independent identically distributed random variables v_i , $i = 1, 2, \dots$, in turn is determined as follows:

$$\Pr\{v_i = m\} = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^m}{m!} dB(x), \quad m = 0, 1, \dots \tag{4.4}$$

Remark 4.3 If the probability distribution $B_{y,0}(x)$ does not exist, then we only have the one-side stochastic inequality $L_n \leq_{st} \sum_{i=1}^{Y_n} v_i$.

Proof Notice that under the assumption that $B(x)$ belongs to the class NBU, we have the inequality $\sum_{i=1}^{Y_n} v_i \geq_{st} L_n$, where v_i , $i = 1, 2, \dots$, is a sequence of independent identically distributed random variables, defined by the probability law $\Pr\{v_i = m\} = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^m}{m!} dB(x)$.

Indeed, under this assumption from (2.21) and (2.24) we obtain that $\kappa_{j,k} \leq_{st} \kappa_{0,1}$ for all $0 \leq j < n$ ($n > 0$) and $k \geq 1$ (for details see Abramov [10]), and then the desired inequality $\sum_{i=1}^{Y_n} v_i \geq_{st} L_n$ follows.

In order to prove the second inequality $L_n \geq_{st} \sum_{i=1}^{X_n} \zeta_i$ let us note the following. Assumption (1.3) means that for all $0 \leq j < n$ ($n > 0$), $k \geq 1$ we have $\mathcal{K}(\vartheta_{j,k}, \chi) < \epsilon$. Then using the same arguments as in the proof of Theorem 2.4 we arrive at the left-side of inequality (4.1) with the offspring generating function defined by inequality (4.2). The remaining part of the proof of the theorem is similar to the corresponding part of the proof of Theorem 2.4. \square

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Appendix A: The order relation \mathcal{C}_λ

Let ξ_1, ξ_2 be two nonnegative random variables, and let $\Xi_1(x)$ and $\Xi_2(x)$ be their probability distribution functions respectively.

Definition A.1 The random variable ξ_1 is said to be less than the random variable ξ_2 in the sense of the relation \mathcal{C}_λ (notation: $\xi_1 \leq_{\mathcal{C}_\lambda} \xi_2$ or $\Xi_1 \leq_{\mathcal{C}_\lambda} \Xi_2$) if for fixed parameter $\lambda > 0$ and all $i = 0, 1, \dots$, there is the inequality

$$\int_0^\infty e^{-\lambda x} x^i \Xi_1(x) dx \geq \int_0^\infty e^{-\lambda x} x^i \Xi_2(x) dx.$$

Relation \mathcal{C}_λ is a partial order relation. It follows from the following lemma.

Lemma A.1 Let $\xi_1 \leq_{\mathcal{C}_\lambda} \xi_2$. Then for the random variables θ_1 and θ_2 given by the probability laws:

$$\Pr\{\theta_k = m\} = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^m}{m!} d\Xi_k(x), \quad k = 1, 2; m = 0, 1, \dots,$$

we have $\theta_1 \leq_{st} \theta_2$. Converse, if $\theta_1 \leq_{st} \theta_2$, then $\xi_1 \leq_{\mathcal{C}_\lambda} \xi_2$.

Proof Indeed, by partial integration we have

$$\int_0^\infty e^{-\lambda x} d\Xi_k(x) = \lambda \int_0^\infty e^{-\lambda x} \Xi_k(x) dx \quad (k = 1, 2),$$

and

$$\begin{aligned} & \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^m}{m!} d\Xi_k(x) \\ &= \lambda \left(\int_0^\infty e^{-\lambda x} \frac{(\lambda x)^m}{m!} \Xi_k(x) dx - \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{m-1}}{(m-1)!} \Xi_k(x) dx \right), \end{aligned}$$

$$(m = 0, 1, \dots; k = 1, 2).$$

Therefore, for all $m = 0, 1, \dots$, and $k = 1, 2$, we obtain

$$\lambda \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^m}{m!} \Xi_k(x) dx = \sum_{i=0}^m \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} d\Xi_k(x),$$

and the result follows. □

Appendix B: The proof of Lemmas 2.2 and 3.1

The only difference between the proofs of these lemmas is that there is an additional Condition (B) in Lemma 3.1. Therefore, we provide the general proof of both Lemmas 2.2 and 3.1 with an appropriate specification of the case studies.

We have

$$\bar{F}(x + y) = \bar{F}(y)\bar{F}(x) + h(x, y),$$

where $h(x, y) = F(x) - \Pr\{\zeta \leq x + y \mid \zeta > y\}$. Then,

$$\int_0^\infty \bar{F}(x + y) dy = \frac{1}{\mu} \bar{F}(x) + \frac{1}{\mu} \Theta(x), \tag{B.1}$$

where

$$\Theta(x) = \mu \int_0^\infty \bar{F}(y)h(x, y) dy. \tag{B.2}$$

Denoting

$$\Psi(x) = \int_0^\infty \bar{F}(x + y) dy = \int_x^\infty \bar{F}(y) dy,$$

from (B.1) we obtain the following differential equation

$$\Psi'(x) + \mu\Psi(x) - \Theta(x) = 0. \tag{B.3}$$

The solution of differential equation (B.3), satisfying the initial condition $\Psi(0) = 1/\mu$, is

$$\Psi(x) = \frac{1}{\mu} e^{-\mu x} + \int_0^x e^{-\mu(x-t)} \Theta(t) dt.$$

Hence,

$$\bar{F}(x) = e^{-\mu x} + \mu \int_0^x e^{-\mu(x-t)} \Theta(t) dt - \Theta(x). \tag{B.4}$$

Let us study some properties of (B.4). Note first that $|\Theta(x)| < \epsilon$. Indeed, according to assumption of this lemma $\sup_{x \geq 0, y \geq 0} |h(x, y)| < \epsilon$, and therefore, as it follows from (B.2),

$$|\Theta(x)| \leq \left(\sup_{x \geq 0, y \geq 0} |h(x, y)| \right) \mu \int_0^\infty \bar{F}(y) dy < \epsilon.$$

Therefore,

$$\begin{aligned} \left| \mu \int_0^x e^{-\mu(x-t)} \Theta(t) dt - \Theta(x) \right| &\leq \left| \mu \int_0^x e^{-\mu(x-t)} \Theta(t) dt \right| + |\Theta(x)| \\ &\leq 2 \sup_{0 \leq t \leq x} |\Theta(t)| < 2\epsilon. \end{aligned} \tag{B.5}$$

Substituting (B.5) for (B.3), for all $x \geq 0$ we obtain

$$|\overline{F}(x) - e^{-\mu x}| < 2\epsilon,$$

and the statement of Lemma 2.2 follows.

Under the assumption that $F(x)$ belongs either to the class NBU or to the class NWU, the function $\Theta(x)$ is either nonnegative or nonpositive. This is because $h(x, y)$ contains the term belonging to one of these classes, and therefore is either nonnegative or nonpositive. Therefore, according to (B.2)

$$\left| \mu \int_0^x e^{-\mu(x-t)} \Theta(t) dt - \Theta(x) \right| \leq |\Theta(x)| < \epsilon.$$

Substituting (B) for (B.3), for all $x \geq 0$ we obtain

$$|\overline{F}(x) - e^{-\mu x}| < \epsilon,$$

and Lemma 3.1 is therefore proved as well.

Appendix C: The proof of Lemma 2.6

Let $\mathcal{T} := \{t_n, n \geq 0\}$ denote the time instants at which customers arrive to the system (a customer can arrive without joining queue), where $t_0 = 0$, and let $\{N(C), C \in \mathcal{B}(\mathbb{R})\}$ denote the Poisson process that consists of the collection of points \mathcal{T} .

Next, let $\{T_n, n \geq 1\}$ denote the time instants at which a new service begins in the system. At these time instants either

- (i) A departure occurs and leaves at least one customer behind the system, or
- (ii) An arrival occurs and finds an empty system.

Finally, let $Q(t)$ denote the number of customers in the system at time t .

Let us find $\Pr\{\vartheta_{j,k} \leq x | \tau_{j,k} = y\}$. Suppose $\xi_m = t_{N(T_m)+j-Q(T_m)} - T_m$. Clearly, ξ_m is the remaining amount of time, after the beginning of the m th service, until a new customer arrives, plus the next $j - Q(T_m) - 1$ interarrival times. The instant T_m is chosen such that $s_{j,k-1} \leq T_m < t_{j,k}$, where we set $s_{j,0} := 0$. (The definition of the time instants $s_{j,k}$ and $t_{j,k}$ is given in the proof of Theorem 2.4.) Denoting $\eta = \inf\{m \geq 1 : \chi_m > \xi_m\}$, where χ_m denote the length of the m th service time, one can see that

$$\Pr\{\vartheta_{j,k} \leq x | \tau_{j,k} = y\} = \Pr\{\chi_\eta - \xi_\eta \leq x | \xi_\eta = y\},$$

and

$$\begin{aligned} & \Pr\{\chi_\eta - \xi_\eta \leq x | \xi_\eta = y\} \\ &= \sum_{n=1}^{\infty} \Pr\{\chi_\eta - \xi_\eta \leq x | \eta = n, \xi_n = y\} \Pr\{\eta = n | \xi_\eta = y\} \\ &= \sum_{n=1}^{\infty} \Pr\{\chi_n - \xi_n \leq x | \chi_m < \xi_m, 1 \leq m \leq n - 1, \chi_n > y, \xi_n = y\} \Pr\{\eta = n | \xi_\eta = y\} \\ &= \sum_{n=1}^{\infty} \frac{\Pr\{\chi_n \leq x + y, \chi_n > y | \chi_m < \xi_m, 1 \leq m \leq n - 1, \xi_n = y\}}{\Pr\{\chi_n > y | \chi_m < \xi_m, 1 \leq m \leq n - 1, \xi_n = y\}} \Pr\{\eta = n | \xi_\eta = y\}. \end{aligned}$$

Notice, that each ξ_n is a function of the Poisson arrival process and the random variables $\chi_1, \chi_2, \dots, \chi_{n-1}$, and therefore the random variable χ_n is independent of the vector $(\xi_1, \chi_1, \xi_2, \chi_2, \dots, \xi_{n-1}, \chi_{n-1}, \xi_n)$, and so

$$\frac{\Pr\{\chi_n \leq x + y, \chi_n > y | \chi_m < \xi_m, 1 \leq m \leq n - 1, \xi_n = y\}}{\Pr\{\chi_n > y | \chi_m < \xi_m, 1 \leq m \leq n - 1, \xi_n = y\}} = B_y(x).$$

Hence,

$$\Pr\{\chi_\eta - \xi_\eta \leq x | \xi_\eta = y\} = B_y(x),$$

or

$$\Pr\{\vartheta_{j,k} \leq x | \tau_{j,k} = y\} = B_y(x),$$

and the statement of Lemma 2.6 is proved.

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