

# LOSSES IN $M/GI/m/n$ QUEUES

VYACHESLAV M. ABRAMOV

ABSTRACT. The  $M/GI/m/n$  queueing system with  $m$  homogeneous servers and the finite number  $n$  of waiting spaces is studied. Let  $\lambda$  be the customers arrival rate, and let  $\mu$  be the reciprocal of the expected service time of a customer. Under the assumption  $\lambda = m\mu$  it is proved that the expected number of losses during a busy period is the same value for all  $n \geq 1$ , while in the particular case of the Markovian system  $M/M/m/n$  the expected number of losses during a busy period is  $\frac{m^m}{m!}$  for all  $n \geq 0$ . Under the additional assumption that the probability distribution function of a service time belongs to the class NBU or NWU, the paper establishes simple inequalities for those expected numbers of losses in  $M/GI/m/n$  queueing systems.

## 1. INTRODUCTION

Analysis of loss queueing systems is very important from both the theoretical and practical points of view. While the multiserver loss queueing system  $M/GI/m/0$  and its network extensions have been intensively studied (see the review paper of Kelly [20], the book of Ross [25] and references in these sources), the information about  $M/GI/m/n$  queueing systems ( $n \geq 1$ ) is very scanty, because explicit results for characteristics of these queueing systems are unknown. (In the present paper, for multiserver queueing systems the notation  $M/GI/m/n$  is used, where  $m$  denotes the number of servers and  $n$  denotes the number of waiting places. Another notation which is also acceptable in the literature is  $M/GI/m/m+n$ .)

From the practical point of view,  $M/GI/m/n$  queueing systems serve as a model for telephone systems, where  $n$  is the maximally possible number of calls that can wait in the line before their service start. The loss probability is one of the most significant performance measures. In the present paper, we study the expected number of losses during a busy period (the characteristic closely related to the stationary loss probability) under the assumption that the arrival rate ( $\lambda$ ) is equal to the maximum service capacity ( $m\mu$ ), which seems to be the most interesting from the theoretical point of view.

There are two main reasons for studying this case.

The first reason is that the case  $\lambda = m\mu$  is a critical case for queueing systems with  $m$  identical servers, i.e. the case associated with critically loaded systems. The theoretical and practical interest in studying heavily loaded loss systems is very high, and there are many results in the literature related to the analysis of the loss probability in heavily loaded systems. The asymptotic results for losses in heavily loaded single server systems ( $n \rightarrow \infty$ ) such as  $M/GI/1/n$  and  $GI/M/1/n$

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and for associated models of telecommunication systems and dams have been studied in [4], [8], [9], [11], [14] and [31]. Heavy-traffic analysis of losses in heavily loaded multiserver systems have been provided in [12], [31], [32] and [33]. The mathematical foundation of heavy traffic theory can be found in the textbook of Whitt [30]. Although the case  $\lambda = m\mu$  is idealistic, it enables us to understand the possible behaviour of the system in certain cases when the values  $\lambda$  and  $m\mu$  are close and approach one another as  $n$  increases to infinity. (Obtaining nontrivial results in the cases  $\lambda < m\mu$  and  $\lambda > m\mu$  is a hard problem, so the analytic investigation of the aforementioned asymptotic behaviour as  $n$  increases to infinity is difficult.)

The second reason is that  $\lambda = m\mu$  is an interesting theoretical case associated with an extension of the following non-trivial property of the symmetric random walk. Let  $X_1, X_2, \dots, X_i, \dots$ , be a sequence of independent and identically distributed random variables taking the values  $\pm 1$  with the equal probability  $\frac{1}{2}$ . Let  $S_0 = 0$ , and  $S_{i+1} = S_i + X_{i+1}$ ,  $i \geq 0$ , be a symmetric random walk, and let  $t = \tau$  be the first time instant after  $t = 0$  when this random walk returns to zero, i.e.  $S_\tau = 0$ . It is known that the expected number of level-crossings through any level  $n \geq 1$  (or  $n \leq -1$ ) is equal to  $\frac{1}{2}$  independently of that level. The mentioning of this fact (but in a slightly different formulation) can be found in Szekely [28], and its proof is given in Wolff [35], p.411. The reformulation of this fact in terms of queueing theory is as follows. Consider  $M/M/1/n$  queueing system with equal arrival and service rates. For this system, the expected number of losses during a busy period is equal to 1 for all  $n \geq 0$ . It has been recently noticed that this property holds true for  $M/GI/1/n$  queueing systems. Namely, it was shown in several recent papers (see Abramov [1], [2], [4], Righter [24], Wolff [36]), that under mutually equal expectations of interarrival and service time, the expected number of losses during a busy period is equal to 1 for all  $n \geq 0$ . Further extension of this property to queueing systems with batch arrivals have been given in Abramov [5], Wolff [36] and Peköz, Righter and Xia [23]. Applications of the aforementioned property of losses can be found in [9] for analysis of lost messages in telecommunication systems and in [11] for optimal control of large dams. Further relevant results associated with the properties of losses have been obtained in the paper by Peköz, Righter and Xia [23]. They solved a characterization problem associated with the properties of losses in  $GI/M/1/n$  queues and established similar properties for  $M/M/m/n$  and  $M^X/M/m/n$  queueing systems. Recently, a similar property related to consecutive losses in busy periods of  $M/GI/1/n$  queueing systems has been discussed in [15]. It follows from the results obtained in this paper that for  $M/GI/1/n$  queueing systems with mutually equal expectations of interarrival and service times, the expected number of losses in series containing at least  $k > 1$  consecutive losses during a busy period generally depends on  $n$ . However, for  $M/M/1/n$  queueing systems with equal arrival and service rates that expected number of consecutive losses during a busy period is the same constant (depending on the value  $k$ ) for all  $n \geq 0$ .

The aim of the present paper is further theoretical contribution in this theory of losses, now to the theory of multiserver loss queueing systems. On the basis of the aforementioned results on losses in  $M/GI/1/n$  and  $M/M/m/n$  queueing systems we address the following open question. *Does the result on losses in  $M/M/m/n$  queueing systems remain true for those  $M/GI/m/n$  too?*

The answer on this question is not elementary. On one hand, under the assumption  $\lambda = m\mu$  the expected numbers of losses in  $M/GI/m/0$  and  $M/GI/m/n$  queueing systems ( $m \geq 2$  and  $n \geq 1$ ) during their busy periods are different. A simple example for this confirmation can be built for  $M/GI/2/1$  queueing systems having the service time distribution  $G(x) = 1 - pe^{-\mu_1 x} - qe^{-\mu_2 x}$ ,  $p + q = 1$ . The analysis of the stationary characteristics for these systems, resulting in an analysis of losses during a busy period, can be provided explicitly. Specifically, the structure of the  $9 \times 9$  Markov chain intensity matrix for the states of the Markov chain associated with an  $M/GI/2/1$  queueing system shows a clear difference between the structure of the stationary probabilities in  $M/GI/2/1$  queues and that in  $M/GI/2/0$  queues given by the Erlang-Sevastyanov formulae. So, the parameters  $p$ ,  $q$ ,  $\mu_1$  and  $\mu_2$  can be chosen such that the expected number of losses during busy periods in these two queueing systems will be different.

On the other hand, the property of losses, which is similar to the aforementioned one, indeed holds. The correctness of this similar property for multiserver  $M/GI/m/n$  queueing systems is proved in the present paper. Namely, we establish the following results.

Let  $L_{m,n}$  denote the number of losses during a busy period of the  $M/GI/m/n$  queueing system, let  $\lambda$ ,  $\mu$  be the arrival rate and, respectively, the reciprocal of the expected service time, and let  $m$ ,  $n$  denote the number of servers and, respectively, the number of waiting places. We will prove that, under the assumption  $\lambda = m\mu$ , the expected number of losses during a busy period of the  $M/GI/m/n$  queueing system,  $EL_{m,n}$ , is the same for all  $n \geq 1$ , which is *not* generally the same as that for the  $M/GI/m/0$  loss queueing system (when  $n = 0$ ). In addition, if the probability distribution function of the service time belongs to the class NBU (New Better than Used), then  $EL_{m,n} = \frac{cm^m}{m!}$ , where a constant  $c \geq 1$  is independent of  $n \geq 1$ . In the opposite case of the NWU (New Worse than Used) service time distribution we correspondingly have  $EL_{m,n} = \frac{cm^m}{m!}$  with a constant  $c \leq 1$  independent of  $n \geq 1$  as well. (The constant  $c$  becomes equal to 1 in the case of exponentially distributed service times.) Recall that a probability distribution function  $\Xi(x)$  of a nonnegative random variable is said to belong to the class NBU if for all  $x \geq 0$  and  $y \geq 0$  we have  $\Xi(x+y) \leq \Xi(x)\Xi(y)$ , where  $\bar{\Xi}(x) = 1 - \Xi(x)$ . If the opposite inequality holds, i.e.  $\bar{\Xi}(x+y) \geq \bar{\Xi}(x)\bar{\Xi}(y)$ , then  $\Xi(x)$  is said to belong to the class NWU.

The proof of the main results of this paper is based on an application of the level-crossing approach to the special type stationary processes. The construction of the level-crossings approach used in this paper is a substantially extended version of that used in the earlier papers by the author (e.g. [1], [3], [6], [7], [10] and [13]) and by Pechinkin [22]. It uses modern geometric methods of analysis and involves an algebraically close system of processes and a nontrivial construction of deleting intervals and merging the ends together with nontrivial applications of the PASTA property.

Throughout the paper, it is assumed that  $m \geq 2$ . (This is not the loss of generality since the case  $m = 1$  is known, see [4], [24] and [36].)

The paper is organized as follows. In Section 2, which is the first part of the paper,  $M/M/m/n$  queueing systems are studied. The results obtained for Markovian systems are then used in Section 3, which is the second part of the paper, in order to study  $M/GI/m/n$  queueing systems. The study in both of Sections 2 and 3 is based on the level-crossing approach. The construction of level-crossings

for  $M/M/m/n$  queueing systems is then developed for  $M/GI/m/n$  queueing systems as follows. The stationary processes associated with these queueing systems is considered, and the stochastic relations between the times spent in state  $m - 1$  associated with  $m - 1$  busy servers during a busy period of  $M/GI/m/n$  ( $n \geq 1$ ) and  $M/GI/m - 1/0$  queueing systems are established. To prove these stochastic relations, some ideas from the paper of Pechinkin [22] are involved to adapt and develop the level-crossing method for the problems of the present paper. The obtained stochastic relations are crucial, and they are then used to prove the main results of the paper in Section 4. In Section 5, possible development of the results for  $M^X/GI/m/n$  queueing systems with batch arrivals is discussed.

## 2. THE $M/M/m/n$ QUEUEING SYSTEM

In this section, the Markovian  $M/M/m/n$  loss queueing system is studied with the aid of the level-crossings approach, in order to establish some relevant properties of this queueing system. Those properties are then developed for  $M/GI/m/n$  queueing systems in the following sections.

Let  $f(j)$ ,  $1 \leq j \leq n + m + 1$ , denote the number of customers arriving during a busy period who, upon their arrival, meet  $j - 1$  customers in the system. It is clear that  $f(1) = 1$  with probability 1. Let  $t_{j,1}, t_{j,2}, \dots, t_{j,f(j)}$  be the instants of arrival of these  $f(j)$  customers, and let  $s_{j,1}, s_{j,2}, \dots, s_{j,f(j)}$  be the instants of the service completions when there remain only  $j - 1$  customers in the system. Notice, that  $t_{n+m+1,k} = s_{n+m+1,k}$  for all  $k = 1, 2, \dots, f(n + m + 1)$ .

For  $1 \leq j \leq n + m$  let us consider the intervals

$$(2.1) \quad (t_{j,1}, s_{j,1}], (t_{j,2}, s_{j,2}], \dots, (t_{j,f(j)}, s_{j,f(j)}].$$

Then, by incrementing index  $j$  we have the following intervals

$$(2.2) \quad (t_{j+1,1}, s_{j+1,1}], (t_{j+1,2}, s_{j+1,2}], \dots, (t_{j+1,f(j+1)}, s_{j+1,f(j+1)}].$$

Delete the intervals of (2.2) from those of (2.1) and merge the ends, that is each point  $t_{j+1,k}$  with the corresponding point  $s_{j+1,k}$ ,  $k = 1, 2, \dots, f(j + 1)$  (see Figure 1).

Then  $f(j + 1)$  has the following properties. According to the property of the lack of memory of the exponential distribution, the residual service time for a service completion, after the procedure of deleting the interval and merging the ends as it is indicated above, remains exponentially distributed with parameter  $\mu \min(j, m)$ . Therefore, the number of points generated by merging the ends within the given interval  $(t_{j,1}, s_{j,1}]$  coincides in distribution with the number of arrivals of the Poisson process with rate  $\lambda$  during an exponentially distributed service time with parameter  $\mu \min(j, m)$ . Namely, for  $1 \leq j \leq m - 1$  we obtain

$$E\{f(j + 1)|f(j) = 1\} = \sum_{u=1}^{\infty} u \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^u}{u!} j \mu e^{-j \mu x} dx = \frac{\lambda}{j \mu}.$$

Considering now a random number  $f(j)$  of intervals (2.1) we have

$$(2.3) \quad E\{f(j + 1)|f(j)\} = \frac{\lambda}{j \mu} f(j).$$

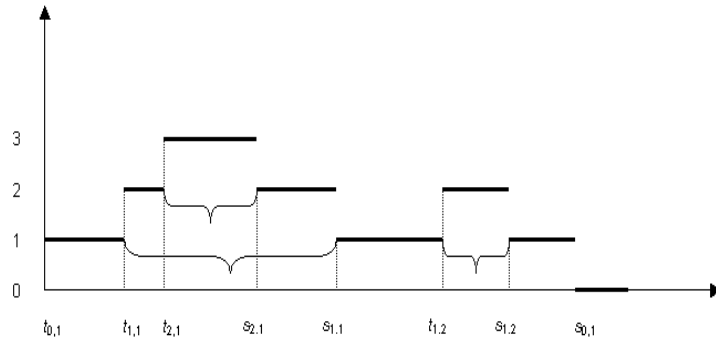


FIGURE 1. Level crossings during a busy period in a Markovian system

Analogously, denoting the load of the system by  $\rho = \frac{\lambda}{m\mu}$ , for  $m \leq j \leq m+n$  we have

$$(2.4) \quad \mathbf{E}\{f(j+1)|f(j)\} = \frac{\lambda}{m\mu}f(j) = \rho f(j).$$

The properties (2.3) and (2.4) mean that the stochastic sequence

$$(2.5) \quad \left\{ f(j+1) \left( \frac{\mu}{\lambda} \right)^j \prod_{i=1}^j \min(i, m), \mathcal{F}_{j+1} \right\}, \quad \mathcal{F}_j = \sigma\{f(1), f(2), \dots, f(j)\},$$

forms a martingale.

It follows from (2.5) that for  $0 \leq j \leq m-1$

$$(2.6) \quad \mathbf{E}f(j+1) = \frac{\lambda^j}{j!\mu^j},$$

and for  $m \leq j \leq m+n$

$$(2.7) \quad \mathbf{E}f(j+1) = \frac{\lambda^m}{m!\mu^m} \rho^{j-m}.$$

For example, when  $\rho = 1$  from (2.7) we obtain the particular case of the result of Peköz, Righter and Xia [23]:  $EL_{m,n} = \mathbf{E}f(n+m+1) = \frac{m^m}{m!}$  for all  $n \geq 0$ , where  $L_{m,n}$  denotes the number of losses during a busy period of the  $M/M/m/n$  queueing system.

Next, let  $B(j)$  be the period of time during a busy cycle of the  $M/M/m/n$  queueing system when there are exactly  $j$  customers in the system. For  $0 \leq j \leq m+n$  we have:

$$(2.8) \quad \lambda \mathbf{E}B(j) = \mathbf{E}f(j+1) = \begin{cases} \frac{\lambda^j}{j!\mu^j}, & \text{for } 0 \leq j \leq m-1, \\ \frac{\lambda^m}{m!\mu^m} \rho^{j-m}, & \text{for } m \leq j \leq m+n. \end{cases}$$

Now, introduce the following notation. Let  $T_{m,n}$  denote the length of a busy period of the  $M/M/m/n$  queueing system, let  $T_{m,0}$  denote the length of a busy period of the  $M/M/m/0$  queueing system with the same arrival and service rates as in the initial  $M/M/m/n$  queueing system, and let  $\zeta_n$  denote the length of a busy period of  $M/M/1/n$  queueing system with arrival rate  $\lambda$  and service rate  $\mu m$ . From (2.6)-(2.8) for the expectation of a busy period of the  $M/M/m/n$  queueing system we have

$$(2.9) \quad \mathbf{E}T_{m,n} = \sum_{j=1}^{n+m} \mathbf{E}B(j) = \sum_{j=1}^{m-1} \frac{\lambda^{j-1}}{j!\mu^j} + \frac{\lambda^{m-1}}{m!\mu^m} \sum_{j=0}^n \rho^j.$$

In turn, for the expectation of a busy period of the  $M/M/m/0$  queueing system we have

$$(2.10) \quad \mathbf{E}T_{m,0} = \sum_{j=1}^m \mathbf{E}B(j) = \sum_{j=1}^m \frac{\lambda^{j-1}}{j!\mu^j},$$

where (2.10) is the particular case of (2.9) where  $n = 0$ .

It is clear that  $T_{m,n}$  contains one busy period  $T_{m-1,0}$ , where the subscript  $m-1$  underlines that there are  $m-1$  servers, and a random number of independent busy periods, which will be called *orbital busy periods*. Denote an orbital busy period by  $\zeta_n$ . (It is assumed that an orbital busy period  $\zeta_n$  starts at instant when an arriving customer finds  $m-1$  servers busy and occupies the  $m$ th server, and it finishes

at the instant when after a service completion there at the first time remain only  $m - 1$  busy servers.) Therefore, denoting the independent sequence of identically distributed orbital busy periods by  $\zeta_n^{(1)}, \zeta_n^{(2)}, \dots$ , we have

$$(2.11) \quad T_{m,n} \stackrel{d}{=} T_{m-1,0} + \sum_{i=1}^{\kappa} \zeta_n^{(i)},$$

where  $\kappa$  is the random number of the aforementioned orbital busy periods and  $\stackrel{d}{=}$  means an equality in distribution. It follows from (2.9), (2.10) and (2.11)

$$(2.12) \quad \mathbb{E} \sum_{i=1}^{\kappa} \zeta_n^{(i)} = \frac{\lambda^{m-1}}{m! \mu^m} \sum_{j=0}^n \rho^j.$$

On the other hand, the expectation of an orbital busy period  $\zeta_n$  is

$$\mathbb{E} \zeta_n = \frac{1}{m\mu} \sum_{j=0}^n \rho^j$$

(this can be easily checked, for example, by the level-crossings method [1], [6] and an application of Wald's identity [16], p. 384), and we obtain

$$(2.13) \quad \mathbb{E} \kappa = \frac{\lambda^{m-1}}{(m-1)! \mu^{m-1}}.$$

Thus,  $\mathbb{E} \kappa$  coincides with the expectation of the number of losses during a busy period in the  $M/M/m/0$  queueing system. In the case  $\rho = 1$  we have  $\mathbb{E} \kappa = \frac{m^m}{m!}$ .

### 3. $M/GI/m/n$ QUEUEING SYSTEMS

In this section, the inequalities between the times spent in the state  $m - 1$  in the  $M/GI/m/n$  ( $n \geq 1$ ) and  $M/GI/m/0$  queueing systems during their busy periods are derived.

Consider two queueing systems:  $M/GI/m/n$  ( $n \geq 1$ ) and  $M/GI/m/0$  both having the same arrival rate  $\lambda$  and probability distribution function of a service time  $G(x)$ ,  $\frac{1}{\mu} = \int_0^\infty x dG(x) < \infty$ . Let  $T_{m,n}(m-1)$  denote the time spent in the state  $m - 1$  during its busy period (i.e. the total time during a busy period when  $m - 1$  servers are occupied) of the  $M/GI/m/n$  queueing system, and let  $T_{m,0}(m-1)$  have the same meaning for the  $M/GI/m/0$  queueing system.

We prove the following lemma.

**Lemma 3.1.** *Under the assumption that the service time distribution  $G(x)$  belongs to the class NBU (NWU),*

$$(3.1) \quad T_{m,n}(m-1) \geq_{st} (\text{resp. } \leq_{st}) T_{m,0}(m-1),$$

*Proof.* . The proof of the lemma is relatively long. In order to make it transparent and easily readable we strongly indicate the steps of this proof given by several propositions (properties). There are also six figures (Figures 2-7) illustrating the constructions in the proof. Each of these figures contain two graphs. The first (upper) of them indicates the initial (or intermediate) possible path of the process (sometimes two-dimensional), while the second (lower) one indicates the part of the path of one or two-dimensional process after a time scaling or specific transformation (e.g. in Figure 5). Arc braces in the graphs indicate the intervals that should be deleted and their ends merged.

Two-dimensional processes are shown as parallel graphs. For example, there are two parallel processes in Figure 3 which are shown in the upper graph, and there are two parallel processes which are shown in the lower graph. The same is in Figures 4, 6 and 7.

For the purpose of the present paper we use strictly stationary processes of order 1 or *strictly 1-stationary processes*. Recall the definition of a strictly stationary process of order  $n$  (see [21], p.206).

**Definition 3.2.** The process  $\xi(t)$  is said to be *strictly stationary of order  $n$*  or *strictly  $n$ -stationary*, if for a given positive  $n < \infty$ , any  $h$  and  $t_1, t_2, \dots, t_n$  the random vectors

$$\left(\xi(t_1), \xi(t_2), \dots, \xi(t_n)\right) \text{ and } \left(\xi(t_1 + h), \xi(t_2 + h), \dots, \xi(t_n + h)\right)$$

have identical joint distributions.

If  $n = 1$  then we have strictly 1-stationary processes satisfying the property:

$$P\{\xi(t) \leq x\} = P\{\xi(t + h) \leq x\}.$$

The probability distribution function  $P\{\xi(t) \leq x\}$  in this case will be called *limiting stationary distribution*.

The class of strictly 1-stationary processes is wider than the class of strictly stationary processes, where it is required that for all finite dimensional distributions

$$P\{(\xi(t_1), \xi(t_2), \dots, \xi(t_k)) \in B_k\} = P\{(\xi(t_1 + h), \xi(t_2 + h), \dots, \xi(t_k + h)) \in B_k\},$$

for any  $h$  and any Borel set  $B_k \subset \mathbb{R}^k$ . The reason of using strictly 1-stationary processes rather than strictly stationary processes themselves is that, the operation of deleting intervals and merging the ends is algebraically close with respect to strictly 1-stationary processes, and it is *not* closed with respect to strictly stationary processes. The last means that if  $\xi(t)$  is a strictly 1-stationary process, then for any  $h > 0$  and arbitrary  $t_0$  the new process

$$\xi_1(t) = \begin{cases} \xi(t), & \text{if } t \leq t_0, \\ \xi(t + h), & \text{if } t > t_0 \end{cases}$$

is also strictly 1-stationary and has the same one-dimensional distribution as the original process  $\xi(t)$ . The similar property is not longer valid for strictly stationary processes. If  $\xi(t)$  is a strictly stationary process, then generally  $\xi_1(t)$  is not strictly stationary.

In the following the prefix ‘strictly’ will be omitted, so strictly stationary and strictly 1-stationary processes will be correspondingly called stationary and 1-stationary processes.

Let us introduce  $m$  independent and identically distributed stationary renewal processes (denoted below  $\mathbf{x}_m(t)$ ) with a renewal period having the probability distribution function  $G(x)$ .

On the basis of these renewal processes we build the stationary  $m$ -dimensional Markov process  $\mathbf{x}_m(t) = \{\xi_1(t), \xi_2(t), \dots, \xi_m(t)\}$ , the coordinates  $\xi_k(t)$ ,  $k = 1, 2, \dots, m$  of which are the residual times to the next renewal times in time moment  $t$ , following in an ascending order.

Let us now consider the two  $(m + 1)$ -dimensional Markov processes corresponding to the  $M/GI/m/n$  ( $n \geq 1$ ) and  $M/GI/m/0$  queueing systems, which are denoted by  $\mathbf{y}_{m,n}(t)$  and  $\mathbf{y}_{m,0}(t)$ . Let  $Q_{m,n}(t)$  denote the stationary queue-length

process (the number of customers in the system) of the  $M/GI/m/n$  queueing system, and let  $Q_{m,0}(t)$  denote the stationary queue-length process corresponding to the  $M/GI/m/0$  queueing system. We have:

$$\mathbf{y}_{m,n}(t) = \left\{ \eta_1^{(m,n)}(t), \eta_2^{(m,n)}(t), \dots, \eta_m^{(m,n)}(t), Q_{m,n}(t) \right\},$$

where

$$\left\{ \eta_{m-\nu_{m,n}(t)+1}^{(m,n)}(t), \eta_{m-\nu_{m,n}(t)+2}^{(m,n)}(t), \dots, \eta_m^{(m,n)}(t) \right\}$$

are the ordered residual service times corresponding to  $\nu_{m,n}(t) = \min\{m, Q_{m,n}(t)\}$  customers in service in time  $t$ , and

$$\left\{ \eta_1^{(m,n)}(t), \eta_2^{(m,n)}(t), \dots, \eta_{m-\nu_{m,n}(t)}^{(m,n)}(t) \right\} = \{0, 0, \dots, 0\}$$

all are zeros. Analogously,

$$\mathbf{y}_{m,0}(t) = \left\{ \eta_1^{(m,0)}(t), \eta_2^{(m,0)}(t), \dots, \eta_m^{(m,0)}(t), Q_{m,0}(t) \right\},$$

only replacing the index  $n$  with 0.

Let us delete all time intervals of the process  $\mathbf{y}_{m,n}(t)$  related to the  $M/GI/m/n$  queueing system ( $n \geq 1$ ) where there are more than  $m-1$  or less than  $m-1$  customers and merge the ends. Remove the last component of the obtained process which is trivially equal to  $m-1$ . Then we get the new  $(m-1)$ -dimensional Markov process (in the following the prefix ‘Markov’ will be omitted and only used in the places where it is meaningful):

$$\widehat{\mathbf{y}}_{m-1,n}(t) = \left\{ \widehat{\eta}_1^{(m,n)}(t), \widehat{\eta}_2^{(m,n)}(t), \dots, \widehat{\eta}_{m-1}^{(m,n)}(t) \right\},$$

the components of which are now denoted by hat. All of the components of this vector are 1-stationary, which is a consequence of the existence of the limiting stationary probabilities of the processes  $\eta_j^{(m,n)}(t)$ ,  $j = 1, 2, \dots, m$  (e.g. Takács [29]) and consequently those of the processes  $\widehat{\eta}_j^{(m,n)}(t)$ ,  $j = 1, 2, \dots, m$ . The joint limiting stationary distribution of the process  $\widehat{\mathbf{y}}_{m-1,n}(t)$  can be obtained by conditioning of that of the processes  $\mathbf{y}_{m,n}(t)$  given  $Q_{m,n}(t) = m-1$ .

The similar operation of deleting intervals and merging the ends, where there are less than  $m-1$  customers in the system, for the process  $\mathbf{y}_{m-1,0}(t)$  is used. We correspondingly have

$$\widehat{\mathbf{y}}_{m-1,0}(t) = \left\{ \widehat{\eta}_1^{(m,0)}(t), \widehat{\eta}_2^{(m,0)}(t), \dots, \widehat{\eta}_{m-1}^{(m,0)}(t) \right\}.$$

We establish the following elementary property related to the  $M/GI/1/n$  queueing systems,  $n=0,1,\dots$

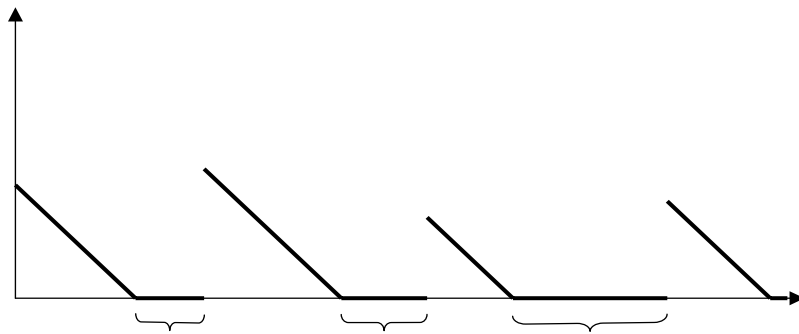
**Property 3.3.**

$$(3.2) \quad \mathbb{P} \left\{ \eta_1^{(1,n)}(t) \in B_1 \mid Q_{1,n}(t) \geq 1 \right\} = \mathbb{P} \{ \mathbf{x}_1(t) \in B_1 \},$$

for any Borel set  $B_1 \subset \mathbb{R}^1$ .

*Proof.* Delete all of the intervals where the server is free and merge the corresponding ends (see Figure 2). Then in the new time scale, the processes all are structured as a stationary renewal process with the length of a period having the probability distribution function  $G(x)$ . Therefore (3.2) follows.  $\square$

**Residual service times in the original  $M/GI/1/n$  queueing system**



**Residual service times of the scaled process in the  $M/GI/1/n$  queueing system**

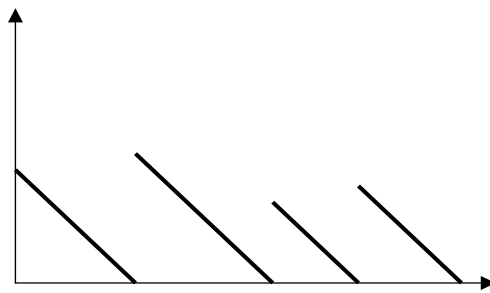


FIGURE 2. Residual service times for the original and scaled processes of the  $M/GI/1/n$  queueing system.

In order to establish similar properties in the case  $m = 2$  let us first study the properties of 1-stationary processes and explain the construction of *tagged server station* which is substantially used in our construction throughout the paper.

*Properties of 1-stationary processes.* Recall (see Definition 3.2) that if  $\xi(t)$  is a 1-stationary process, then for any  $h$  and  $t_0$  the probability distributions of  $\xi(t_0)$  and  $\xi(t_0 + h)$  are the same. The result remains correct (due to the total probability formula) if  $h$  is replaced by random variable  $\vartheta$  with some given probability distribution, which is assumed to be independent of the process  $\xi$ . Namely, we have:

$$\begin{aligned}
 \mathbb{P}\{\xi(t_0 + \vartheta) \leq x\} &= \int_{-\infty}^{\infty} \mathbb{P}\{\xi(t_0 + h) \leq x\} d\mathbb{P}\{\vartheta \leq h\} \\
 (3.3) \qquad \qquad \qquad &= \mathbb{P}\{\xi(t_0) \leq x\} \int_{-\infty}^{\infty} d\mathbb{P}\{\vartheta \leq h\} \\
 &= \mathbb{P}\{\xi(t_0) \leq x\}.
 \end{aligned}$$

That is,  $\xi(t_0)$  and  $\xi(t_0 + \vartheta)$  have the same distribution.

The above property will be used for the following *construction of the sequence of 1-stationary processes*  $\xi^{(1)}(t)$ ,  $\xi^{(2)}(t)$ ,  $\dots$ , having identical one-dimensional distributions.

Let  $\xi^{(0)}(t) = \xi(t)$  be a 1-stationary process, let  $t_1$  be an arbitrary point, and let  $\vartheta_1$  be a random variable with some given probability distribution, which is independent of the process  $\xi^{(0)}(t)$ . Let us build a new process  $\xi^{(1)}(t)$  as follows. Put

$$\xi^{(1)}(t) = \begin{cases} \xi^{(0)}(t), & \text{for all } t < t_1, \\ \xi^{(0)}(t + \vartheta_1), & \text{for all } t \geq t_1. \end{cases}$$

Since the probability distributions of  $\xi(t)$  and  $\xi(t + \vartheta_1)$  are the same for all  $t \geq t_1$ , then the processes  $\xi(t)$  and  $\xi^{(1)}(t)$  have the same one-dimensional distributions, and  $\xi^{(1)}(t)$  is a 1-stationary process as well.

With a new point  $t_2$  and a random variable  $\vartheta_2$ , which is assumed to be independent of the process  $\xi^{(0)}(t)$  and random variable  $\vartheta_1$  (therefore, it is also independent of the process  $\xi^{(1)}(t)$ ) by the same manner one can build the new 1-stationary process  $\xi^{(2)}(t)$ . Specifically,

$$\xi^{(2)}(t) = \begin{cases} \xi^{(1)}(t), & \text{for all } t < t_2, \\ \xi^{(1)}(t + \vartheta_2), & \text{for all } t \geq t_2. \end{cases}$$

The new process  $\xi^{(2)}(t)$  has the same one-dimensional distribution as the processes  $\xi^{(0)}(t)$  and  $\xi^{(1)}(t)$ . The procedure can be infinitely continued, and one can obtain the infinite family of 1-stationary processes, having the same one-dimensional distribution.

The points  $t_1, t_2, \dots$  in the above construction are assumed to be some fixed (non-random) points. However, the construction also remains correct in the case of random points  $t_0, t_1, \dots$  of Poisson process, since according to the PASTA property [34] the limiting stationary distribution of a 1-stationary process in a point of a Poisson arrival coincides with the limiting stationary distribution of the same 1-stationary process in an arbitrary non-random point.

*Tagged server station.* Consider a stationary queueing system  $M/GI/m/n$ , which is referred to as *main server station*, and in addition to this queueing system introduce another one containing a server station in order to register specific arrivals, for example losses or, say, customers waiting their service in the main system. This server station is called *tagged server station*. The main idea of introducing tagged server station is to decompose the main system as follows. Assume that along with a Poisson stream of arrivals of customers occupying servers in the main system, there is another Poisson stream of arrivals of customers in the tagged server system. For instance, the losses in the main system can be supposed to occupy the tagged server station. In fact, application of a tagged server station is wider than that, and its aim is a proper decomposition of the original system into main and tagged systems for appropriate study of the properties of losses.

Another idea of using tagged server stations is a proper application of the PASTA property as follows. At the moment of arrival of a customer in the tagged server station, the stationary characteristics in the main server station remain the same. Specifically, the distributions of residual service times in servers of the main station at the moment of arrival of a customer in the tagged station coincides with the usual stationary distributions of these residual service times.

Let us now formulate and prove a property similar to Property 3.3 for  $m = 2$ . We have the following.

**Property 3.4.** *For the  $M/GI/2/0$  queueing system we have:*

$$(3.4) \quad \mathbb{P}\{\widehat{\mathbf{y}}_{1,0}(t) \in B_1\} = \mathbb{P}\{\mathbf{x}_1(t) \in B_1\},$$

*Proof.* In order to simplify the explanation in this case, let us consider two auxiliary stationary one-dimensional processes  $\zeta_{1,0}(t)$  and  $\zeta_{2,0}(t)$ . The first process describes a residual service time in the first server, and the second one describes a residual service time in the second server. If the  $i$ th server ( $i = 1, 2$ ) is free in time  $t$ , then we set  $\zeta_{i,0}(t) := 0$ . The stationary processes  $\zeta_{1,0}(t)$  and  $\zeta_{2,0}(t)$  are assumed to be equivalent each other in the sense that if at the moment of arrival of a customer both of servers are free, then each of these servers can be occupied with the equal probability  $\frac{1}{2}$ .

Thus we have two alternating stationary renewal processes having service periods and idle times.

Let us call the first server station a *tagged server station*. Then we have two systems, one of which is a system consisting of a tagged server station with one Poisson input and the  $M/GI/1/0$  queueing system with another Poisson input. According to elementary properties of Poisson processes, these two Poisson inputs can be supposed to be independent. (There is the stationary probability that an arriving customer occupies the first server, which is tagged. So, original Poisson process can be thinned into two independent Poisson processes.)

Let  $\tau$  be the time moment when an arriving customer occupies the tagged server station. Then, according to the PASTA property,

$$(3.5) \quad \mathbb{P}\{\zeta_{2,0}(\tau) \leq x\} = \mathbb{P}\{\zeta_{2,0}(t) \leq x\},$$

where  $t$  is an arbitrary fixed point, and the probability distribution function of  $\zeta_{2,0}(t)$  in this point coincides with the distribution of residual service time in specific  $M/GI/1/0$  system with some specific value of parameter of Poisson process, which is not important here. On the other hand, the process  $\zeta_{2,0}(t)$  is stationary and

Markov. Therefore from (3.5) for any  $h > 0$  we have

$$(3.6) \quad \mathbb{P}\{\zeta_{2,0}(\tau + h) \leq x\} = \mathbb{P}\{\zeta_{2,0}(t + h) \leq x\} = \mathbb{P}\{\zeta_{2,0}(t) \leq x\}.$$

Let  $\chi$  denotes the service time of the customer, who arrives at the aforementioned time moment  $\tau$  occupying the tagged server station. The random variable  $\chi$  is independent of the process  $\zeta_{2,0}(t)$ . Therefore, according to the properties of 1-stationary processes,

$$(3.7) \quad \mathbb{P}\{\zeta_{2,0}(\tau + \chi) \leq x\} = \mathbb{P}\{\zeta_{2,0}(\tau) \leq x\}.$$

The only difference between (3.7) and the aforementioned basic property is that the time moment  $\tau$  is random, while in the basic property it is not. However keeping in mind (3.6), this modified equation (3.7) follows by the same derivation as in (3.3).

Therefore, from (3.5),

$$\mathbb{P}\{\zeta_{2,0}(\tau + \chi) \leq x\} = \mathbb{P}\{\zeta_{2,0}(t) \leq x\}.$$

In our construction we let the first server be tagged server station. Clearly, that nothing would be changed if we assumed that the second server was a tagged server station while the first server main, since the servers were equivalent. Thus, *in any arrival or the service completion time instant in one server (which can be called tagged), the residual service time in another server has the same stationary distribution.* This fact is used in constructions below.

Now considering the stationary  $M/GI/2/0$  queueing system, let us delete the time intervals where the both servers are simultaneously free, and merge the corresponding ends (see Figure 3). The new processes are denoted by  $\tilde{\zeta}_{1,0}(t)$  and  $\tilde{\zeta}_{2,0}(t)$ . This two-dimensional 1-stationary process characterizes the system where in any time  $t$  at least one of two servers is busy. Consider the event of arrival of a customer in the stationary system when only one server is busy. Let  $\tau^*$  be the moment of this arrival, and let  $\tau^{**}$  denote the moment of the first service completion in one of two servers following after the moment  $\tau^*$ . Then, at the endpoint  $\tau^{**}$  of the interval  $[\tau^*, \tau^{**})$  the distribution of the residual service time will be the same as that at the moment  $\tau^*$  (due to the established fact that at the end of a service completion in one server, the distribution of a residual service time in another server must coincide with the stationary distribution of a residual service time).

The additional details here are as follows. There can be different events associated with the points  $\tau^*$  and  $\tau^{**}$ . For example, at the moment  $\tau^*$  an arriving customer can be accepted by one of the servers, while the service completion at the moment  $\tau^{**}$  can be either in the same server or in another server. If time moments  $\tau^*$  and  $\tau^{**}$  are associated with the same server (for example, the moment of service beginning and service completion in the first server) then we speak about residual service times in another server (in the given case in the second one). If time moments  $\tau^*$  and  $\tau^{**}$  are associated with different servers (say,  $\tau^*$  is the service beginning in the first server, but  $\tau^{**}$  is the service completion in the second one), then we speak about residual service times in different servers (in this specific case we speak about residual service time in the second server at the time moment  $\tau^*$  and residual service time in the first server at the time moment  $\tau^{**}$ ).

Deleting the interval  $[\tau^*, \tau^{**})$  and merging the ends  $\tau^*$  and  $\tau^{**}$  (see Figure 4) we obtain the following structure of the 1-stationary process  $\hat{\mathbf{y}}_{1,0}(t)$ .

In the points where idle intervals are deleted and the ends are merged we have renewal points: one of periods is finished and another is started. In the other points

where the intervals of type  $[\tau^*, \tau^{**})$  are deleted and their ends are merged we have the points of ‘interrupted’ renewal processes, where because  $\tau^*$  is the moment of Poisson arrival, according to the PASTA property the stationary distribution of the residual time started from the merged point  $\tau^*$  until the next renewal point coincides with the limiting stationary distribution of the renewal process, i.e. with the distribution of  $\mathbf{x}_1(t)$ . (Notice, that the intervals of type  $[\tau^*, \tau^{**})$  are an analogue of the intervals  $[s_{1,k}, t_{1,k})$  considered in the Markovian case in Section 2.)

By amalgamating the residual service times of the first and second servers given in the the lower graph in Figure 4, one can built a typical one-dimensional 1-stationary process  $\hat{\mathbf{y}}_{1,0}(t)$ , the limiting stationary distribution of which coincides with that of  $\mathbf{x}_1(t)$ . (see Figure 5).

Therefore the processes  $\hat{\mathbf{y}}_{1,0}(t)$  and  $\mathbf{x}_1(t)$  have the identical one-dimensional distribution, and relation (3.4) is proved.  $\square$

Let us develop Property 3.4 to the case  $m = 3$  and then to the case of an arbitrary  $m > 1$  for the  $M/GI/m/0$  queueing systems. Namely, we have the following.

**Property 3.5.** *For the  $M/GI/m/0$  queueing system we have:*

$$(3.8) \quad \mathbb{P}\{\hat{\mathbf{y}}_{m-1,0}(t) \in B_{m-1}\} = \mathbb{P}\{\mathbf{x}_{m-1}(t) \in B_{m-1}\},$$

where  $B_{m-1}$  is an arbitrary Borel set of  $\mathbb{R}^{m-1}$ .

*Proof.* The proof will be concentrated in the case  $m = 3$  for the 1-stationary process  $\hat{\mathbf{y}}_{2,0}(t)$ , which is associated with the paths of the  $M/GI/3/0$  queueing system where only two servers are busy. Then the result will be concluded for an arbitrary  $m \geq 2$  by induction.

Prior studying this case, we first study the specific case of the  $M/GI/2/0$  queueing system by considering the paths when the both servers are busy. Then using arguments of the proof of Property 3.4 enables us to extend that specific result related to the  $M/GI/2/0$  queueing systems to the 1-stationary process  $\hat{\mathbf{y}}_{2,0}(t)$  of the  $M/GI/3/0$  queueing system.

As in the proof of Property 3.4 in the specific case of the  $M/GI/2/0$  queueing system considered here, we will study the stationary one-dimensional processes  $\zeta_{1,0}(t)$  and  $\zeta_{2,0}(t)$ . However the idea of the present proof generally differs from that of the proof of Property 3.4. Here we do not call the first (or second) server a tagged server station to use decomposition. We simply use the fact established in the proof of Property 3.4 that at the moment of arrival or service completion of a customer in one server, the distribution of a residual service time in another server will coincide with the stationary distribution of a residual service time in this server. (The same idea was used in the proof Property 3.4.)

The present proof explicitly uses the fact that the class of 1-stationary processes is algebraically closed with respect to the operations of deleting intervals and merging the ends, which was mentioned before.

Let us delete the idle intervals of the process  $\zeta_{1,0}(t)$  and merge the ends. Then we get a stationary renewal process as in the above case  $m = 1$  (Property 3.3).

After deleting the same time intervals in the second stationary process  $\zeta_{2,0}(t)$  and merging the ends, the process will be transformed as follows. Let  $t^*$  be a moment of Poisson arrival when a customer occupies the first server. (Recall that owing to the known properties of Poisson process, the stream of arrival to each of  $i$  servers ( $i = 1, 2$ ) is Poisson.) Then, according to the PASTA property,  $\zeta_{2,0}(t^*) = \zeta_{2,0}(t)$  in

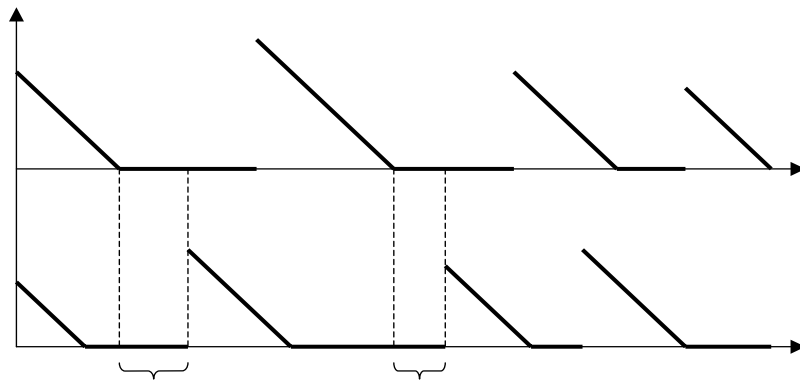
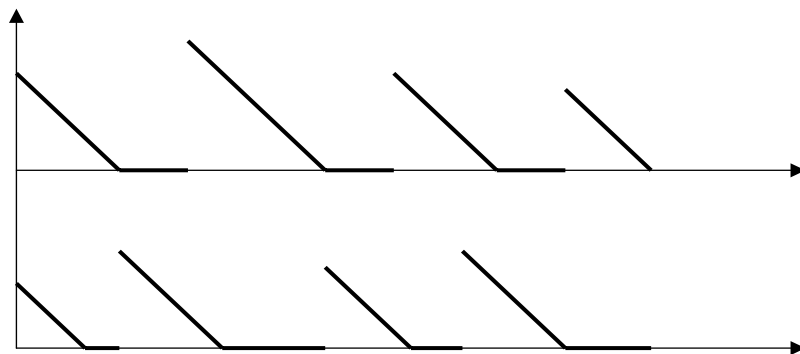
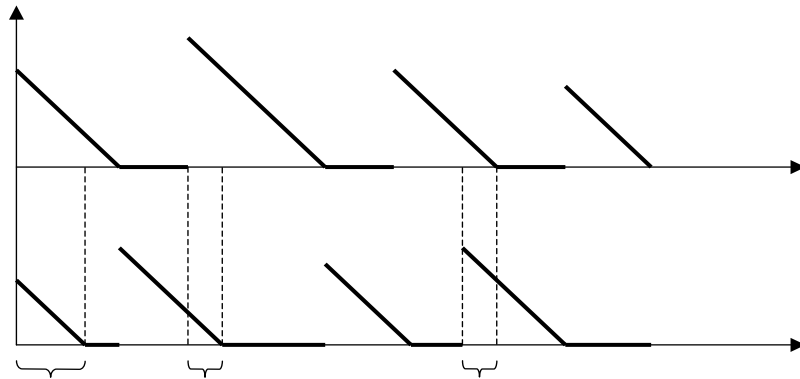
Residual service times in the original  $M/GI/2/0$  queueing systemResidual service times of the scaled process in the  $M/GI/2/0$  queueing system after deleting the intervals when both of the servers are idle, and merging the ends

FIGURE 3. Residual service times for the original and scaled processes of the  $M/GI/2/0$  queueing system after deleting intervals where both of the servers are free, and merging the ends.

**Residual service times of the scaled process in the  $M/GI/2/0$  queueing system after deleting the intervals when both of the servers are idle, and merging the ends**



**Residual service times of the scaled process in the  $M/GI/2/0$  queueing system after deleting the intervals when both of the servers are idle and both are busy, and merging the ends**

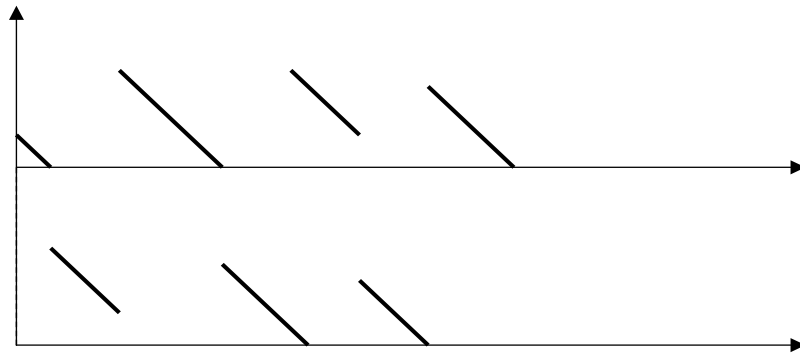


FIGURE 4. Residual service times for the original and scaled processes of the  $M/GI/2/0$  queueing system after deleting intervals where both of the servers are free and both are busy, and merging the ends.

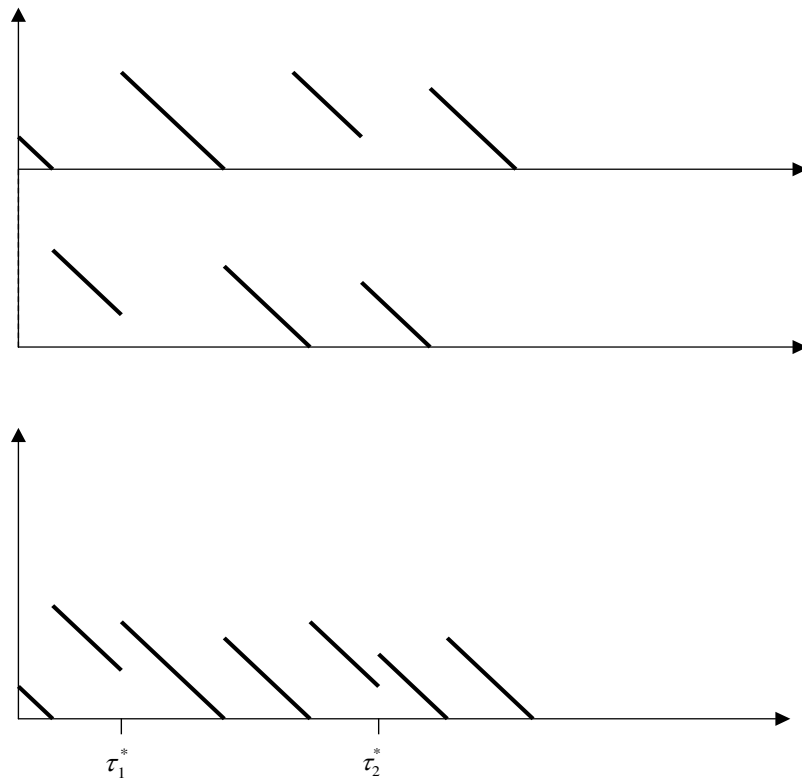


FIGURE 5. A typical 1-stationary process of residual service times is obtained by amalgamating residual service times of the first and second servers.  $\tau_1^*$  and  $\tau_2^*$  are the points where the intervals of type  $[\tau^*, \tau^{**})$  are deleted, and the ends are merged

distribution. Therefore after deleting all of the idle intervals of the second server and merging the ends, after the first time scaling (i.e. removing corresponding time intervals, see Figure 6) instead of the initial 1-stationary process  $\zeta_{2,0}(t)$  we obtain the new 1-stationary process with the equivalent one-dimensional distribution. This process is denoted by  $\tilde{\zeta}_{2,0}(t)$ .

Notice, that the process  $\tilde{\zeta}_{2,0}(t)$  is obtained from the process  $\zeta_{2,0}(t)$  by constructing a sequence of 1-stationary processes described above.

Then we have the two-dimensional process the first component of which is  $\mathbf{x}_1(t)$  and the second one is  $\tilde{\zeta}_{2,0}(t)$ . For our convenience this first component is provided with upper index, and the two-dimensional vector looks now as  $\{\mathbf{x}_1^{(1)}(t), \tilde{\zeta}_{2,0}(t)\}$ .

Let us repeat the above procedure, deleting the remaining idle intervals of the second server and merging the ends. We get the 1-stationary process being equivalent in the distribution to the stationary renewal process  $\mathbf{x}_1(t)$ , which is denoted now  $\mathbf{x}_1^{(2)}(t)$ .

Upon this (final) time scaling the first process  $\mathbf{x}_1^{(1)}(t)$  is transformed as follows. Let  $t^{**}$  be a random point of Poisson arrival when the second server is occupied. Applying the PASTA property once again, for the first component of the process we obtain that  $\mathbf{x}_1^{(1)}(t^{**})$  coincides in one-dimensional distribution with  $\mathbf{x}_1^{(1)}(t)$ . Thus, after deleting the entire idle intervals and merging the ends, we finally obtain the two-dimensional process  $\{\mathbf{x}_1^{(1)}(t), \mathbf{x}_1^{(2)}(t)\}$  each component of which has the same one-dimensional distribution as this of the process  $\mathbf{x}_1(t)$ . The dynamic of this time scaling is shown in Figure 7.

The next result that we prove now is independence of the processes  $\mathbf{x}_1^{(1)}(t)$  and  $\mathbf{x}_1^{(2)}(t)$ . To prove this independence, we again use the PASTA property and existence of the limiting stationary distributions independently of the initial conditions (see [29]).

Indeed, in each point of the Poisson arrival where one process changes its value, the other process has the same distribution as  $\mathbf{x}_1(t)$ , and then in an arbitrary point  $t$  each of the processes has the same distribution as  $\mathbf{x}_1(t)$ . Then numbering the merged points of the process  $\mathbf{x}_1^{(1)}(t)$  as  $t_1^{**}, t_2^{**}, \dots$ , we have:

$$\begin{aligned}
 \text{P}\left\{\mathbf{x}_1^{(1)}(t) \leq x_1, \mathbf{x}_1^{(2)}(t) \leq x_2\right\} &= \lim_{j \rightarrow \infty} \text{P}\left\{\mathbf{x}_1^{(1)}(t_j^{**}) \leq x_1, \mathbf{x}_1^{(2)}(t) \leq x_2\right\} \\
 (3.9) \qquad \qquad \qquad &= \lim_{j \rightarrow \infty} \text{P}\left\{\mathbf{x}_1^{(1)}(t_j^{**}) \leq x_1\right\} \text{P}\left\{\mathbf{x}_1^{(2)}(t) \leq x_2\right\} \\
 &= \text{P}\{\mathbf{x}_1(t) \leq x_1\} \text{P}\{\mathbf{x}_1(t) \leq x_2\}.
 \end{aligned}$$

The first equality of (3.9) is a consequence of the PASTA property given from the above construction. The second equality of (3.9) is a consequence of the existence of the limiting stationary one-dimensional distribution of the processes  $\mathbf{x}_1^{(1)}(t)$  and  $\mathbf{x}_1^{(2)}(t)$ . The point  $t$  is fixed, while the sequence of points  $t_j^{**}$  increases unboundedly. Since the limiting distribution is independent of point  $t$  we have the second equality of (3.9). The last equality of (3.9) is, again, a consequence of the PASTA property and the fact that each of the processes  $\mathbf{x}_1^{(1)}(t)$  and  $\mathbf{x}_1^{(2)}(t)$  has the same limiting stationary distribution as the process  $\mathbf{x}_1(t)$ .

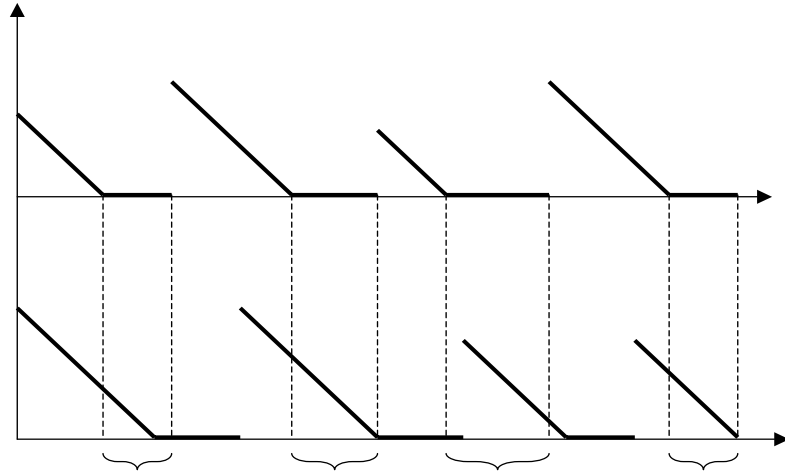
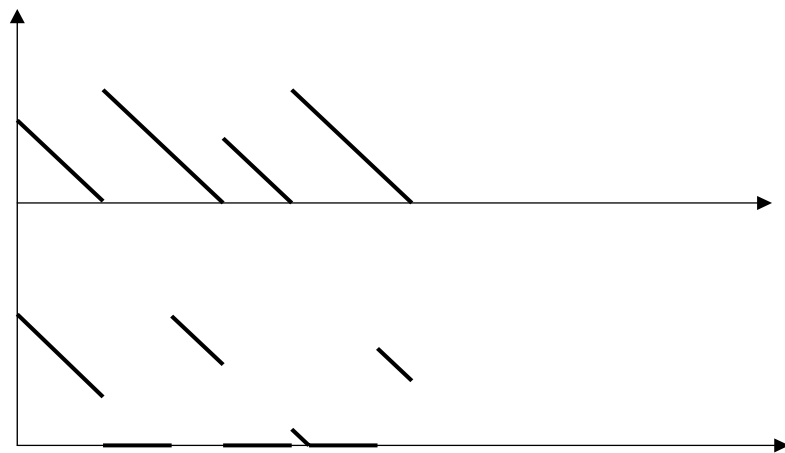
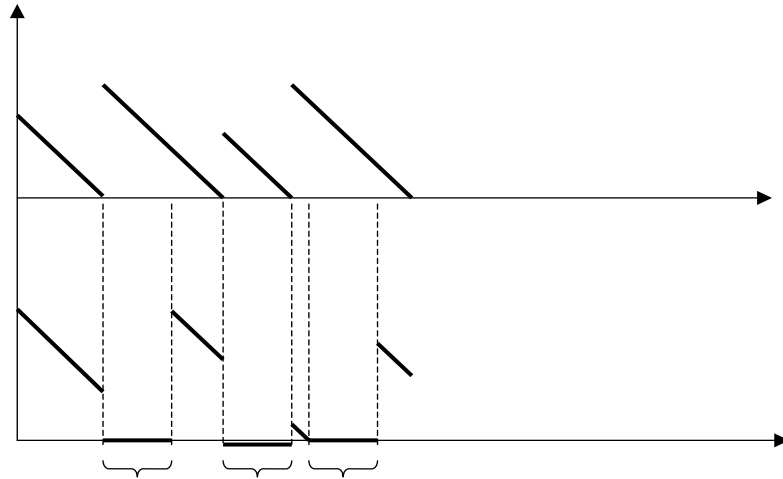
The processes  $\zeta_{1,0}(t)$  and  $\zeta_{2,0}(t)$ The transformed processes  $\zeta_{1,0}(t)$  and  $\zeta_{2,0}(t)$  after deleting the intervals and merging the ends

FIGURE 6. The dynamic of time scaling for a queueing system with two servers after deleting the idle intervals in the first server and merging the ends

The processes  $x_1(t)$  and  $\tilde{\zeta}_{2,0}(t)$



The processes  $x_1^{(1)}(t)$  and  $x_1^{(2)}(t)$  obtained from the processes  $x_1(t)$  and  $\tilde{\zeta}_{2,0}(t)$  by the scaling procedure

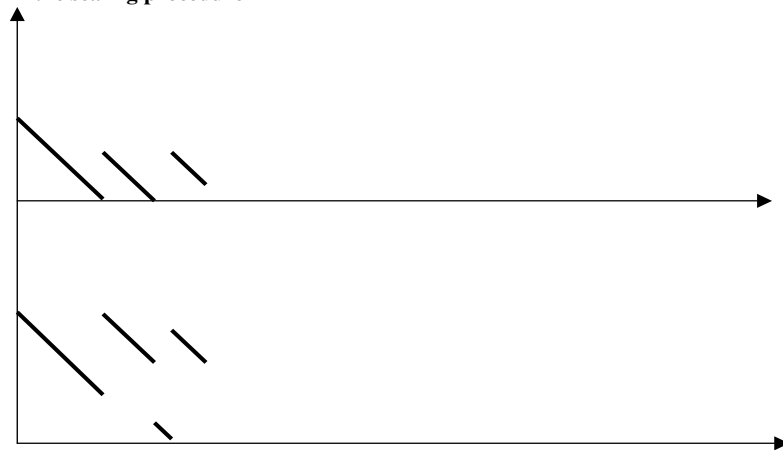


FIGURE 7. The dynamic of time scaling for a queueing system with two servers after deleting the idle intervals in the second server and merging the ends

Now, using the arguments of the proof of Property 3.4 one can easily extend the result obtained now for  $M/GI/2/0$  queueing system to the  $M/GI/3/0$  queueing system, and thus prove (3.8) for the  $M/GI/3/0$  queueing system.

Similarly to the proof of Property 3.4, let us introduce the processes  $\zeta_{1,0}(t)$ ,  $\zeta_{2,0}(t)$  and  $\zeta_{3,0}(t)$  of the residual service times in the first, second and third servers correspondingly. These processes all are assumed to have the same stationary distribution of residual times, which respects to the scheme where an arriving customer occupies one of available free servers with equal probability. Let us call the first server of the  $M/GI/3/0$  queueing system a tagged server station. So, we decompose the original system into the  $M/GI/2/0$  and tagged queueing system. According to the above result, where after deleting idle intervals and merging the corresponding ends in the  $M/GI/2/0$  queueing system we obtained the process which was equivalent in distribution to the stationary process  $\mathbf{x}_2(t)$ . This stationary distribution will remain the same in all random points of arrivals and service completions in the tagged server station. So, after deleting and merging the corresponding intervals in tagged server station, we arrive at the same stationary distribution, which is the same as the distribution of  $\mathbf{x}_2(t)$ . Finally, it should be noted that, because all of the processes  $\zeta_{1,0}(t)$ ,  $\zeta_{2,0}(t)$  and  $\zeta_{3,0}(t)$  are assumed to have the same distribution, then any of the servers can be called tagged and we arrive at the same result. So, the result for  $m = 3$  follows.

This induction becomes clear for an arbitrary  $m \geq 2$  as well, where the original  $M/GI/m/0$  system can be decomposed into the  $M/GI/m - 1/0$  queueing system and a tagged server station.  $\square$

Now we will establish a connection between the processes  $\hat{\mathbf{y}}_{m-1,n}(t)$ ,  $\hat{\mathbf{y}}_{m-1,0}(t)$  and  $\mathbf{x}_{m-1}(t)$ . We start from the case  $m = 2$ .

**Property 3.6.** *Under the assumption that the probability distribution function  $G(x)$  belongs to the class NBU (NWU) we have*

$$(3.10) \quad \mathbb{P}\{\hat{\mathbf{y}}_{1,n}(t) \leq x\} \leq (\text{resp. } \geq) \mathbb{P}\{\mathbf{x}_1(t) \leq x\}.$$

*Proof.* Along with the 1-stationary processes  $\hat{\mathbf{y}}_{1,n}(t)$ , let us introduce another 1-stationary processes  $\hat{\mathbf{Y}}_{2,n}(t)$ . This last process is related to the same  $M/GI/2/n$  queueing system as the process  $\hat{\mathbf{y}}_{1,n}(t)$ , and is obtained by deleting time intervals when there are more than two or less than two customers in the system, and merging the ends.

Using the same arguments as in the proof of Property 3.5 one can prove that the components of this process are generated by independent 1-stationary processes and having the same distribution. Indeed, involving as earlier in the proof of Property 3.4 the processes  $\zeta_1(t)$  and  $\zeta_2(t)$  having the same distribution, one can delete intervals where the system is empty and merge the ends. Apparently, the new processes  $\tilde{\zeta}_1(t)$  and  $\tilde{\zeta}_2(t)$  obtained after this procedure will have the same stationary distribution. (However, it will be shown later that the limiting stationary distribution of these one-dimensional processes differs from such the distribution obtained after the similar procedure for the  $M/GI/2/0$  system, and, therefore, have the one-dimensional distribution different than that of the process  $\mathbf{x}_1(t)$ .)

Let us go back to the initial process  $\mathbf{y}_{2,n}(t)$ ,  $n \geq 1$ , to delete the time intervals where the both servers are free and merge the corresponding ends. We also remove the last component corresponding to the queue-length  $Q_{2,n}(t)$ . (The exact value of

the queue-length  $Q_{2,n}(t)$  is irrelevant here and is not used in our analysis.) In the new time scale we obtain the two-dimensional process  $\tilde{\mathbf{y}}_{2,n}(t)$ .

Similarly to the proof of relation (3.4) we have time moments  $\tau^*$  and  $\tau^{**}$ . The first of them is a moment of arrival of a customer at the system with one busy and one free server, and the second one is the following after  $\tau^*$  moment of service completion of a customer when there remain one busy server only. The time interval  $[\tau^*, \tau^{**})$  is an *orbital busy period*. (The concept of orbital busy period is defined in Section 2 for Markovian systems. For  $M/GI/m/n$  queueing systems this concept is the same.) It can contain queueing customers waiting for their service. Let  $t^{\text{begin}}$  be a moment of arrival of a customer during the orbital busy period  $[\tau^*, \tau^{**})$  who occupies a waiting place, and let  $t^{\text{end}}$  be the following after  $t^{\text{begin}}$  moment of time when after the service completion the queue space becomes empty again. A period of time  $[t^{\text{begin}}, t^{\text{end}})$  is called *queueing period*. (Note that for the  $M/GI/m/n$  queueing system, the intervals of type  $[\tau^*, \tau^{**})$  are an analogue of the intervals of type  $[s_{m-1,k}, t_{m-1,k})$  in the Markovian queueing system  $M/M/m/n$ , and the intervals of type  $[t^{\text{begin}}, t^{\text{end}})$  are an analogue of the intervals of type  $[s_{m,k}, t_{m,k})$  in that Markovian queueing system  $M/M/m/n$ .)

All customers of *queueing periods*, i.e. those arrived during orbital busy period can be considered as customers arriving in a tagged server station. At the moment of  $t^{\text{begin}}$ , which is an instant of a Poisson arrival, the two-dimensional distribution of the random vector  $\tilde{\mathbf{y}}_{2,n}(t^{\text{begin}})$  coincides with the stationary distribution of the random vector  $\hat{\mathbf{Y}}_{2,n}(t)$ . However, in the point  $t^{\text{end}}$ , the probability distribution of  $\tilde{\mathbf{y}}_{2,n}(t^{\text{end}})$  is different from the stationary distribution of  $\hat{\mathbf{Y}}_{2,n}(t)$ , because this specific time instant  $t^{\text{end}}$  coincides with a service beginning in one of servers of the main system. Therefore, deleting the interval  $[t^{\text{begin}}, t^{\text{end}})$  and merging the end leads to the change of the distribution.

More specifically, at the time instant  $t^{\text{end}}$  one of the components of the vector  $\hat{\mathbf{Y}}_{2,n}(t)$ , say the first one, is a random variable having the probability distribution  $G(x)$ . Then, another component, i.e. the second one, because of the aforementioned properties of 1-stationary processes, coincides in distribution with  $\tilde{\zeta}_{1,n}$  (or  $\tilde{\zeta}_{2,n}$ ), which is a component of the stationary process  $\hat{\mathbf{Y}}_{2,n}(t)$ . Indeed, let customers arriving in a busy system and waiting in the queue be assigned to the tagged server station. At the moment of Poisson arrival  $t^{\text{begin}}$  of a customer in the tagged server station, the two-dimensional Markov process associated with the main queueing system has the same distribution as the vector  $\hat{\mathbf{Y}}_{2,n}(t)$ , i.e. two-dimensional distribution coinciding with the joint distribution of  $\tilde{\zeta}_{1,n}$  and  $\tilde{\zeta}_{2,n}$ . Then, at the moment of the service completion  $t^{\text{end}}$ , which coincides with the moment of service completion in one of two servers, the probability distribution function of the residual service time in another server, where the service is being continued, coincides with the distribution of a component of the vector  $\hat{\mathbf{Y}}_{2,n}(t)$ , i.e. with the distribution of  $\tilde{\zeta}_{1,n}$ .

If the probability distribution function  $G(x)$  belongs to the class NBU, then the 1-stationary process  $\tilde{\mathbf{y}}_{2,n}(t)$  satisfies the property  $\tilde{\mathbf{y}}_{2,n}(t^{\text{begin}}) \leq_{st} \tilde{\mathbf{y}}_{2,n}(t^{\text{end}})$ . If  $G(x)$  belongs to the class NWU, then the opposite inequality holds:  $\tilde{\mathbf{y}}_{2,n}(t^{\text{end}}) \leq_{st} \tilde{\mathbf{y}}_{2,n}(t^{\text{begin}})$ . (The stochastic inequality between vectors means the stochastic inequality between their corresponding components.)

The above stochastic inequalities are between random values of the process  $\tilde{\mathbf{y}}_{2,n}(t)$  in the different time instants  $t^{\text{begin}}$  and  $t^{\text{end}}$ . Our further task is to compare two different processes  $\tilde{\mathbf{y}}_{2,n}(t)$  and  $\tilde{\mathbf{y}}_{2,0}(t)$ . The first of these processes is associated with the  $M/GI/m/n$  queueing system, while the second one is associated with the  $M/GI/m/0$  queueing system. The idea of comparison is very simple. Suppose that both queueing systems are started at zero, i.e. consider the paths of these systems when the both of them are not in steady state, and compare the Markov processes associated with these systems. For the non-stationary processes we will use the same notation  $\tilde{\mathbf{y}}_{2,n}(t)$  and  $\tilde{\mathbf{y}}_{2,0}(t)$  understanding that it is spoken about usual (not stationary) Markov processes. The notation for time moments such as  $t^{\text{begin}}$  and  $t^{\text{end}}$  is now associated with these usual (i.e. non-stationary) processes as well. We will consider the Markov processes associated with  $M/GI/2/n$  and  $M/GI/2/0$  queueing systems in the same probability space. In the time interval  $[0, t^{\text{begin}})$  the paths of the Markov processes  $\tilde{\mathbf{y}}_{2,n}(t)$  and  $\tilde{\mathbf{y}}_{2,0}(t)$  coincide ( $n \neq 0$ ). However, after deleting the interval  $[t^{\text{begin}}, t^{\text{end}})$  and merging the ends, then in the end point  $t^{\text{begin}}$  the values of the processes  $\tilde{\mathbf{y}}_{2,n}(t)$  and  $\tilde{\mathbf{y}}_{2,0}(t)$  will be different. Indeed, in the case of the process  $\tilde{\mathbf{y}}_{2,0}(t)$ , which is associated with the  $M/GI/m/0$  queueing system,  $t^{\text{begin}}$  and  $t^{\text{end}}$  is the same point, and the value of Markov processes will be the same after replacing the points  $t^{\text{begin}}$  with  $t^{\text{end}}$ . However, in the case of the process  $\tilde{\mathbf{y}}_{2,n}(t)$  associated with  $M/GI/m/n$  queueing system, the values in these points will be different with probability 1, and consequently, because of the inequality  $\tilde{\mathbf{y}}_{2,n}(t^{\text{begin}}) \leq_{st} \tilde{\mathbf{y}}_{2,n}(t^{\text{end}})$  we will have  $\tilde{\mathbf{y}}_{2,0}(t^{\text{begin}}) \leq_{st} \tilde{\mathbf{y}}_{2,n}(t^{\text{begin}})$  in the case when  $G(x)$  belongs to the class NBU. If  $G(x)$  belongs to the class NWU, we will have the opposite inequality:  $\tilde{\mathbf{y}}_{2,n}(t^{\text{begin}}) \leq_{st} \tilde{\mathbf{y}}_{2,0}(t^{\text{begin}})$ .

Therefore, after deleting all the intervals of the type  $[t^{\text{begin}}, t^{\text{end}})$  from the original Markov process we obtain new Markov process, and in the case when  $G(x)$  belongs either to the class NBU or to the class NWU one can apply the theorem of Kalmykov [17] (see also [19]) to compare these two Markov processes. In the case where  $G(x)$  belongs to the class NBU, all the path of the Markov process, associated with  $M/GI/2/n$  is not smaller (in stochastic sense) than that path of the Markov process, associated with  $M/GI/2/0$ . If  $G(x)$  belongs to the class NWU, then the opposite stochastic inequality holds between two different Markov processes. Apparently, the same stochastic inequalities remain correct if we speak about stationary Markov processes. Nothing is changed if we let  $t$  to increase to infinity and arrive at stationary distributions. So, under the assumption that  $G(x)$  belongs to the class NBU, for the stationary processes we obtain  $\hat{\mathbf{y}}_{1,0}(t) \leq_{st} \hat{\mathbf{y}}_{1,n}(t)$ . In other words, due to the fact that  $\hat{\mathbf{y}}_{1,0}(t) =_{st} \mathbf{x}_1(t)$ , we obtain that  $\mathbf{x}_1(t) \leq_{st} \hat{\mathbf{y}}_{1,n}(t)$ . In the case where  $G(x)$  belongs to the class NWU, the opposite inequality holds.

The arguments of the proof given for  $m = 2$  remain correct for an arbitrary  $m \geq 2$ . The proof given by induction uses decomposition of the original system into the main system and a tagged server station as above. The further arguments for stochastic comparison of Markov processes are also easily extended for the case of an arbitrary  $m \geq 2$ .  $\square$

From the above results for the Markov processes the statement of the lemma follows. The stochastic inequalities between  $T_{m,n}(m-1)$  and  $T_{m,0}(m-1)$  follow by the coupling arguments. The lemma is completely proved.  $\square$

4. THEOREMS ON LOSSES IN  $M/GI/m/n$  QUEUEING SYSTEMS

The results obtained in the previous section enable us to establish theorems for the number of losses in  $M/GI/m/n$  queueing systems during their busy periods.

**Theorem 4.1.** *Under the assumption  $\lambda = m\mu$ , the expected number of losses during a busy period of the  $M/GI/m/n$  queueing system is the same for all  $n \geq 1$ .*

*Proof.* Consider the system  $M/GI/m/n$  under the assumption  $\lambda = m\mu$ , and similarly to the construction in the proof of Lemma 3.1 let us delete all the intervals where the number of customers in the system is less than  $m$ , and merge the corresponding ends. The process obtained is denoted  $\hat{\mathbf{y}}_{m,n}(t)$ . This is the 1-stationary process of orbital busy periods.

The stationary departure process, together with the arrival Poisson process of rate  $\lambda$  and the number of waiting places  $n$  describes the stationary  $M/G/1/n$  queue-length process (with generally dependent service times). As soon as a busy period is finished (in our case it is an orbital busy period, see Section 2 for the definition), the system immediately starts a new busy period by attaching a new customer into the system. This unusual situation arises because of the construction of the process. There are no idle periods, and servers all are continuously busy. Thus, the busy period, which is considered here, is one of the busy periods attached one after another.

Let  $T$  be a large period of time, and during that time there are  $K(T)$  busy periods of the  $M/G/1/n$  queueing system (which does not contain idle times as mentioned). Let  $L(T)$  and  $\nu(T)$  denote the number of lost and served customers during time  $T$ . We have the formula:

$$(4.1) \quad \lim_{T \rightarrow \infty} \frac{1}{\mathbf{E}K(T)} \left( \mathbf{E}L(T) + \mathbf{E}\nu(T) \right) = \lim_{T \rightarrow \infty} \frac{1}{\mathbf{E}K(T)} \left( \lambda T + \mathbf{E}K(T) \right),$$

the proof of which is given below.

Relationship (4.1) has the following explanation. The left-hand side term  $\mathbf{E}L(T) + \mathbf{E}\nu(T)$  is the expectation of the number of lost customers plus the expectation of the number of served customers during time  $T$ , and the right-hand side term  $\lambda T + \mathbf{E}K(T)$  is the expectation of the number of arrivals during time  $T$  plus the expected number of attached customers.

Relationship (4.1) can be proved by renewal arguments as follows.

There are  $m$  independent copies  $\mathbf{x}_1^{(1)}(t), \mathbf{x}_1^{(2)}(t), \dots, \mathbf{x}_1^{(m)}(t)$  of the stationary renewal process, which model the process  $\hat{\mathbf{y}}_{m,n}(t)$ . (In fact, we have  $m$  1-stationary processes, which have the same distributions as  $m$  independent renewal processes  $\mathbf{x}_1^{(1)}(t), \mathbf{x}_1^{(2)}(t), \dots, \mathbf{x}_1^{(m)}(t)$ .) Let  $1 \leq i \leq m$ , and let  $C_1, C_2, \dots, C_{K_i(T)}$  be such points of busy period starts associated with the renewal process  $\mathbf{x}_1^{(i)}(t)$  (one of those  $m$  independent and identically distributed renewal processes), where  $K_i(T)$  denotes the total number of these regeneration point indexed by  $i$ . Denote also by  $z_1, z_2, \dots, z_{K_i(T)}$  the corresponding lengths of busy periods, by  $\ell_1, \ell_2, \dots, \ell_{K_i(T)}$  the corresponding number of losses during these  $K_i$  busy periods, and by  $n_1, n_2, \dots, n_{K_i(T)}$  the corresponding number of served customers during these busy periods. Let  $T_i = z_1 + z_2 + \dots + z_{K_i(T)}$ , let  $L_i(T) = \ell_1 + \ell_2 + \dots + \ell_{K_i(T)}$  and let  $\nu_i(T) = n_1 + n_2 + \dots + n_{K_i(T)}$ .

Since at the moments  $C_1, C_2, \dots, C_{K_i(T)}$  of the busy period starts the distribution of the above stationary Markov process of residual times is the same, then the numbers of losses  $\ell_1, \ell_2, \dots, \ell_{K_i(T)}$  and, respectively, the numbers of served customers  $n_1, n_2, \dots, n_{K_i(T)}$  during each of these busy periods have the same distributions, and one can apply the renewal reward theorem.

By the renewal reward theorem we have:

$$(4.2) \quad \lim_{T \rightarrow \infty} \frac{1}{mEK_i(T)} \left( EL_i(T) + E\nu_i(T) \right) = \lim_{T \rightarrow \infty} \frac{1}{mEK_i(T)} \left( \lambda ET_i + mEK_i(T) \right).$$

Taking into account that

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{EK(T)}{EK_i(T)} &= m, \\ \lim_{T \rightarrow \infty} \frac{EL(T)}{EL_i(T)} &= 1, \\ \lim_{T \rightarrow \infty} \frac{E\nu(T)}{E\nu_i(T)} &= 1, \end{aligned}$$

and

$$\lim_{T \rightarrow \infty} \frac{ET_i}{T} = 1,$$

because of the correspondence between the left- and right-hand sides, from (4.2) we arrive at (4.1). Thus, we bypass the fact that the times between departures are dependent, and thus (4.1) is actually obtained by application of the renewal reward theorem by a usual manner, like in the case of independent times between departures (see e.g. Ross [26], Karlin and Taylor [18]).

Together with (4.1) we have

$$(4.3) \quad \lim_{T \rightarrow \infty} \frac{1}{EK(T)} E\nu(T) = \lim_{T \rightarrow \infty} \frac{1}{EK(T)} m\mu T.$$

Let us now introduce the following notation. Let  $\zeta_n$  denote the length of an orbital busy period, and let  $L_n$  and  $\nu_n$  correspondingly denote the number of lost and served customers during that orbital busy period. Using the arguments of [5], we prove that  $EL_n = 1$  for all  $n \geq 1$ .

Indeed, from (4.1) and (4.3) we have the equations:

$$(4.4) \quad EL_n + E\nu_n = \lambda E\zeta_n + 1,$$

$$(4.5) \quad E\nu_n = m\mu E\zeta_n.$$

The substitution  $\lambda = m\mu$  into the system of equations (4.4) and (4.5) yields  $EL_n = 1$ .

Hence, during an orbital busy period there is exactly one lost customer in average for any  $n \geq 0$ . To finish the proof we need in a more deeper analysis. First, we should find the expected number of queueing periods during one orbital busy period. For this purpose, one can use the similar construction by deleting all the intervals when the number of customers in the system is not greater than  $m$ , and merge the corresponding ends. The obtained process is denoted  $\hat{\mathbf{y}}_{m+1,n}(t)$ , and this is one stationary process of queueing periods following one after another.

The structure of the process  $\hat{\mathbf{y}}_{m+1,n}(t)$  is similar to that of the process  $\hat{\mathbf{y}}_{m,n}(t)$ . The process  $\hat{\mathbf{y}}_{m+1,n}(t)$  describes the stationary  $M/GI/1/n - 1$  queueing system the service times of which are generally dependent. As soon as one busy period in this system is finished, a new customer starting a new busy period is immediately

attached into the system. Thus, the only difference between the processes  $\widehat{\mathbf{y}}_{m,n}(t)$  and  $\widehat{\mathbf{y}}_{m+1,n}(t)$  is that the numbers of waiting places differ by value of parameter  $n$ . Therefore, using the similar notation and arguments, one arrive at the conclusion that the expected number of losses per queueing period is equal to 1 as well. Therefore, in long-run period of time, the number of queueing periods and orbital busy periods is the same in average. So, there is exactly one queueing period per orbital busy period in average.

Therefore, during a long-run period the number of events that the different Markov processes  $\widehat{\mathbf{y}}_{m-1,n}(t)$  change their values after deleting queueing periods and merging the ends (as exactly explained in the proof of Lemma 3.1) is the same in average for all  $n \geq 1$ , and the stationary characteristics of all of these Markov processes  $\widehat{\mathbf{y}}_{m-1,n}(t)$ , given for different values  $n=1,2,\dots$ , are the same. Hence, the expectation  $ET_{m,n}(m-1)$  is the same for all  $n=1,2,\dots$  as well. (Recall that  $T_{m,n}(m-1)$  denote the total time during a busy period when  $m-1$  servers are occupied.)

Hence, using Wald's identity connecting  $ET_{m,n}(m-1)$  with  $EL_{m,n}$  (the expected number of losses during a busy period) we arrive at the desired result, since  $ET_{m,n}(m-1)$  and the expectation of the number of orbital busy periods during a busy period of  $M/GI/m/n$  both are the same for all  $n \geq 1$ .  $\square$

Application of Lemma 3.1 and the arguments of Theorem 4.1 enables us to prove the following result.

**Theorem 4.2.** *Let  $\lambda = m\mu$ . Then, under the assumption that  $G(x)$  belongs to the class NBU, for the number of losses in  $M/GI/m/n$  queueing systems,  $n \geq 1$ , we have*

$$(4.6) \quad EL_{m,n} = \frac{cm^m}{m!},$$

where the constant  $c \geq 1$  depends on  $m$  and the probability distribution  $G(x)$  but is independent of  $n$ .

Under the assumption that  $G(x)$  belongs to the class NWU we have (4.6) but with constant  $c \leq 1$ .

*Proof.* Notice first, that for the expected number of losses in  $M/GI/m/0$  queueing systems we have

$$EL_{m,0} = \frac{m^m}{m!}$$

This result follows immediately from the Erlang-Sevastyanov formula [27], so that the expected number of losses during a busy period of the  $M/GI/m/0$  queueing system is the same that this of the  $M/M/m/0$  queueing system. The expected number of losses during a busy period of the  $M/M/m/0$  queueing system,  $EL_{m,0} = \frac{m^m}{m!}$ , has also been established in Section 2.

In the case where  $G(x)$  belongs to the class NBU according to Lemma 3.1 we have  $ET_{m,n}(m-1) \geq ET_{m,0}(m-1)$ , and therefore, the expected number of orbital busy periods in the  $M/GI/m/n$  queueing system ( $n \geq 1$ ) is not smaller than this in the  $M/GI/m/0$  queueing system. Therefore, repeating the proof of Theorem 4.1 leads to the inequality  $EL_{m,n} \geq EL_{m,0}$  and consequently to the desired result. If  $G(x)$  belongs to the class NWU, then we have the opposite inequalities, and finally the corresponding result stated in the formulation of the theorem.  $\square$

## 5. BATCH ARRIVALS

The case of batch arrivals is completely analogous to the case of ordinary (non-batch) arrivals. In the case of a Markovian  $M^X/M/m/n$  queueing system one can also apply the level-crossing method to obtain equations analogous to (2.6)-(2.13). The same arguments as in Sections 3 and 4 in an extended form can be used for  $M^X/GI/m/0$  queueing systems.

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SCHOOL OF MATHEMATICAL SCIENCES, MONASH UNIVERSITY, BUILDING 28M, CLAYTON CAMPUS, WELLINGTON ROAD, VIC-3800, AUSTRALIA

*E-mail address:* vyacheslav.abramov@sci.monash.edu.au